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Bath-induced Control of Two-qubit Entanglement under Markovian Noises

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***Abstract*— An entanglement control strategy is presented for two-qubit quantum systems against Markovian noises. This proposal is realized by a tunable coupling between qubits which is induced by varying the parameters of an intermediate squeezed field. Its applications to the independent and collective amplitude damping decoherence channels and their mixture show that entanglement can be efficiently enhanced.**

I. INTRODUCTION

Quantum entanglement [1], [2], [3] is a unique quantum phenomenon in which the states of multiple quantum objects are correlated and the operations on one object will disturb the rest ones. Quantum entanglement is critical to the realization of high-speed quantum computation and high-security quantum communication. However, in reality, the quantum systems exposed in environments [4] may suffer from decoherence effects that may lead to the deterioration of quantum entanglement [5]. Hence, the active protection of quantum entanglement against decoherence effects becomes important.

In existing entanglement-protection strategies, a collective decoherence channel (e.g., a common heat bath) for all subsystems [6], [7] is often assumed. In this special case, a subspace of quantum states, called decoherence-free subspace, that is unaffected by decoherence can be found to storage entangled states. However, in more general cases, the entanglement would tend to disappear, especially via independent decoherence channels [8] (i.e., each subsystem is coupled to independent modes in the environment). Even worse, a combination of independent decoherence channels and collective decoherence channels may destroy the decoherence-free subspace and lead to the failure of entanglement-protection. This paper shows that in such cases entanglement can be partially protected at least for two-qubit systems by coupling the two qubits with an auxiliary manipulatable squeezed field. Here, "qubit" refers to the simplest unit of quantum information which is physically implemented by a two-level quantum-mechanical system. The paper is organized as follows: Section II introduces the model and derives the effective two-qubit Hamiltonian by tracing out the degrees of freedom of the auxiliary field in the dispersive regime. Section III proposes the entanglement control under three kinds of decoherence channels, and more general cases are discussed in Section IV. Conclusions and perspectives are drawn in Section V.

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II. MODEL FORMULATION

This paper discusses a system of two (identical) qubits that are indirectly interacted with each other through an intermediate single-mode squeezed field, e.g., a pair of two-state atoms in a single-mode optical cavity [9], [10] or two superconducting qubits both coupled to a one-dimensional transmission line resonator [11]. Under the rotating-wave approximation, the total Hamiltonian of the two qubits and the single-mode field can be expressed as:

$$\begin{aligned} H &= \omega_c a^\dagger a + \sum_{j=1}^{2} \frac{\omega_a}{2}\sigma_{zj} + \sum_{j=1}^{2} g(a^\dagger \sigma_{-j} + a\sigma_{+j}) \\ &\quad + \xi\left[e^{-i(\Omega t+\phi_0)}a^{\dagger 2} + e^{i(\Omega t+\phi_0)}a^2\right], \end{aligned} \tag{1}$$

where the Planck constant $\hbar$ has been assumed to be 1. The first term in H is the free Hamiltonian of the field with ω_c being the oscillating frequency. a and $a^\dagger$ are the annihilation and creation operators of the field mode. The second term in H denotes the free Hamiltonian of the two qubits, where ω_a is determined by the energy gap between the two eigen states of a single qubit and σ_{zj} is the z-axis Pauli matrix of the qubit j. The third term in H represents the interaction between the two qubits and the field with the coupling strength $g \in \mathbb{R}$. $\sigma_{\pm j} = \sigma_{xj} \pm i\sigma_{yj}$ denote the ladder operators of the qubit j. The last term in H represents the squeezed effects of the field, where *the effective amplitude ξ, the frequency Ω, and the initial phase ϕ_0 of the squeezed field are all tunable parameters in our strategy*. Such a controllable squeezed field could be realized in optical cavities (see, e.g., [12] and [13]), and our recent work shows that it is also realizable in superconducting circuits (see Sec. V of Ref. [14]).

Under the dispersive-detuning condition: $\Delta = \omega_a - \omega_c, |\xi| \gg |g|$, the Hamiltonian H can be diagonalized by the following unitary transformation [14], [15]:

$$U = \exp\left[\frac{g}{\Delta}\sum_{j=1}^{2}(a\sigma_{+j} - a^\dagger\sigma_{-j})\right].$$

In fact, expanding the exponential terms to the first two orders of g/Δ, we can obtain:

$$\begin{aligned} UHU^\dagger &\approx \omega_c a^\dagger a + \xi e^{-i(\Omega t+\phi_0)} a^{\dagger 2} + \xi e^{i(\Omega t+\phi_0)} a^2 \\ &+ \sum_{j=1}^{2} \left[\frac{\tilde{\omega}_a}{2} + \frac{4g^2}{\Delta^2} (\xi e^{-i(\Omega t+\phi_0)} a^{\dagger 2} + h.c.) + \frac{4g^2}{\Delta} a^\dagger a \right] \sigma_{zj} \\ &+ \sum_{j=1}^{2} \left[\left(\frac{2g\xi}{\Delta} a^\dagger + \frac{g^2\xi}{\Delta^2} \right) e^{-i(\Omega t+\phi_0)} \sigma_{+j} + h.c. \right] \\ &+\mu_1 (e^{-i(\Omega t+\phi_0)} \sigma_{+1}\sigma_{+2} + e^{i(\Omega t+\phi_0)} \sigma_{-1}\sigma_{-2}) \\ &+\mu_2(\sigma_{+1}\sigma_{-2} + \sigma_{-1}\sigma_{+2}), \end{aligned}$$

where

$$\tilde{\omega}_a = \omega_a + \frac{4g^2}{\Delta}, \quad \mu_1 = \frac{2g^2}{\Delta^2}\xi, \quad \mu_2 = \frac{g^2}{\Delta}, \tag{2}$$

and $h.c.$ means Hermitian conjugate. The Hamiltonian can be reduced by adiabatically eliminating the degrees of freedom of the field mode:

$$\begin{aligned} \tilde{H}_A^{\text{eff}} &= \sum_{j=1}^{2} \frac{\omega_a}{2} \sigma_{zj} + \mu_2(\sigma_{+1}\sigma_{-2} + \sigma_{-1}\sigma_{+2}) \\ &\quad +\mu_1 (e^{-i(\Omega t+\phi_0)} \sigma_{+1}\sigma_{+2} + e^{i(\Omega t+\phi_0)} \sigma_{-1}\sigma_{-2}). \end{aligned}$$

Here, we have omitted all the single-qubit terms induced by the interaction between qubits and the field mode due to the fact that:

$$\frac{\omega_a}{2} \gg \frac{g^2}{\Delta}, \frac{g\xi}{\Delta}, \frac{g^2\xi}{\Delta^2},$$

under the dispersive-detuning condition $\Delta \gg g$ and $\omega_a \gg \xi$ (from experimental parameters in optical cavities and superconducting circuits). In the interaction picture, $\tilde{H}_A^{\text{eff}}$ can be further expressed as:

$$\begin{aligned} H_A^{\text{eff}} &= \left(e^{i\sum_{j=1}^2 \frac{\omega_a t}{2} \sigma_{zj}} \right) \tilde{H}_A^{\text{eff}} \left(e^{-i\sum_{j=1}^2 \frac{\omega_a t}{2} \sigma_{zj}} \right) \\ &= \mu_1 \left(e^{-i\phi_0} \sigma_{+1}\sigma_{+2} + e^{i\phi_0} \sigma_{-1}\sigma_{-2} \right) \\ &\quad +\mu_2 \left(\sigma_{+1}\sigma_{-2} + \sigma_{-1}\sigma_{+2} \right), \end{aligned} \tag{3}$$

when the frequency Ω is set to be $2\omega_a$.

According to Eq. (2), one can continuously tune the coupling strength μ_1 by varying the amplitude ξ of the intermediate squeezed field. Thus, in the following discussions, μ_1 and ϕ_0 are taken as control parameters.

III. Three Special Decoherence Channels

Besides the single-mode control field, the qubits also interact with decoherence channels. The *ab initio* model for this open quantum system can be constructed from the Schrödinger equation of the composition of the two-qubit system plus their environment. Since the environment is generally an infinite-dimensional system, it is extremely difficult to analyze the composite system. In this regard, the environmental freedoms are usually averaged out when the Born-Markov approximation is satisfied, leaving the dynamics of the two-qubit system represented by the so-called master equation [16].

This section will consider three kinds of decoherence channels, namely, the independent amplitude damping decoherence channel, the collective amplitude damping decoherence channel and a mixture of them. More general cases will be discussed in the next section.

A. Independent amplitude damping decoherence channel

The independent amplitude damping decoherence channel can be expressed as the following master equation:

$$\dot{\rho} = -i[H_A^{\text{eff}}, \rho] + \Gamma \sum_{j=1}^{2} \mathscr{D}[\sigma_{-j}]\rho, \tag{4}$$

where $[A,B] = AB - BA$ and the superoperator $\mathscr{D}[L]\rho$ is defined as:

$$\mathscr{D}[L]\rho = L\rho L^\dagger - \frac{1}{2}L^\dagger L\rho - \frac{1}{2}\rho L^\dagger L;$$

$\Gamma > 0$ represents the relaxation rate of each qubit. Such decoherence channels can describe two qubits in an environment when they are spatially so separated that their surrounding environments are completely independent. For the example of two atoms in an optical cavity, decoherence channels on two atoms can be taken as independent channels when the distance between two atoms is far larger than the resonant wavelength of a single atom [9].

In this work, we will use the concurrence $C(\rho)$:

$$C(\rho) = \max\{\lambda_1 - \lambda_2 - \lambda_3 - \lambda_4, 0\}, \tag{5}$$

to measure the quantum entanglement between two qubits [17], where λ_i's are the square roots of the eigenvalues, in decreasing order, of the matrix:

$$\mathscr{M}(\rho) = \rho(\sigma_{y1}\sigma_{y2})\rho^*(\sigma_{y1}\sigma_{y2}).$$

ρ^* in the above equation is the complex conjugate of ρ.

The concurrence of an uncontrolled system always decreases to zero when coupled to independent amplitude damping decoherence channels [8]. This fact can be shown from the solution $\rho(t)$ of Eq. (4) for $H_A^{\text{eff}} = 0$, which decays to the two-qubit ground state $\rho_\infty^u = |00\rangle\langle 00|$ when $t \to \infty$, where the superscript "u" refers to the "uncontrolled" qubit system. Since $\rho_\infty^u = |00\rangle\langle 00|$ is a separable state, i.e., a non-entangled state with zero concurrence, the entanglement between the two qubits is completely lost.

However, the entanglement can be partially protected via the intermediate squeezed field, because the solution $\rho(t)$ of Eq. (4) tends to a stationary state

$$\rho_\infty = \left(\frac{2\mu_1\Gamma}{4\mu_1^2 + \Gamma^2} \right) \rho_m + \left(1 - \frac{2\mu_1\Gamma}{4\mu_1^2 + \Gamma^2} \right) \rho_s, \tag{6}$$

as a convex combination of a maximally-entangled state, i.e., an entangled state with maximum concurrence,

$$\rho_m = \frac{1}{2} \begin{pmatrix} 1 & & & e^{-i\left(\phi_0 - \frac{\pi}{2}\right)} \\ & 0 & & \\ & & 0 & \\ e^{i\left(\phi_0 - \frac{\pi}{2}\right)} & & & 1 \end{pmatrix}, \tag{7}$$

and a diagonal separable state

$$\rho_s = \text{diag}(1-3\beta, \beta, \beta, \beta),$$

where

$$\beta = \frac{1}{8}\left(1-\sqrt{1-\left(\frac{4\mu_1\Gamma}{4\mu_1^2+\Gamma^2}\right)^2}\right).$$

The corresponding stationary concurrence and fidelity between the stationary state ρ_∞ and the maximally-entangled state ρ_m are, respectively,

$$\begin{aligned} C(\rho_\infty) &= \max\left\{\frac{2\mu_1(\Gamma-\mu_1)}{4\mu_1^2+\Gamma^2}, 0\right\}, \\ F(\rho_\infty) &= \text{tr}(\rho_m\rho_\infty) = \frac{\mu_1(\Gamma-\mu_1)}{4\mu_1^2+\Gamma^2}+\frac{1}{2}. \end{aligned} \tag{8}$$

It can be verified from Eq. (8) that the maximum concurrence and fidelity

$$C_{\max} = \frac{\sqrt{5}-1}{4} \approx 0.31, \quad F_{\max} = \frac{\sqrt{5}+3}{8} \approx 0.65 \tag{9}$$

are achieved when the control parameter μ_1 is tuned to be:

$$\mu_1 = \frac{1}{\sqrt{5}+1}\Gamma, \tag{10}$$

which is realizable in a low-decay open quantum system (e.g., a low-decay atom-optical system [10] or a superconducting circuit-QED system [11]).

It should be pointed out that, as shown in Eq. (9), the fidelity between the stationary state ρ_∞ and the maximally-entangled state ρ_m (≤ 0.65) may not be strong enough to be applied in quantum information processing. Nonetheless, the proposed strategy is still effective for entanglement-protection. Since the maximum fidelity exceeds 0.5, we can in principle asymptotically obtain the maximally-entangled states by the entanglement-purification strategies [18], i.e., to purify the given quantum state by quantum measurements and unitary operations to increase the proportion of the maximally-entangled state.

B. Collective amplitude damping decoherence channel

The master equation for collective amplitude damping decoherence channel is as follows:

$$\dot{\rho} = -i[H_A^{\text{eff}}, \rho] + \Gamma\mathscr{D}[S_-]\rho, \tag{11}$$

where the collective lowering operator

$$S_- = \sigma_{-1} + \sigma_{-2},$$

and $\Gamma > 0$ denotes the collective damping rate. Such a decoherence channel can be used to describe the case in which the two qubits are strongly coupled to each other. For the example of two atoms in an optical cavity, a collective decoherence channel can be obtained when the distance between the two atoms is far shorter than the resonant wave length of a single atom [9], i.e., when the two atoms are so close and strongly coupled to each other that they see almost the same environment.

In absence of controls, i.e., $H_A^{\text{eff}} = 0$ in Eq. (11), the stationary state of the two-qubit system

$$\rho_\infty^u = (1-\lambda)\tilde{\rho}_m + \lambda\rho_0 \tag{12}$$

is a convex combination of the maximally-entangled state

$$\tilde{\rho}_m = \frac{1}{2}\begin{pmatrix} 0 & & & \\ & 1 & -1 & \\ & -1 & 1 & \\ & & & 0 \end{pmatrix} \tag{13}$$

and the two-qubit ground state $\rho_0 = |00\rangle\langle 00|$. The weight $\lambda \in [0,1]$ is determined by the initial density matrix $\rho(t_0)$:

$$\lambda = \text{tr}\left[\left(\frac{1}{4}\sigma_{z1}\sigma_{z2}\right)\rho(t_0)\right] + \frac{\sqrt{2}}{2}\text{tr}(\Omega_{23}^x\rho(t_0)) + \frac{3}{4},$$

where

$$\Omega_{23}^x = \frac{1}{\sqrt{2}}\begin{pmatrix} & & & 0 \\ & & 1 & \\ & 1 & & \\ 0 & & & \end{pmatrix}.$$

The concurrence of ρ_∞^u is $C(\rho_\infty^u) = 1-\lambda$.

In presence of controls, the stationary state of Eq. (11) is the following convex combination

$$\rho_\infty = q\tilde{\rho}_m + p\rho_m + (1-q-p)\tilde{\rho}_s, \tag{14}$$

of two maximally-entangled states $\tilde{\rho}_m$ and ρ_m defined in Eqs. (7) and (13), and a diagonal separable state

$$\tilde{\rho}_s = \text{diag}\left(\tilde{\beta}_1+\tilde{\beta}_2, \frac{1}{2}-\tilde{\beta}_2, \frac{1}{2}-\tilde{\beta}_2, \tilde{\beta}_2-\tilde{\beta}_1\right),$$

where

$$\tilde{\beta}_1 = \frac{\lambda\Gamma^2}{2(\Gamma^2+3\mu_1^2)}, \quad \tilde{\beta}_2 = \frac{\lambda(\Gamma^2+2\mu_1^2)}{2(\Gamma^2+3\mu_1^2)}.$$

The weights q and p in Eq. (14) are, respectively,

$$q = 1-\frac{7}{6}\lambda + \frac{\Gamma^2-3\mu_1^2}{\Gamma^2+3\mu_1^2}\lambda, \quad p = \frac{2\Gamma\mu_1\lambda}{\Gamma^2+3\mu_1^2}.$$

When the control parameter μ_1, which is used to adjust the stationary entanglement, is in the range:

$$\frac{\lambda-\sqrt{-9\lambda^2+22\lambda-12}}{6-5\lambda} \leq \frac{\mu_1}{\Gamma} \leq \frac{\lambda+\sqrt{-9\lambda^2+22\lambda-12}}{6-5\lambda},$$

the stationary concurrence under control exceeds that of the uncontrolled system, i.e.,

$$C(\rho_\infty) = \frac{(\Gamma^2+2\mu_1^2+2\Gamma\mu_1)\lambda}{\Gamma^2+3\mu_1^2} - 1 \geq C(\rho_\infty^u) = 1-\lambda.$$

Note that the above interval of the control parameter μ_1 is non-empty only when $\lambda \in \left[\frac{11}{9}-\frac{1}{9}\sqrt{13}, 1\right]$. Otherwise, our strategy cannot improve the stationary entanglement. Calculations also show that when $\lambda \in \left[\frac{11}{9}-\frac{1}{9}\sqrt{13}, 1\right]$ the controlled stationary concurrence $C(\rho_\infty)$ is maximally valued when

$$\mu_1 = \frac{2}{\sqrt{13}+1}\Gamma, \tag{15}$$

and the corresponding maximum value is

$$C_{\max} = \frac{\sqrt{13}+5}{6}\lambda - 1. \tag{16}$$

The above analysis shows that the stationary state of the uncontrolled system may stay entangled under the collective amplitude damping decoherence channel. This is the starting point of some existing dissipation-induced entanglement-protection strategies (see, e.g., [6], [7]). The analysis also exhibits evident enhancement of the stationary entanglement by our strategy when λ is large enough. The resulting controlled stationary entangled state (14) is indeed a mixture of two maximally-entangled states $\tilde{\rho}_m$ and ρ_m, where $\tilde{\rho}_m$ is induced by the collective decoherence channel and ρ_m comes from our strategy. When $\tilde{\rho}_m$ is less dominant in the uncontrolled stationary state, i.e., the parameter λ is large, the tradeoff between $\tilde{\rho}_m$ and ρ_m may increase the final entanglement. In the opposite situation, such a tradeoff may not increase the final entanglement and our strategy does not work.

C. Mixed amplitude damping decoherence channel

The analysis in subsection III-B shows that the dissipation-induced strategy may outperform our strategy for a perfect collective decoherence channel. However, in laboratory the decoherence channel is impossible to be perfectly collective. For the example of two atoms in an optical cavity, a perfect collective amplitude damping decoherence channel is available only when the distance between the two atoms is far shorter than the resonant wavelength of a single atom, which is actually impossible in reality. The present atom trapping and cooling techniques can only hold two atoms approximately at the distance of the same order of the resonant wavelength of the atom (see, e.g., Ref. [9]). Thus, the resulting decoherence channel is between an independent amplitude damping decoherence channel and a collective amplitude damping decoherence channel, as shown in the following master equation:

$$\begin{aligned}\dot{\rho} &= -i[H_A^{\text{eff}},\rho] + \sum_{i=1}^{2}\Gamma\mathscr{D}[\sigma_{-j}]\rho \\ &\quad +\Gamma_{12}\left(\sigma_{-1}\rho\sigma_{+2} - \frac{1}{2}\{\sigma_{+2}\sigma_{-1},\rho\}\right) \\ &\quad +\Gamma_{12}\left(\sigma_{-2}\rho\sigma_{+1} - \frac{1}{2}\{\sigma_{+1}\sigma_{-2},\rho\}\right), \end{aligned} \tag{17}$$

where $0 < \Gamma_{12} < \Gamma$.

It can be verified that the stationary state of the uncontrolled system (i.e., $H_A^{\text{eff}} = 0$) is the separable two-qubit ground state $\rho_\infty^u = |00\rangle\langle 00|$, implying that entanglement will be completely lost in absence of control. The stationary behavior of this kind of mixed decoherence channels is just the same as the independent decoherence channel. Further, when we set the parameter μ_1 as in Eq. (10), we can obtain the same maximum concurrence and fidelity in Eq. (9).

IV. Generalized Decoherence Channels

Considering the case that the system undergoes simultaneously the relaxation and dephasing decoherence channels, we can obtain the following master equation:

$$\dot{\rho} = -i[H_A^{\text{eff}},\rho] + \mathscr{L}(\rho), \tag{18}$$

where

$$\begin{aligned}\mathscr{L}(\rho) &= \sum_{i,j=x,y}\Bigg(A_{ij}\left[\sigma_{i1}\rho\sigma_{j1} - \frac{1}{2}\{(\sigma_{j1}\sigma_{i1}),\rho\}\right] \\ &\quad +C_{ij}\left[\sigma_{i2}\rho\sigma_{j2} - \frac{1}{2}\{\sigma_{j2}\sigma_{i2},\rho\}\right] \\ &\quad +B_{ij}\left[\sigma_{i1}\rho\sigma_{j2} - \frac{1}{2}\{\sigma_{j2}\sigma_{i1},\rho\}\right] \\ &\quad +B_{ij}^*\left[\sigma_{j2}\rho\sigma_{i1} - \frac{1}{2}\{\sigma_{i1}\sigma_{j2},\rho\}\right]\Bigg) \\ &\quad +A_{zz}[\sigma_{z1}\rho\sigma_{z1} - \rho] + C_{zz}[\sigma_{z2}\rho\sigma_{z2} - \rho] \\ &\quad +B_{zz}\left[\sigma_{z1}\rho\sigma_{z2} - \frac{1}{2}\{\sigma_{z2}\sigma_{z1},\rho\}\right] \\ &\quad +B_{zz}^*\left[\sigma_{z2}\rho\sigma_{z1} - \frac{1}{2}\{\sigma_{z1}\sigma_{z2},\rho\}\right],\end{aligned}$$

and $\{A,B\} = AB + BA$.

In order to simplify the discussions, we introduce the so-called coherence vector picture (see, e.g., [19], [20], [21]). Let the inner product $\langle X,Y\rangle = \text{tr}(X^\dagger Y)$ and define the following matrix basis for all two-qubit matrices:

$$\begin{gathered}\left\{\tfrac{1}{2}I_{4\times 4},\, \Omega_{14}^x,\, \Omega_{14}^y,\, \Omega_{23}^x,\, \Omega_{23}^y,\, \tfrac{1}{2}\sigma_{x1},\, \tfrac{1}{2}\sigma_{y1},\right. \\ \tfrac{1}{2}\sigma_{x2},\, \tfrac{1}{2}\sigma_{y2},\, \tfrac{1}{2}\sigma_{x1}\sigma_{z2},\, \tfrac{1}{2}\sigma_{z1}\sigma_{x2},\, \tfrac{1}{2}\sigma_{y1}\sigma_{z2}, \\ \left.\tfrac{1}{2}\sigma_{z1}\sigma_{y2},\, \Omega_{14}^z, \Omega_{23}^z, \tfrac{1}{2}\sigma_{z1}\sigma_{z2}\right\},\end{gathered} \tag{19}$$

where Ω_{14}^x, Ω_{14}^y, Ω_{23}^x, Ω_{23}^y, Ω_{14}^z, Ω_{23}^z are defined as:

$$\Omega_{14}^x = \begin{pmatrix} & & & \frac{1}{\sqrt{2}} \\ & & 0 & \\ & 0 & & \\ \frac{1}{\sqrt{2}} & & & \end{pmatrix},\ \Omega_{14}^y = \begin{pmatrix} & & & \frac{-i}{\sqrt{2}} \\ & & 0 & \\ & 0 & & \\ \frac{i}{\sqrt{2}} & & & 0 \end{pmatrix},$$

$$\Omega_{23}^x = \begin{pmatrix} & & & 0 \\ & & \frac{1}{\sqrt{2}} & \\ & \frac{1}{\sqrt{2}} & & \\ 0 & & & \end{pmatrix},\ \Omega_{23}^y = \begin{pmatrix} & & & 0 \\ & & \frac{-i}{\sqrt{2}} & \\ & \frac{i}{\sqrt{2}} & & \\ 0 & & & \end{pmatrix},$$

$$\Omega_{14}^z = \begin{pmatrix} \frac{1}{\sqrt{2}} & & & \\ & 0 & & \\ & & 0 & \\ & & & \frac{-1}{\sqrt{2}} \end{pmatrix},\ \Omega_{23}^z = \begin{pmatrix} 0 & & & \\ & \frac{1}{\sqrt{2}} & & \\ & & \frac{-1}{\sqrt{2}} & \\ & & & 0 \end{pmatrix}.$$

Under this matrix basis, the system density matrix can be expanded as:

$$\rho = \frac{1}{4}I_{4\times 4} + \sum_{i=1}^{15} m_i\Omega_i,$$

where Ω_i, $i = 1,\cdots,15$, are the traceless matrices in Eq. (19) and $m_i = \text{tr}(\Omega_i\rho)$.

Let $m=(m_1,\cdots,m_{15})^T$. The master equation (18) can be rewritten as:

$$\dot{m}=8\mu_1\cos\phi_0 O_1 m+8\mu_1\sin\phi_0 O_2 m+8\mu_2 O_3 m+Dm+g,$$

where O_1, O_2, O_3 are the adjoint representation matrices (see Ref. [19]) of the non-local Hamiltonians:

$$\begin{aligned} H_1&=\tfrac{1}{4}(\sigma_{x1}\sigma_{x2}-\sigma_{y1}\sigma_{y2}),\\ H_2&=\tfrac{1}{4}(\sigma_{x1}\sigma_{y2}+\sigma_{y1}\sigma_{x2}),\\ H_3&=\tfrac{1}{4}(\sigma_{x1}\sigma_{x2}+\sigma_{y1}\sigma_{y2}),\end{aligned}$$

and "$Dm+g$" is the coherence vector representation of the Lindblad term $\mathscr{L}(\rho)$ in Eq. (18) with $D\le 0$ and g a constant vector.

Further, let

$$\begin{aligned} m^p &= (m_{14}^x,m_{14}^y,m_{23}^x,m_{23}^y)^T,\\ m^\eta &= (m_{14}^z,m_{23}^z,m_{zz})^T, \\ m^\varepsilon &= (m_{x0},m_{y0},m_{0x},m_{0y},m_{xz},m_{zx},m_{yz},m_{zy})^T,\end{aligned} \tag{20}$$

where

$$m_{14}^\alpha=\mathrm{tr}(\Omega_{14}^\alpha\rho),\, m_{23}^\beta=\mathrm{tr}(\Omega_{23}^\beta\rho),\ \alpha,\beta=x,y,z,$$
$$m_{\alpha\beta}=\mathrm{tr}\left[\left(\tfrac{1}{2}\sigma_{\alpha 1}\sigma_{\beta 2}\right)\rho\right],\ \alpha,\beta=0,x,y,z,$$

and $\sigma_{0j}=I_{2\times 2}$, $j=1,2$ is the 2×2 identity matrix acting on the qubit j. Then, corresponding to m^p, m^η, m^ε, the matrices D, g, O_i can be expressed as the following block forms:

$$D=\begin{pmatrix} D^{pp} & D^{p\eta} & \\ D^{\eta p} & D^{\eta\eta} & \\ & & D^\varepsilon\end{pmatrix},\ g=\begin{pmatrix}0\\ g^\eta\\ 0\end{pmatrix},$$

$$O_i=\begin{pmatrix} 0 & O_i^\eta & \\ -O_i^{\eta\,T} & 0 & \\ & & O_i^\varepsilon\end{pmatrix},\ i=1,2,3.$$

If we define

$$Q_1=\frac{1}{\sqrt{2}}\begin{pmatrix}0&0&1&-1\\1&1&0&0\\0&0&1&1\\-1&1&0&0\end{pmatrix},\, Q_2=\frac{1}{\sqrt{2}}\begin{pmatrix}1&1&0\\1&-1&0\\0&0&1\end{pmatrix},$$

the matrix blocks in D, g, O_i can be calculated as:

$$D^{pp}=-2Q_1^{-1}\begin{pmatrix} d_1 & 2B_{zz}^{\mathrm{Re}} & -C_{xy}^{\mathrm{Re}} & -A_{xy}^{\mathrm{Re}}\\ 2B_{zz}^{\mathrm{Re}} & d_2 & -A_{xy}^{\mathrm{Re}} & -C_{xy}^{\mathrm{Re}}\\ -C_{xy}^{\mathrm{Re}} & -A_{xy}^{\mathrm{Re}} & d_3 & -2B_{zz}^{\mathrm{Re}}\\ -A_{xy}^{\mathrm{Re}} & -C_{xy}^{\mathrm{Re}} & -2B_{zz}^{\mathrm{Re}} & d_4\end{pmatrix}Q_1,$$

$$D^{p\eta}=2Q_1^{-1}\begin{pmatrix} B_{yy}^{\mathrm{Im}} & B_{xx}^{\mathrm{Im}} & -B_{yx}^{\mathrm{Re}}\\ -B_{xx}^{\mathrm{Im}} & -B_{yy}^{\mathrm{Im}} & -B_{xy}^{\mathrm{Re}}\\ B_{yx}^{\mathrm{Im}} & -B_{xy}^{\mathrm{Im}} & B_{yy}^{\mathrm{Re}}\\ -B_{xy}^{\mathrm{Im}} & B_{yx}^{\mathrm{Im}} & B_{xx}^{\mathrm{Re}}\end{pmatrix}Q_2,$$

$$D^{\eta p}=2Q_2^{-1}\begin{pmatrix} -B_{yy}^{\mathrm{Im}} & B_{xx}^{\mathrm{Im}} & -B_{yx}^{\mathrm{Im}} & B_{xy}^{\mathrm{Im}}\\ -B_{xx}^{\mathrm{Im}} & B_{yy}^{\mathrm{Im}} & B_{xy}^{\mathrm{Im}} & -B_{yx}^{\mathrm{Im}}\\ -B_{yx}^{\mathrm{Re}} & -B_{xy}^{\mathrm{Re}} & B_{yy}^{\mathrm{Re}} & B_{xx}^{\mathrm{Re}}\end{pmatrix}Q_1,$$

$$D^{\eta\eta}=-2Q_2^{-1}\begin{pmatrix} d_5 & 0 & 0\\ 0 & d_6 & 0\\ -2C_{xy}^{\mathrm{Im}} & -2A_{xy}^{\mathrm{Im}} & d_5+d_6\end{pmatrix}Q_2,$$

$$g^\eta=2Q_2^{-1}\begin{pmatrix}A_{xy}^{\mathrm{Im}}\\ C_{xy}^{\mathrm{Im}}\\ 0\end{pmatrix},\, O_1^\eta=\begin{pmatrix}0&0&0\\-1&0&0\\0&0&0\\0&0&0\end{pmatrix}$$

$$O_2^\eta=\begin{pmatrix}1&0&0\\0&0&0\\0&0&0\\0&0&0\end{pmatrix},\, O_3^\eta=\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\\0&-1&0\end{pmatrix},$$

where M^{Re} (M^{Im}) is the real (imaginary) part of the complex coefficient M, and

$$d_1=A_{yy}+C_{xx}+A_{zz}+C_{zz},\ d_2=A_{xx}+C_{yy}+A_{zz}+C_{zz},$$
$$d_3=A_{yy}+C_{yy}+A_{zz}+C_{zz},\ d_4=A_{xx}+C_{xx}+A_{zz}+C_{zz},$$
$$d_5=A_{xx}^{\mathrm{Re}}+A_{yy}^{\mathrm{Re}},\ d_6=C_{xx}^{\mathrm{Re}}+C_{yy}^{\mathrm{Re}}.$$

Here, the matrix blocks D^ε, O_i^ε do not affect the stationary state of Eq. (18), so we omit them.

With the above preparations, we can present our main results:

Proposition 1: Let $v=(v_1,v_2,v_3,v_4)^T$, $\xi=(\xi_1,\xi_2,\xi_3)^T$ be the solution of the linear equation:

$$\begin{pmatrix} D^{pp} & D^{p\eta}+O_A^\eta\\ -\left(O_A^\eta\right)^T+D^{\eta p} & D^{\eta\eta}\end{pmatrix}\begin{pmatrix}v\\ \xi\end{pmatrix}=\begin{pmatrix}0\\ -g^\eta\end{pmatrix}, \tag{21}$$

where

$$O_A^\eta=8\mu_1\cos\phi_0 O_1^\eta+8\mu_1\sin\phi_0 O_2^\eta+8\mu_2 O_3^\eta,$$

then the controlled stationary state

$$\begin{aligned}\rho_\infty &= \sqrt{2(v_1^2+v_2^2)}\rho_{1m}+\sqrt{2(v_3^2+v_4^2)}\rho_{2m}\\ &\quad+\left(1-\sqrt{2(v_1^2+v_2^2)}-\sqrt{2(v_3^2+v_4^2)}\right)\tilde{\tilde{\rho}}_s\end{aligned}$$

is a linear combination of two maximally entangled states

$$\rho_{1m}=\frac{1}{2}\begin{pmatrix}1&&&e^{-i\phi_1}\\&0&&\\&&0&\\e^{i\phi_1}&&&1\end{pmatrix},\ \phi_1=\arctan\left(\frac{v_2}{v_1}\right),$$

and

$$\rho_{2m}=\frac{1}{2}\begin{pmatrix}0&&&0\\&1&e^{-i\phi_2}&\\&e^{i\phi_2}&1&\\0&&&0\end{pmatrix},\ \phi_2=\arctan\left(\frac{v_4}{v_3}\right),$$

and a diagonal separable state

$$\begin{aligned}\tilde{\tilde{\rho}}_s &= \mathrm{diag}\left(\frac{1}{4}+\frac{\xi_1}{\sqrt{2}}+\frac{\xi_3}{2},\frac{1}{4}+\frac{\xi_2}{\sqrt{2}}-\frac{\xi_3}{2}\right.\\ &\quad\left.\frac{1}{4}-\frac{\xi_2}{\sqrt{2}}-\frac{\xi_3}{2},\frac{1}{4}-\frac{\xi_1}{\sqrt{2}}+\frac{\xi_3}{2}\right).\end{aligned}$$

The corresponding stationary concurrence is:

$$C(\rho_\infty) = \max\{G_1, G_2, 0\},$$

where

$$G_1 = \sqrt{2(v_1^2+v_2^2)} - \sqrt{\left(\frac{1}{2}+\xi_3\right)^2 - 2\xi_1^2},$$

$$G_2 = \sqrt{2(v_3^2+v_4^2)} - \sqrt{\left(\frac{1}{2}-\xi_3\right)^2 - 2\xi_2^2}.$$

Proposition 1 is a generalization of the results obtained in subsections III-A and III-B. Actually, for the independent amplitude damping decoherence channel, we have:

$$v_1 = \frac{\sqrt{2}\mu_1\Gamma}{4\mu_1^2+\Gamma^2}\sin\phi_0,\ v_2 = -\frac{\sqrt{2}\mu_1\Gamma}{4\mu_1^2+\Gamma^2}\cos\phi_0,$$
$$v_3 = v_4 = 0,$$
$$\xi_1 = 0,\ \xi_2 = \sqrt{2}\xi_3 = \frac{\sqrt{2}\Gamma^2}{8\mu_1^2+2\Gamma^2},$$

while, for the collective amplitude damping decoherence channel, we have:

$$v_1 = \frac{\sqrt{2}\Gamma\mu_1\lambda}{\Gamma^2+3\mu_1^2}\sin\phi_0,\ v_2 = -\frac{\sqrt{2}\Gamma\mu_1\lambda}{\Gamma^2+3\mu_1^2}\cos\phi_0,$$
$$v_3 = -\frac{1}{\sqrt{2}}\left(1-\frac{7}{6}\lambda+\frac{\Gamma^2-3\mu_1^2}{\Gamma^2+3\mu_1^2}\lambda\right),\ v_4 = 0,$$
$$\xi_1 = 0,\ \xi_2 = \frac{\sqrt{2}\lambda\Gamma^2}{2(\Gamma^2+3\mu_1^2)},\ \xi_3 = \lambda - \frac{1}{2} - \frac{\lambda\mu_1^2}{\Gamma^2+3\mu_1^2}.$$

V. CONCLUSIONS

In summary, we propose a two-qubit entanglement control strategy to protect entanglement against Markovian noises. In this strategy, an auxiliary controllable squeezed field is introduced to couple the two qubits. By varying the control parameters, e.g., the amplitude, the frequency and the initial phase, of this squeezed field, one can tune the coupling between the two qubits to control the stationary entanglement. Under special conditions, our entanglement control strategy exhibits evident enhancement of the stationary entanglement compared with the uncontrolled systems.

Although the proposed strategy can remarkably recover entanglement from decoherence, the resulting stationary entanglement may still be insufficient in quantum information processing (see, for example, the stationary entanglement in subsection III-A is below 0.31). Additional entanglement purification processes are required to increase the stationary entanglement, which may tremendously complexify the total experimental systems. More efficient strategies are still to be pursued in future studies.

REFERENCES

[1] D. D'Alessandro, *Introduction to Quantum Control and Dynamics*, Chapman & Hall, Boca Raton; 2007.

[2] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information*, Cambridge University Press, Cambrige, England; 2000.

[3] N. Yamamoto, K. Tsumura, and S. Hara, Feedback Control of Quantum Entanglement in a Two-spin System, *Automatica*, vol. 43, 2007, pp. 981-992.

[4] H. P. Breuer and F. Petruccione, *The Theory of Open Quantum Systems*, Oxford University Press, Oxford; 2007.

[5] M. P. Almeida, F. de Melo, M. Hor-Meyll, A. Salles, S. P. Walborn, P. H. Souto Ribeiro, and L. Davidovich, Environment-Induced Sudden Death of Entanglement, *Science*, vol. 316, 2007, pp. 579-582.

[6] D. Braun, Creation of Entanglement by Interaction with a Common Heat Bath, *Phys. Rev. Lett.*, vol. 89, 2002, pp. 277901.

[7] M. B. Plenio and S. F. Huelga, Entangled Light from White Noise, *Phys. Rev. Lett.*, vol. 88, 2002, pp. 197901.

[8] A. R. R. Carvalho, F. Mintert, S. Palzer, and A. Buchleitner, Entanglement Dynamics Under Decoherence: From Qubits to Qudits, *Eur. Phys. J. D*, vol. 41, 2007, pp. 425-432.

[9] Z. Ficek and R. Tanas, Entangled States and Collective Nonclassical Effects in Two-atom Systems, *Phys. Rep.*, vol. 372, 2002, pp. 369-443.

[10] J. M. Raimond, M. Brune, and S. Haroche, Manipulating Quantum Entanglement with Atoms and Photons in a Cavity, *Rev. Mod. Phys.*, vol. 73, 2001, pp. 565-582.

[11] A. Blais, J. Gambetta, A. Wallraff, D. I. Schuster, S. M. Girvin, M. H. Devoret, and R. J. Schoelkopf, Quantum-information Processing with Circuit Quantum Electrodynamics, *Phys. Rev. A*, vol. 75, 2007, pp. 032329.

[12] C. J. Villas-Boas, N. G. de Almeida, R. M. Serra, and M. H. Y. Moussa, Squeezing Arbitrary Cavity-field States through Their Interaction with a Single Driven Atom, *Phys. Rev. A*, vol. 68, 2003, pp. 061801(R).

[13] N. G. de Almeida, R. M. Serra, C. J. Villas-Boas, and M. H. Y. Moussa, Engineering Squeezed States in High-Q Cavities, *Phys. Rev. A*, vol. 69, 2004, pp. 035802.

[14] J. Zhang, Y.-X Liu, C. W. Li, T. J. Tarn, F. Nori, Protecting Entanglement in Superconducting Qubits, *http://arxiv.org/abs/0808.0395*.

[15] A. Blais, R. S. Huang, A. Wallraff, S. M. Girvin, and R. J. Schoelkopf, Cavity Quantum Electrodynamics for Superconducting Electrical Circuits: An Architecture For Quantum Computation, *Phys. Rev. A*, vol. 69, 2004, pp. 062320.

[16] R. P. Puri, *Mathematical Methods of Quantum Optics*, Springer Verlag, New York; 2001.

[17] W. K. Wootters, Entanglement of Formation of an Arbitrary State of Two Qubits, *Phys. Rev. Lett.*, vol. 80, 1998, pp. 2245-2248.

[18] W. Dür and H. J. Briegel, Entanglement Purification and Quantum Error Correction, *Rep. Prog. Phys.*, vol. 70, 2007, pp. 1381-1424.

[19] R. Alicki and K. Lendi, *Quantum Dynamical Semigroup and Applications*, Springer-Verlag, New York; 1985.

[20] C. Altafini, Controllability Properties for Finite Dimensional Quantum Markovian Master Equations, *J. Math. Phys.*, vol. 44, 2003, pp. 2357-2372.

[21] J. Zhang, R. B. Wu, C. W. Li, T. J. Tarn, and J. W. Wu, Asymptotically Noise Decoupling for Markovian Open Quantum Systems, *Phys. Rev. A*, vol. 75, 2007, pp. 022324.