Proceedings of the
47th IEEE Conference on Decision and Control
Cancun, Mexico, Dec. 9-11, 2008

Conservativeness of State-Dependent Riccati Inequality :Effect of Free Parameters of State-Dependent Coefficient Form

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***Abstract*— Recently, nonlinear H_∞ control theory has been paid attention. The solvable condition of nonlinear H_∞ control problem is given by the Hamilton Jacobi Inequality (HJI). State-Dependent Riccati Inequality (SDRI) is one of approaches to solve the HJI. The SDRI contains State-Dependent Coefficient (SDC) form of a nonlinear system. The SDC form is not unique, so free parameters of it is considered. If bad SDC form is chosen, then there is no solution of SDRI.**

In this paper, the relationship between free parameters and SDRI is clarified. The free parameters are generated when SDRI is derived from HJI. And they affect the conservativeness of SDRI. Then new method of design free parameters which reduces the conservativeness of SDRI is proposed. Finally, numerical examples to verify the effect of this method is shown.

I. INTRODUCTION

Linear H_∞ Control Theory has become a remarkably popular tool in engineering applications because there are many convenience tools (MATLAB, etc.) to solve it. On the other hand, even though a lot of theoretical developments of Nonlinear H_∞ Control Theory have been done [1][2][3], applications are very few, since a useful method of solving it has not been established yet.

In order to solve Nonlinear H_∞ Control Problems, we have to deal with a kind of partial differential inequalities called Hamillton-Jacobi Inequality (HJI). For Linear H_∞ Control Problems, we can design the linear H_∞ controller easily by solving a familiar Algebraic Riccati Inequality (ARI), but it turns out to be much more complicate to derive nonlinear H_∞ controller due to a necessity on dealing with the HJI. Since HJI is a partial differential inequality, it is quite hard to solve HJI analytically.

Numerical solutions of HJI have been researched. One of the researches is approximate solution of HJI using Taylor Expansion around a equilibrium point [4]. This approximate solution shows a good behavior around the equilibrium point, but not away from that point. On the other hand, there is a way using nonlinear matrix inequality which is so-called State-Dependent Riccati Inequality(SDRI)[5][6][7]. For a nonlinear system, Lu and Doyle showed SDRI issues [5]. If there exists a positive definite matrix $P(x)$ which is a solution of SDRI and also exists a positive definite scalar function $V(x)$ satisfying $\partial V/\partial x = 2x^T P(x)$ (a integrability condition), then such the $V(x)$ is a positive definite solution of the HJI. By solving the point-wise ARI, they got a set of point-wise solutions and also an approximate continuous solution $P(x)$.

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For these methods which use SDRI to solve HJI, there is a problem that State-Dependent Coefficient (SDC) form of nonlinear system is not unique. This problem means that there are many representations of $A(x)$ satisfying $f(x) = A(x)x$. In other words, free parameters is considered in SDC form[8][9]. Since the solution of SDRI depends on choice of SDC form. If bad SDC form is chosen, there is no solution. It is very important to choose a good SDC form to solve SDRI. But, naturally HJI dosen't depends on this free parameters.

In this paper we focus on the free parameters of SDC form. First, we introduce a representation of free parameters of SDC form. And then, we clarify that free parameters of SDC form affect the conservativeness of SDRI. In addition, we introduce new method of design free parameters which reduces the conservativeness of SDRI. Finally, we show numerical examples to verify the effect of this method.

II. PRELIMINARIES

A. Linear H_∞ Control Problem

Let us consider the following linear system S_l

$$S_l \begin{cases} \dot{x} = Ax + B_1 w + B_2 u \\ z = C_1 x + D_{12} u \end{cases} \tag{1}$$

where w is an unknown disturbance, z is a controlled output, u is a control input to be chosen. Objectives of Linear H_∞ Control Problem are to find a state feedback controller that achieves closed-loop stability and makes L_2-gain from w to z less than or equal to γ . For an easy formulation of control input, let us assume $C_1^T D_{12} = 0,\ D_{12}^T D_{12} = I$. Then the control input is given by

$$u = -B_2^T P x \tag{2}$$

where P is a positive definite matrix which satisfies following Algebraic Riccati Inequality (ARI)

$$PA + A^T P + P\left(\frac{1}{\gamma^2} B_1 B_1^T - B_2 B_2^T\right) P + C_1^T C_1 < 0. \tag{3}$$

B. Nonlinear H_∞ Control Problem

Let us consider the following nonlinear system S_{nl}

$$S_{nl} \begin{cases} \dot{x} &= f(x) + g_1(x) w + g_2(x) u \\ z &= h_1(x) + j_{12}(x) u \end{cases} \tag{4}$$

where w, z, u is the same as (1). And objectives of Nonlinear H_∞ Control Problem are also the same as linear one. Refer to [1], under standard assumptions $h_1^T j_{12} = 0$ and $j_{12}^T j_{12} = I$, an optimal feedback control law is given by

$$u(x) = -\frac{1}{2} g_2^T(x) \frac{\partial V}{\partial x^T} \tag{5}$$

where $V(x)$ is a positive definite solution of Hamilton Jacobi Inequality(HJI)

$$\frac{\partial V}{\partial x} f + \frac{1}{4} \frac{\partial V}{\partial x} \left(\frac{1}{\gamma^2} g_1 g_1^T - g_2 g_2^T \right) \frac{\partial V}{\partial x^T} + h_1^T h_1 + \varepsilon x^T x \leq 0 \tag{6}$$

for some positive ε.

C. State-Dependent Riccati Inequality

Let us define as follows

$$f(x) = A(x)x,\ g_1(x) = B_1(x),\ g_2(x) = B_2(x)$$
$$h_1(x) = C_1(x)x,\ j_{12}(x) = D_{12}(x),$$

then the nonlinear system S_{nl} is transformed into SDC form

$$S_{nl} \begin{cases} \dot{x} = A(x)x + B_1(x)w + B_2(x)u \\ z = C_1(x)x + D_{12}(x)u \end{cases} . \tag{7}$$

With assumption

$$\frac{\partial V}{\partial x^T} = 2P(x)x, \tag{8}$$

the HJI becomes State-Dependent Riccati Inequality(SDRI)

$$P(x)A(x) + A^T(x)P(x) + \frac{1}{\gamma^2} P(x)B_1(x)B_1^T(x)P(x) - P(x)B_2(x)B_2^T(x)P(x) + C_1^T(x)C_1(x) < 0. \tag{9}$$

For this SDRI, a nonlinear H_∞ control input u is given by

$$u = -\frac{1}{2} g_2^T(x) \frac{\partial V}{\partial x^T} = -B_2^T(x)P(x)x. \tag{10}$$

D. Solving SDRI via LMI

When SDRI(9) is fixed with a state x, it is a same inequality as ARI(3) with variable P. To solve this matrix inequality, (3) is transformed into LMI. By using Schur Complement and a variable transformation $X = P^{-1}$, (3) becomes

$$\begin{bmatrix} AX + XA^T + \frac{1}{\gamma^2} B_1 B_1^T - B_2 B_2^T & XC_1^T \\ C_1 X & -I \end{bmatrix} < 0 \tag{11}$$

which is a LMI with respect to a variable X. If we ignore the integrability condition(8), we get state-dependent solution $P(x)$ by solving (11) at each state. One solution which satisfies (8) is a constant solution P_c which satisfies ARIs at several states simultaneously.

E. Existence of Free Parameters of SDC Form

We can represent $f(x)$ as below.

$$f(x) = A(x)x = (A(x) + E(x))\, x \tag{12}$$

$E(x) \in \mathbb{R}^{n \times n}$ is any matrix that satisfies

$$E(x)x = \mathbf{0}. \tag{13}$$

Lemma 1: Although we can represent $h_1(x)$ as well as $f(x)$, the nonlinearity of $C_1(x)$ is transformed into $A(x)$ by using coordinate transformation.

Let us consider coordinate transformation

$$\tilde{x} = T(x) = \begin{bmatrix} h_1^\perp(x) \\ h_1(x) \end{bmatrix}. \tag{14}$$

$h^\perp(x)$ is any function which is independent of $h(x)$. In other words, the rank of $\partial T(x)/\partial x$ should be n. And $x = T^{-1}(\tilde{x})$. By using coordinate transformation, nonlinear system(4) is transformed into

$$S_{nl} \begin{cases} \dot{\tilde{x}} &= \frac{\partial T(x)}{\partial x} \left(f(x) + g_1(x)w + g_2(x)u \right) \\ &:= \tilde{f}(\tilde{x}) + \tilde{g_1}(\tilde{x})w + \tilde{g_2}(\tilde{x})u \\ \\ z &= \begin{bmatrix} \mathbf{0} & I \end{bmatrix} \tilde{x} + j_{12}(x)u \\ &:= C_1 \tilde{x} + \tilde{j_{12}}(\tilde{x})u \end{cases} \tag{15}$$

As we can see, the controlled output z is represented in a linear expression with this transformed system. And the $f(x)$ is transformed into $\tilde{f}(\tilde{x})$ which includes the nonlinearity of z. From now on, we only focus on SDC form of $f(x)$.

III. REPRESENTATION OF FREE PARAMETERS OF SDC FORM

We introduce a representation which clarifies free parameters of SDC form.

Theorem 1: Let $A_0(x)$ be one of state-dependent coefficient matrices of $f(x)$, $x \in \mathbb{R}^n$ such that

$$f(x) = A_0(x)x. \tag{16}$$

All $A(x)$ which satisfies

$$\forall x \neq \mathbf{0},\ f(x) = A(x)x \tag{17}$$

can be represented by

$$A(x) = A_0(x) + M_a(x)\Theta(x). \tag{18}$$
$$\Theta(x) := \begin{bmatrix} x^T/|x| \\ \Theta_p(x) \end{bmatrix}, \Theta_p x = \mathbf{0}, \Theta_p \in \mathbb{R}^{n-1 \times n}$$
$$M_a(x) := \begin{bmatrix} \mathbf{0} & M_{ap}(x) \end{bmatrix}, M_{ap}(x) \in \mathbb{R}^{n \times n-1}$$

The first column of $M_a(x) \in \mathbb{R}^{n \times n}$ must be $\mathbf{0}$. Anather elements (M_{ap}) are free parameters. And $\Theta(x) \in \mathbb{R}^{n \times n}$ is combined rotation matrices which rotate x_1 axis to the direction of x. For detail of $\Theta(x)$, see the appendix A.

Proof: Sufficiency: Let $A(x)$ be (18). $A(x)x$ becomes

$$A(x)x = A_0(x)x + M_a(x)\Theta(x)x = A_0(x)x = f(x) \tag{19}$$
$$\because M_a(x)\Theta(x)x = 0.$$

So $A(x)x$ is a state-dependent coefficient matrix of $f(x)$.

Necessity: Let $A(x)$ be a state-dependent coefficient matrix of $f(x)$. To represent $A(x)$, $M_a(x)$ should be

$$M_a(x) = \{A(x) - A_0(x)\}\,\Theta^T(x). \tag{20}$$

The first column of $M_a(x)$ is $\mathbf{0}$, because the first column of $\Theta^T(x)$ is $x/|x|$ and $A_1(x)x - A_0(x)x = \mathbf{0}$. Since inverse matrix of rotation matrix is transpose matrix of itself,

$$\begin{aligned}&A_0(x) + M_a(x)\Theta(x)\\ =&A_0(x) + \{A(x) - A_0(x)\}\,\Theta^T(x)\Theta(x)\\ =&A(x).\end{aligned} \tag{21}$$

So we can represent all of $A(x)$ by (18). ■

IV. EFFECT OF FREE PARAMETERS ON CONSERVATIVENESS OF SDRI

Nevertheless HJI dosen't depend on free parameters, SDRI which is derived from HJI depends on these. In this section, we clarify relationship between HJI, SDRI and free parameters. And then we propose a new method of design free parameters.

A. Conservativeness of SDRI

With SDC form and assumption (8) , SDRI is derived from HJI as follows,

$$HJI : x^T G(x)x < 0 \tag{22}$$
$$\Leftarrow \forall z \in \mathbb{R}^n : z^T G(x)z < 0 \tag{23}$$
$$\Leftrightarrow G(x) < 0 : SDRI \tag{24}$$
$$\begin{aligned}G(x) :=&P(x)A(x) + A^T(x)P(x)\\ &+ P(x)\left\{\frac{1}{\gamma^2}B_1(x)B_1^T(x) - B_2(x)B_2^T(x)\right\}P(x)\\ &+ C_1^T(x)C_1(x).\end{aligned} \tag{25}$$

Note that HJI and SDRI are not equal. SDRI is sufficient condition of HJI. In other words, SDRI is more conservative than HJI. So, even if the solution of HJI exists , the solution of SDRI dose not necessarily exist.

B. Effect of free parameters

Here, let us consider M_a. By substituting (18) for $A(x)$, we have

$$G(x) = G_0(x) + \mathrm{He}\{P(x)M_a(x)\Theta(x)\} \tag{26}$$
$$\begin{aligned}G_0(x) :=&P(x)A_0(x) + A_0^T(x)P(x)\\ &+ P(x)\left\{\frac{1}{\gamma^2}B_1(x)B_1^T(x) - B_2(x)B_2^T(x)\right\}P(x)\\ &+ C_1^T(x)C_1(x)\end{aligned} \tag{27}$$

where $\mathrm{He}\{J\}$ means $J + J^T$. Sice $M_a(x)\Theta(x)x = 0$,

$$x^T G(x)x = x^T G_0(x)x \tag{28}$$
$$z^T G(x)z \neq z^T G_0(x)z. \tag{29}$$

Then we realize that the effect of free parameters is generated at (23).

C. Relationship between conservativeness and free parameters

Sice, $\Theta(x)$ is full rank,

$$SDRI : G(x) < 0 \Leftrightarrow \Theta(x)G(x)\Theta^T(x) < 0. \tag{30}$$

It's expanded to

$$\begin{aligned}\Theta(x)G(x)\Theta^T(x) &= \begin{bmatrix}\frac{x^T}{|x|}\\ \Theta_p(x)\end{bmatrix} G(x) \begin{bmatrix}\frac{x}{|x|} & \Theta_p^T(x)\end{bmatrix}\\ &= \begin{bmatrix}\frac{x^T G(x)x}{|x|^2} & *\\ * & *\end{bmatrix} < 0.\end{aligned} \tag{31}$$

The $(1, 1)$ element of (31) is HJI. It implies that SDRI includes HJI and other extra conditions. Furthermore, by substituting (26) for $G(x)$,

$$\begin{aligned}\Theta(x)G(x)\Theta^T(x) =&\Theta(x)G_0(x)\Theta(x)\\ &+ \mathrm{He}\{\Theta(x)P(x)M_a(x)\}.\end{aligned} \tag{32}$$

The nodes which include M_a are

$$\begin{aligned}&\mathrm{He}\{\Theta(x)P(x)M_a(x)\}\\ =&\begin{bmatrix}0 & \frac{x^T P(x)M_{ap}(x)}{|x|}\\ \frac{M_{ap}^T(x)P(x)x}{|x|} & \mathrm{He}\{\Theta_p(x)P(x)M_{ap}(x)\}\end{bmatrix}.\end{aligned} \tag{33}$$

Now we realize that free parameters(M_a) effect on extra conditions without HJI.

D. Method of setting free parameters

Let us consider the design of M_a which reduces conservativeness of SDRI.

Theorem 2: Let us define $A(x)$ as (18) , $G_0(x)$ as (27) and $P(x)$ as (8). If we define $M_a(x)$ as follows

$$\begin{aligned}M_a(x) =& -P^{-1}(x)\Theta^T(x)\left\{\begin{bmatrix}1 & \mathbf{0}\\ \mathbf{0} & I_p/2\end{bmatrix}\Theta(x)G_0(x)\Theta^T(x)\right.\\ &\left. + I\right\}\begin{bmatrix}0 & \mathbf{0}\\ \mathbf{0} & I_p\end{bmatrix}\end{aligned} \tag{34}$$

where $I_p \in \mathbb{R}^{n-1\times n-1}$ is a identitiy matrix, then HJI (6) and SDRI (9) are equal.

Proof: By substituting (34) for $M_a(x)$, (33) becomes

$$\begin{aligned}&\mathrm{He}\{\Theta(x)P(x)M_a(x)\}\\ =& -\mathrm{He}\left\{\begin{bmatrix}1 & \mathbf{0}\\ \mathbf{0} & I_p/2\end{bmatrix}\Theta(x)G_0(x)\Theta^T(x)\begin{bmatrix}0 & \mathbf{0}\\ \mathbf{0} & I_p\end{bmatrix}\right\}\\ &-\begin{bmatrix}0 & \mathbf{0}\\ \mathbf{0} & 2I_p\end{bmatrix}\\ =& -\mathrm{He}\left\{\begin{bmatrix}0 & \frac{x^T G_0(x)\Theta_p^T(x)}{|x|}\\ \mathbf{0} & \frac{\Theta_p(x)G_0(x)\Theta_p^T(x)}{2}\end{bmatrix}\right\} - \begin{bmatrix}0 & \mathbf{0}\\ \mathbf{0} & 2I_p\end{bmatrix}\\ =& -\begin{bmatrix}0 & \frac{x^T G_0(x)\Theta_p^T(x)}{|x|}\\ \frac{\Theta_p^T G_0(x)x}{|x|} & \Theta_p(x)G_0(x)\Theta_p^T(x)\end{bmatrix} - \begin{bmatrix}0 & \mathbf{0}\\ \mathbf{0} & 2I_p\end{bmatrix}\end{aligned}$$

So (32) becomes

$$\Theta(x)G_0(x)\Theta(x) + \mathrm{He}\{\Theta(x)P(x)M_a(x)\} = \begin{bmatrix} \frac{x^TG_0(x)x}{|x|^2} & 0 \\ 0 & -2I_p \end{bmatrix}. \tag{35}$$

Then HJI (22) and SDRI (30) are equal. ■

V. METHOD OF SEARCHING A BETTER SOLUTION

To solve SDRI, it should be transformed into LMI. By substituting (18) for $A(x)$, (11) become

$$\begin{bmatrix} lmi_{11} & XC_1^T \\ C_1X & -I \end{bmatrix} < 0 \tag{36}$$

$$lmi_{11} := \mathrm{He}\{(A_0 + M_a\Theta)X\} + \frac{1}{\gamma^2}B_1B_1^T - B_2B_2^T.$$

Unfortunately, (34) includes $P(x) = X^{-1}(x)$ which is a solution of SDRI. So if we substitute (34) for $M_a(x)$ then (36) dosen't become LMI.

Then let us use iteration to search a better solution $P(x)$. Initial value of $P(x)$ should be calculated without $M_a(x)$. And then a better solution is calculated by iteration with $M_a(x)$ which is derived with last $P(x)$.

VI. SIMULATION

We consider constant solution P_c which satisfies SDRI at local area. Let us solve SDRI wia LMI which is satisfied at several point in local area , and minimize γ.

A. Considered System

$$S_{nl}\begin{cases} \dot{x} = \begin{bmatrix} -\sin(2x_1) \\ x_1 - x_2 - 2x_1^3 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} w + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u \\ z = \begin{bmatrix} u \\ x_1/10 \\ x_2 \end{bmatrix} \end{cases}$$

$$x = [x_1,\ x_2]^T, x_0 = [-1.0,\ 1.0]^T,$$
$$w = 3\sin(\pi t)$$

The SDC form is selected as

$$A(x) = A_0(x) + M_a(x)\Theta(x),$$

$$A_0(x) := \begin{bmatrix} -\frac{\sin(2x_1)}{x_1} & 0 \\ 1 - 2x_1^2 & -1 \end{bmatrix},$$

$$\Theta(x) = \begin{cases} \frac{1}{|x|}\begin{bmatrix} x_1 & x_2 \\ -x_2 & x_1 \end{bmatrix} & (x \neq 0) \\ \mathbf{0} & (x = 0) \end{cases},$$

$$B_1(x) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, B_2(x) = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

$$C_1(x) = \begin{bmatrix} 0 & 0 \\ 1/10 & 0 \\ 0 & 1 \end{bmatrix}, D_{12}(x) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

The considered points where LMIs are solved simultaneously are

$$x_d = \{0.5[i_1,\ i_2]^T \mid -2 \leq i_j \leq 2,\ i_j \in \mathbb{Z}\} \tag{37}$$

B. ARI

Let us solve LMI(11) at the origin. The constant solution P_c and minimam γ are obtained as

$$P_0 = \begin{bmatrix} 13.1 & 19.1 \\ 19.1 & 28.2 \end{bmatrix}$$
$$\gamma_0 = 0.452.$$

The simulation results are shown P_0 in Fig.1, Fig.2. HJI isn't satisfied in large area, so states can not be converged.

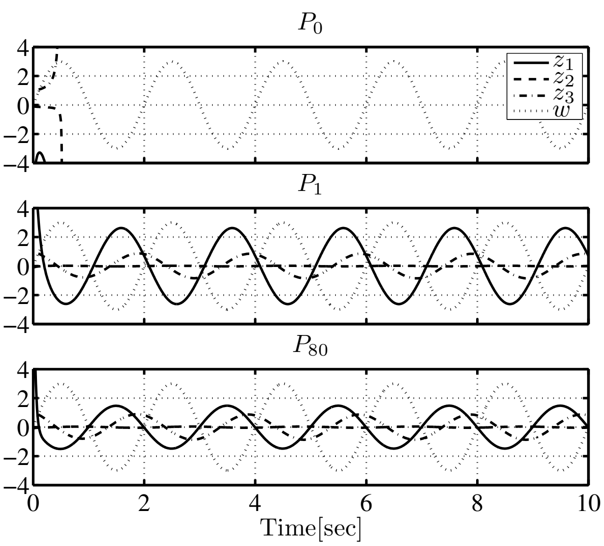

Fig. 1. Simulation Result

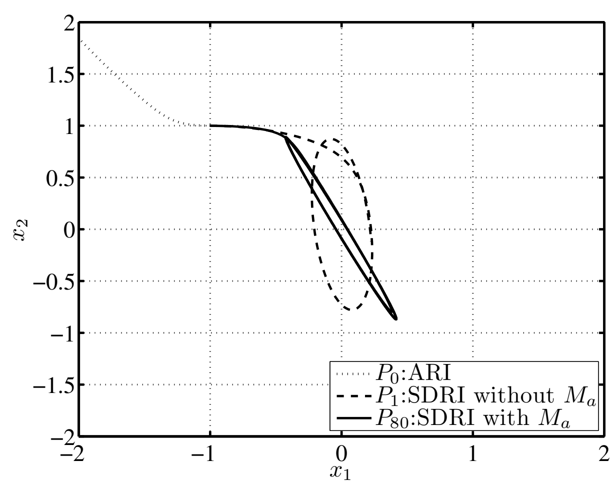

Fig. 2. Trajectory

C. SDRI without M_a at several states

Let us solve LMIs(11) without M_a at (37). We can get

$$P_1 = \begin{bmatrix} 10.2 & -0.745 \\ -0.745 & 0.963 \end{bmatrix}$$
$$\gamma_1 = 1.42.$$

The simulation results are shown P_1 in Fig.1, Fig.2. As we can see, the system converges to the origin.

D. SDRI with M_a at several states

Let us solve LMIs(11) with M_a(34) at (37). The initial value of P is P_1. 80 times iteration derives,

$$P_{80} = \begin{bmatrix} 31.8 & 14.5 \\ 14.5 & 7.27 \end{bmatrix}$$
$$\gamma_{80} = 0.768.$$

γ_{80} is smaller than γ_1. The simulation results are shown P_{80} in Fig.1, Fig.2. As we can see, the system converges to the origin. And the effect of disturbance is smaller Than P_1.

VII. CONCLUSION

In this paper, the relation between free parameters of SDC form and SDRI is clarified. First, we introduced a representation of free parameters of SDC form. And then, we clarified that free parameters of SDC form affect the conservativeness of SDRI. In addition, we introduced new method of design free parameters which reduces the conservativeness of SDRI. Finally, we showed numerical examples to verify the effect of this method.

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APPENDIX A

Hereunder the structure of combined rotation matrices are presented.

$$\Theta(x) = \prod_{i=2}^{n} \Theta_i(x).$$

$\Theta_i(x)$ is a rotation matrix at $x_1 - x_i$ plane, i.e.

$$\Theta_i(x) = \begin{bmatrix} \cos\theta_i & \mathbf{0} & \sin\theta_i & \mathbf{0} \\ \mathbf{0} & I_{p1} & \mathbf{0} & \mathbf{0} \\ -\sin\theta_i & \mathbf{0} & \cos\theta_i & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & I_{p2} \end{bmatrix}$$

where $I_{p1} \in \mathbb{R}^{(i-2)\times(i-2)}$ and $I_{p2} \in \mathbb{R}^{(n-i)\times(n-i)}$ are identity matrices. Let us choose trigonometric function as follows,

$$(\cos\theta_i, \sin\theta_i) := \begin{cases} (t_{i+1}/t_i, x_i/t_i) & t_i \neq 0 \\ (1, 0) & t_i = 0 \end{cases}$$
$$t_i := \begin{cases} \sqrt{x_1^2 + \sum_{k=i}^{n} x_k^2} & 2 \le i \le n \\ x_1 & i = n+1. \end{cases}$$

The first row of $\Theta(x)$ is calculated to

$$\begin{aligned} \vartheta_{11} &= \prod_{k=2}^{n} \cos\theta_k = \frac{t_3}{t_2}\frac{t_4}{t_3}\cdots\frac{t_{n+1}}{t_n} = \frac{t_{n+1}}{t_2} = \frac{x_1}{|x|} \\ \vartheta_{12} &= \sin\theta_2 = \frac{x_2}{|x|} \\ \vartheta_{13} &= \sin\theta_3\cos\theta_2 = \frac{x_3}{t_3}\frac{t_3}{t_2} = \frac{x_3}{|x|} \\ &\vdots \\ \vartheta_{1i} &= \sin(\theta_i)\prod_{k=2}^{i-1}\cos(\theta_k) = \frac{x_i}{t_i}\frac{t_3}{t_2}\cdots\frac{t_i}{t_{i-1}} = \frac{x_i}{|x|}. \end{aligned}$$

So the first row of $\Theta(x)$ is $x^T/|x|$. And $\Theta^{-1}(x) = \Theta(x)^T$, because $\Theta_i^{-1}(x) = \Theta_i^T(x)$.