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Local Mode Dependent Decentralized Control of Uncertain Markovian Jump Large-scale Systems

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Abstract— **This paper is concerned with the robust stabilization of a class of stochastic large-scale systems. The uncertainties satisfy integral quadratic constraints. The random parameter is a Markov process. A sufficient condition is developed to design stabilizing decentralized controllers which use local system states and local system operation modes to produce local control inputs. The sufficient condition is given in terms of a set of rank constrained linear matrix inequalities.**

I. Introduction

Markovian jump large-scale systems are a class of large-scale systems where the system parameters are governed by a finite-state Markov process. The values of the Markov process indicate which parameters are used by the system at the current time. Many practical examples that can be modeled as large-scale systems are found in fields such as flexible communications networks, economic systems, and power systems [1], [2]. For the large-scale systems that can be decomposed into a set of interconnected dynamic subsystems, decentralized control has proven to be one of the most useful techniques [1]. However, the design of decentralized controllers is challenging as the controllers can only use partial information of the system. Even worse, the dynamics of the subsystems are also affected by other subsystems [3].

In this paper we consider a class of uncertain Markovian jump large-scale systems. In such a system, each subsystem is uncertain, and these uncertainties are termed the local uncertainties. The interconnections among the subsystems are also treated as uncertainties and termed the interconnection uncertainties. All the uncertainties are described using integral quadratic constraints [4]. This problem formulation was originally introduced in [2], [5] and was inspired by applications in the area of power systems.

In [2], a lossless S-procedure for Markovian jump systems was derived and used to establish a necessary and sufficient condition for the absolute stabilization of uncertain Markovian jump large-scale systems with decentralized state feedback controllers. The condition was given in terms of generalized algebraic Riccati equations, and served as a basis for an optimization procedure for constructing an optimal controller which attained the minimum of a worst-case performance index. The output feedback version of this stabilization problem was studied in [5]. A necessary and sufficient condition was established in terms of generalized algebraic Riccati equations and inequalities. The suboptimal controllers can be constructed through an optimization problem subject to a set of rank constrained linear matrix inequalities. This paper continues the study of uncertain Markovian jump large-scale systems initiated in [2], [5].

What makes the work in this paper different from that in [2], [5] is the form of the decentralized controllers we design. In [2], [5], the global operation mode of the large-scale system is used in the controllers. As a result, the number of the controllers for each subsystem is equal to the number of the operation modes of the large-scale system, and hence greater than the number of the operation modes of the subsystems they control. Moreover, the controller has to change its operation mode even if the subsystem it controls does not change. In addition, an important underlying assumption required to implement such a controller is that the operation mode of the large-scale system must be known to every controller. Such controllers are called global mode dependent controllers. In this paper, we stabilize the large-scale system using the controllers that change their operation mode only when the subsystems they control change operation mode. By using such a control scheme, the broadcast of the global operation mode of the large-scale system is no longer needed for the decentralized controllers. We refer to this type of decentralized controller as a local mode dependent controller. Compared with the control techniques in [2], [5], the technique developed in this paper is obviously easier to implement and is more efficient in that it significantly reduces communication overheads.

The local mode dependent decentralized control design procedure proposed in this paper can be roughly stated as follows. First, one designs the global mode dependent controllers, which can stabilize a somewhat larger class of uncertain Markovian jump large-scale systems containing the class of the uncertain systems we originally wanted to stabilize as a special case. These decentralized global mode dependent controllers are obtained based upon the techniques in [2]. Second, the local mode dependent controllers are designed to be the limit (as time approaches infinity) of the conditional mean value of the global mode dependent control gains conditioned on the corresponding local subsystem modes. Third, the local mode dependent controllers are implemented and used to stabilize the large-scale system. A sufficient condition is provided in terms of a set of rank constrained linear matrix inequalities to design the desired decentralized controllers. It is also noted that the local mode dependent controllers which are designed using our approach

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are the limiting minimum variance estimates of the global mode dependent controllers as time approaches infinity.

Notation: $\mathbb{R}^+$ denotes the set of positive real numbers. $\mathbb{R}^n$, $\mathbb{R}^{n\times m}$, and $\mathbb{S}^+$ denote, respectively, the n-dimensional Euclidean space, the set of $n \times m$ real matrices, and the set of real symmetric positive definite matrices. The notation $X \geq Y$ (respectively, $X > Y$) where X and Y are real symmetric matrices, means that $X - Y$ is positive semi-definite (respectively, positive definite). I is the identity matrix of compatible dimensions. The superscript "T" denotes the transpose for vectors or matrices. $\mathrm{rank}(\cdot)$ is the rank of a matrix. $\mathrm{diag}(A_1, \ldots, A_n)$ stands for the block diagonal matrix with $A_1, \ldots, A_n$ on the main diagonal. $\|\cdot\|$ refers to the Euclidean norm for vectors and the induced 2-norm for matrices. Moreover, let $(\Omega, \mathcal{F}, \mathrm{Pr})$ be a complete probability space. $\mathrm{Pr}(\cdot)$ is the probability measure, and $\mathrm{E}(\cdot)$ denotes the mathematical expectation operator.

II. Problem Formulation

Consider an uncertain Markovian jump large-scale system consisting of N subsystems. The ith subsystem is described by

$$\mathcal{S}_i : \begin{cases} \dot{x}_i(t) = A_i(\eta_i(t))x_i(t) + B_i(\eta_i(t))u_i(t) \\ \qquad\qquad + E_i(\eta_i(t))\xi_i(t) + L_i(\eta_i(t))r_i(t) \\ \zeta_i(t) = H_i(\eta_i(t))x_i(t) \end{cases} \tag{1}$$

where $i \in \mathcal{N} \triangleq \{1, 2, \ldots, N\}$ signals that $\mathcal{S}_i$ is the ith subsystem of the large-scale system, $x_i(t) \in \mathbb{R}^{n_i}$ is the system state of subsystem $\mathcal{S}_i$, $u_i(t) \in \mathbb{R}^{m_i}$ is the control input, $\xi_i(t) \in \mathbb{R}^{p_i}$ is the local uncertainty input, $\zeta_i(t) \in \mathbb{R}^{q_i}$ is the uncertainty output, $r_i(t) \in \mathbb{R}^{s_i}$ is the interconnection input, which describes the effect of other subsystems $\mathcal{S}_j$, $j \neq i$, on $\mathcal{S}_i$ due to the interconnection between subsystem $\mathcal{S}_i$ and other subsystems $\mathcal{S}_j$, $j \neq i$. $\eta_i(t)$ is a random process denoting the operation mode of subsystem $\mathcal{S}_i$ and takes values in the finite state space $\mathcal{M}_i \triangleq \{1, 2, \ldots, M_i\}$. The initial condition of subsystem $\mathcal{S}_i$ is given by $x_{i0} \in \mathbb{R}^{n_i}$ and $\eta_{i0} \in \mathcal{M}_i$.

The mechanism of mode change for the large-scale system is described by the random process $\eta(t)$, which is the operation mode of the large-scale system, and depends on (and also determines) the operation modes of the subsystems. That is, there is a bijective mapping between $\eta(t)$ and η_i, $i \in \mathcal{N}$. It is assumed that the random process $\eta(t)$ is a homogeneous, stationary, ergodic continuous-time Markov process defined on a complete probability space $(\Omega, \mathcal{F}, \mathrm{Pr})$ and takes values in $\mathcal{M} \triangleq \{1, 2, \ldots, M\}$ where $\max_{i\in\mathcal{N}} M_i \leq M \leq \prod_{i=1}^N M_i$. The state transition rate matrix of $\eta(t)$ is given by $\mathbf{Q} = (q_{\mu\nu}) \in \mathbb{R}^{M\times M}$, in which $q_{\mu\nu} \geq 0$ if $\nu \neq \mu$, and $q_{\mu\mu} \triangleq -\sum_{\nu=1,\nu\neq\mu}^M q_{\mu\nu}$.

The uncertainties and interconnections in the system (1) are described by

$$\xi_i(t) = \phi_i^\xi(t, \zeta_i(t))$$
$$r_i(t) = \phi_i^r(t, \zeta_1(t), \ldots, \zeta_{i-1}(t), \zeta_{i+1}(t), \ldots, \zeta_N(t))$$

and are assumed to satisfy the following integral quadratic constraints [6], [2], [5].

Definition 1: Given a set of matrices $\bar{S}_i \in \mathbb{S}^+$, $i \in \mathcal{N}$. A collection of uncertainty inputs $\xi_i(t)$, $i \in \mathcal{N}$, is an admissible local uncertainty for the large-scale system if there exists a sequence $\{t_l\}_{l=1}^\infty$ such that $t_l \to \infty$, $t_l \geq 0$ and

$$\mathrm{E}\left(\int_0^{t_l} \left[\|\zeta_i(t)\|^2 - \|\xi_i(t)\|^2\right] dt \mid x_{i0}, \eta_{i0}\right) \geq -x_{i0}^T \bar{S}_i x_{i0} \tag{2}$$

for all l and for all $i \in \mathcal{N}$. The set of the admissible local uncertainties is denoted by Ξ.

Definition 2: Given a set of matrices $\tilde{S}_i \in \mathbb{S}^+$, $i \in \mathcal{N}$. The subsystems of the large-scale system are said to have admissible interconnections to other subsystems if there exists a sequence $\{t_l\}_{l=1}^\infty$ such that $t_l \to \infty$, $t_l \geq 0$ and

$$\mathrm{E}\left(\int_0^{t_l} \left[\sum_{j=1,j\neq i}^N (\|\zeta_j(t)\|^2) - \|r_i(t)\|^2\right] dt \mid x_{i0}, \eta_{i0}\right) \geq -x_{i0}^T \tilde{S}_i x_{i0} \tag{3}$$

for all l and for all $i \in \mathcal{N}$. The set of the admissible interconnection uncertainties is denoted by Π.

The objective of the paper is to design a decentralized local mode dependent state-feedback controller of the form

$$u_i(t) = K_i(\eta_i(t))x_i(t) \tag{4}$$

for uncertain system (1), (2), (3), such that the closed-loop system is robustly stochastically stable.

It is worthwhile to emphasize that the operation mode of the controller (4) coincides with the local operation mode of the subsystem while the operation mode of the controllers proposed in [2], [5] is the global operation mode of the large-scale system.

Definition 3: The closed-loop system corresponding to the uncertain system (1), (2), (3) with the controller (4) is said to be robustly stochastically stable if there exists a constant $c_1 \in \mathbb{R}^+$ such that $x_i(\cdot) \in \mathbb{L}_2[0, \infty)$, $i \in \mathcal{N}$, and

$$\sum_{i=1}^N \mathrm{E}\left(\int_0^\infty \|x_i(t)\|^2 dt \mid x_{i0}, \eta_{i0}\right) \leq c_1 \sum_{i=1}^N \|x_{i0}\|^2 \tag{5}$$

for any initial conditions x_{i0}, η_{i0}, any admissible local uncertainty $\xi_i(t)$ and any admissible interconnection $r_i(t)$, $i \in \mathcal{N}$.

Remark 1: The robust stochastic stability of the closed-loop large-scale system defined above is equivalent to the absolute stability of the closed-loop system considered in [2].

III. Controller Design

This section presents the main results of the paper. Our controller design technique is based on the decentralized global mode dependent control, which is prescribed in [2], [5]. However, it is different in that our controller uses the local operation mode whilst those in [2], [5] use the global operation mode. In Section III-A, the relation between the operation modes of the subsystems and that of the large-scale system is studied. In Section III-B, a sufficient condition is developed to ensure that our controller solves the problem under consideration. A sufficient condition is provided in Section III-C for the design of a global mode dependent controller for the large-scale system. Then a local mode dependent controller design technique based on the controller in Section III-C is proposed in Section III-D. In Section III-E, we combine all the conditions together and formulate them into a feasible problem for a set of rank constrained linear matrix inequalities. A summary of the design procedure is given in Section III-F.

A. Operation Modes

Let $\mathcal{M}_{\mathrm{p}}$ be a non-empty subset of the set $\mathcal{M}_1 \times \mathcal{M}_2 \times \cdots \times \mathcal{M}_N$, which defines the set of admissible operating patterns of the subsystems. The introduction of the pattern set $\mathcal{M}_{\mathrm{p}}$ allows some restrictions to be imposed on the operation modes of the subsystems. In addition, the number of the elements in the pattern set $\mathcal{M}_{\mathrm{p}}$ is equal to the number of the operation modes of the large-scale system. So bijective functions exist between $\mathcal{M}_{\mathrm{p}}$ and $\mathcal{M}$.

Suppose the bijective function $\Psi : \mathcal{M}_{\mathrm{p}} \to \mathcal{M}$ with $\mu = \Psi(\mu_1, \mu_2, \ldots, \mu_N)$, and its inverse $\Psi^{-1} : \mathcal{M} \to \mathcal{M}_{\mathrm{p}}$ with $(\mu_1, \mu_2, \ldots, \mu_N) = \Psi^{-1}(\mu)$ are given. Then we can further define functions $\Psi_i^{-1} : \mathcal{M} \to \mathcal{M}_i$ with $\mu_i = \Psi_i^{-1}(\mu)$.

Let us first consider an example with such operating mode restrictions.

Example 1: Suppose $N = 3$, $M_i = 2$, $i = 1, 2, 3$. The set $\mathcal{M}_1 \times \mathcal{M}_2 \times \mathcal{M}_3$ denotes all the possible operating patterns of the subsystems, and has 8 elements. However, if some restrictions are imposed on the operation modes, say $\mu_1 = \mu_2 = \mu_3$, then there are only two patterns, which are $(1, 1, 1)$ and $(2, 2, 2)$. Hence the large-scale system has 2 operation modes, and we can let $\mathcal{M} = \{1, 2\}$. Now let $\mathcal{M}_{\mathrm{p}} = \{(1, 1, 1), (2, 2, 2)\}$ be the admissible operating pattern set. Then the bijective function Ψ may be given by $\mu = \Psi(\mu_1, \mu_2, \mu_3) = \mu_1$. Moreover $\mu_i = \Psi_i^{-1}(\mu) = \mu$, $i = 1, 2, 3$.

B. Design Methodology

Instead of studying the stability of the uncertain system (1), (2), (3), (4) directly, we first study the stability of a new class of uncertain systems that contain the uncertain system in (1), (2), (3), (4). The reason we adopt such an approach is that the design technique for this new class of uncertain systems has been well studied.

Consider a class of uncertain large-scale systems given by

$$\begin{cases} \dot{\tilde{x}}_i(t) = \tilde{A}_i(\eta(t))\tilde{x}_i(t) + \tilde{B}_i(\eta(t)) \left[\tilde{u}_i(t) + \tilde{\xi}_i^u(t)\right] \\ \qquad\qquad + \tilde{E}_i(\eta(t))\tilde{\xi}_i(t) + \tilde{L}_i(\eta(t))\tilde{r}_i(t) \\ \tilde{\zeta}_i(t) = \tilde{H}_i(\eta(t))\tilde{x}_i(t) \end{cases} \tag{6}$$

where $\tilde{A}_i(\mu) = A_i(\mu_i)$, $\tilde{B}_i(\mu) = B_i(\mu_i)$, $\tilde{E}_i(\mu) = E_i(\mu_i)$, $\tilde{L}_i(\mu) = L_i(\mu_i)$, $\tilde{H}_i(\mu) = H_i(\mu_i)$, and $\mu_i = \Psi_i^{-1}(\mu)$. The uncertainty inputs $\tilde{\xi}_i(t)$ and $\tilde{r}_i(t)$ are, respectively, the same as $\xi_i(t)$ and $r_i(t)$ in (1), that is, $\tilde{\xi}_i(t) \in \Xi$ and $\tilde{r}_i(t) \in \Pi$. Moreover, $\tilde{\xi}_i^u(t)$ is the uncertainty in the control input, and is described by a function of $\tilde{x}(t)$ and $\eta(t)$ of the form

$$\tilde{\xi}_i^u(t) = \phi_i^u(t, \tilde{x}_i(t), \eta(t))$$

which satisfies the following integral quadratic constraint.

Definition 4: Given $S_i \in \mathbb{S}^+$ and $\beta_i^u(\mu) \in \mathbb{R}^+$, $i \in \mathcal{N}$, $\mu \in \mathcal{M}$. A collection of uncertainty inputs $\tilde{\xi}_i^u(t)$, $i \in \mathcal{N}$, is an admissible uncertainty input for the large-scale system in (6) if there exists a sequence $\{t_l\}_{l=1}^{\infty}$ such that $t_l \to \infty$, $t_l \geq 0$ and

$$\begin{aligned} &\mathrm{E}\left(\int_0^{t_l} \left(\beta_i^u(\eta(t)) \left\|\tilde{x}_i(t)\right\|^2 - \left\|\tilde{\xi}_i^u(t)\right\|^2\right) dt \mid x_{i0}, \eta_{i0}\right) \\ &\geq -x_{i0}^T S_i x_{i0} \end{aligned} \tag{7}$$

for all l and for all $i \in \mathcal{N}$. The set of the admissible uncertainty inputs is denoted by Ξ^u.

Remark 2: The fact that the uncertainty input $\tilde{\xi}_i^u(t)$ depends on $\tilde{x}_i(t)$ and $\eta(t)$ is justified by the form of the control input $\tilde{u}_i(t)$, which depends on $\tilde{x}_i(t)$ and $\eta(t)$ as well. Note that due to $\tilde{\xi}_i^u(t)$ being a function of the global mode process $\eta(t)$, the subsystems of the system (6) are also governed by the global mode process $\eta(t)$.

Associated with the uncertain system (6), (2), (3), (7), is the cost function

$$\begin{aligned} J \triangleq \sum_{i=1}^{N} \mathrm{E}\bigg(\int_0^{\infty} &\left[\tilde{x}_i^T(t) Q_i(\eta(t)) \tilde{x}_i(t)\right. \\ &\left. + \tilde{u}_i^T(t) R_i(\eta(t)) \tilde{u}_i(t)\right] dt \mid x_{i0}, \eta_{i0}\bigg) \end{aligned} \tag{8}$$

where $Q_i(\mu) \in \mathbb{S}^+$, $R_i(\mu) \in \mathbb{S}^+$, $i \in \mathcal{N}$, $\mu \in \mathcal{M}$.

Consider a decentralized global mode dependent state-feedback controller of the form

$$\tilde{u}_i(t) = \tilde{K}_i(\eta(t))\tilde{x}_i(t) \tag{9}$$

such that the closed-loop system in (6), (2), (3), (7), (9), is robustly stochastically stable and the cost function in (8) satisfies $J < c$ for some $c \in \mathbb{R}^+$.

The following result provides a sufficient condition for when the controller in (4) will stabilize the uncertain system (1) if controller (9) stabilizes the system (6).

Theorem 1: Suppose controller (9) stochastically stabilizes the uncertain large-scale system (6) subject to the constraints (2), (3), (7). If the controller gains $K_i(\cdot)$ in (4) are chosen so that

$$\left\|\tilde{K}_i(\mu) - K_i(\mu_i)\right\|^2 \leq \beta_i^u(\mu) \tag{10}$$

where $\mu_i = \Psi_i^{-1}(\mu)$ for all $i \in \mathcal{N}$, $\mu \in \mathcal{M}$, then the controller in (4) stochastically stabilizes the uncertain large-scale system (1) subject to the constraints (2), (3).

Proof: The idea here follows that in [6], [7]. Define $\Delta_i(\mu) \triangleq \tilde{K}_i(\mu) - K_i(\mu_i)$ where $\mu_i = \Psi_i^{-1}(\mu)$. Then (10) implies $\|\Delta_i(\mu)\|^2 \leq \beta_i^u(\mu)$. Consider a particular uncertainty in the control input of the form $\tilde{\xi}_i^u(t) = -\Delta_i(\eta(t))\tilde{x}_i(t)$. We have

$$\left\|\tilde{\xi}_i^u(t)\right\|^2 \leq \beta_i^u(\eta(t)) \left\|\tilde{x}_i(t)\right\|^2.$$

So (7) holds for any $S_i \in \mathbb{S}^+$. That is, $\tilde{\xi}_i^u(t)$ is an admissible uncertainty input for the system (6). Also, we have

$$\begin{aligned}\tilde{u}_i(t) + \tilde{\xi}_i^u(t) &= \left[\tilde{K}_i(\eta(t)) - \Delta_i(\eta(t))\right]\tilde{x}_i(t)\\ &= K_i(\eta_i(t))\tilde{x}_i(t)\end{aligned}$$

which is of the same form as in (4). Hence the class of the systems modeled by (1), (2), (3), (4) is a subclass of the class of the systems modeled by (6), (2), (3), (7), (9). Therefore, the stability of system (6) implies that of system (1). ■

To stabilize uncertain large-scale system (1), (2), (3), using controller (4) instead of controller (9) leads to at least two benefits. Firstly, knowing the local operation modes is enough. This can be seen from the fact that the operation mode of controller (4) is the same as that of the subsystem $\mathcal{S}_i$ that (4) controls. In contrast, controller (9) uses the global operation mode of the large-scale system. In other words, controller (4) changes its mode only when the subsystem $\mathcal{S}_i$ does, while controller (9) will change its mode if any subsystem does. So, the controller in (4) designed for subsystem $\mathcal{S}_i$ does not need to know the operation modes of the other subsystems.

Secondly, a fewer number of controllers need to be implemented. If controller (4) is used, we need to implement M_i controllers for subsystem $\mathcal{S}_i$. So the total number of the controllers to be implemented is $\sum_{i=1}^N M_i$. If we use the controller (9) as was proposed in the control algorithm presented in [2], then M controllers are needed for each subsystem. So the total number of the controllers is NM. Obviously, $\sum_{i=1}^N M_i \leq NM$. In particular, when no restrictions are imposed on the operating patterns of the subsystems, the number of the controllers in (4) is much less than the number of controllers in (9).

C. Design of Global Mode Dependent Controllers

In this section, a sufficient condition is established for the design of a stabilizing controller of the form (9). This condition, together with Theorem 1, provides a basis for the design of a local mode dependent stabilizing controller of the form (4).

Theorem 2: 1) If there exist constants $\tau_i \in \mathbb{R}^+$, $\theta_i \in \mathbb{R}^+$, $\tau_i^u \in \mathbb{R}^+$, $i \in \mathcal{N}$, such that the coupled algebraic Riccati equations

$$\begin{aligned}&\tilde{A}_i^T(\mu)X_i(\mu) + X_i(\mu)\tilde{A}_i(\mu) + \sum_{\nu=1}^{M} q_{\mu\nu}X_i(\nu) + Q_i(\mu)\\ &+ \left[\tau_i + \sum_{j=1, j\neq i}^{N}\theta_j\right]\tilde{H}_i^T(\mu)\tilde{H}_i(\mu) + \tau_i^u\beta_i^u(\mu)I\\ &- X_i(\mu)\left[\tilde{B}_i(\mu)R_i^{-1}(\mu)\tilde{B}_i^T(\mu) - \frac{1}{\tau_i^u}\tilde{B}_i(\mu)\tilde{B}_i^T(\mu)\right.\\ &\left. - \frac{1}{\tau_i}\tilde{E}_i(\mu)\tilde{E}_i^T(\mu) - \frac{1}{\theta_i}\tilde{L}_i(\mu)\tilde{L}_i^T(\mu)\right]X_i(\mu) = 0; \quad (11)\end{aligned}$$

have solutions $X_i(\mu) \in \mathbb{S}^+$ for all $i \in \mathcal{N}$, $\mu \in \mathcal{M}$. Then the controller (9) given by

$$\tilde{K}_i(\mu) = -R_i^{-1}(\mu)\tilde{B}_i^T(\mu)X_i(\mu) \quad (12)$$

robustly stabilizes the uncertain system (6) subject to the constraints (2), (3), (7), and leads to the cost bound

$$J \leq \sum_{i=1}^{N} x_{i0}^T\left[X_i(\eta_{i0}) + \tau_i\bar{S}_i + \theta_i\tilde{S}_i + \tau_i^u S_i\right]x_{i0}.$$

2) The claim in part 1 remains true if instead of the ARE (11) one uses the coupled algebraic Riccati inequalities obtained by replacing the "=" sign in (11) with "<".

Proof: The idea here follows that in [2], [5]. Define a controlled output for the system (6) as

$$\tilde{z}_i(t) \triangleq \begin{bmatrix} Q_i^{\frac{1}{2}}(\eta(t)) \\ 0 \end{bmatrix}\tilde{x}_i(t) + \begin{bmatrix} 0 \\ R_i^{\frac{1}{2}}(\eta(t)) \end{bmatrix}\tilde{u}_i(t).$$

Then the cost function (8) can be rewritten as

$$J = \sum_{i=1}^{N} \mathrm{E}\left(\int_0^\infty \|\tilde{z}_i(t)\|^2\,dt \mid x_{i0}, \eta_{i0}\right).$$

The rest of the proof for part 1) is similar to the proof of Theorem 1 in [2] (or that of Theorem 12 in [5]). The proof for part 2) follows the same lines except the LMI version of the bounded real lemma is used. ■

Remark 3: If the initial operation mode of the large-scale system is random and there exist a transition group of measure-preserving set transformations $\Gamma_s : \Omega \to \Omega$ such that $\eta(t, \Gamma_s\omega) = \eta(t+s, \omega)$ almost surely for all $t, s \geq 0$ [8, Chapter XI]. Then the sufficient condition in part 1) of Theorem 2 is also necessary. The reason is that the lossless S-procedure developed in Lemma 2 of [2] becomes valid. The proof follows similar lines to that of Theorem 1 in [2].

Remark 4: As explained in Theorem 2 of [2], if the infimum of

$$\inf_{\substack{\tau_i,\theta_i,\tau_i^u,X_i(\cdot)\\ \text{subject to (11)}}} \sum_{i=1}^{N} x_{i0}^T\left[X_i(\eta_{i0}) + \tau_i\bar{S}_i + \theta_i\tilde{S}_i + \tau_i^u S_i\right]x_{i0}$$

is obtained at τ_i^*, θ_i^*, τ_i^{u*}, $X_i^*(\mu)$, $i \in \mathcal{N}$, $\mu \in \mathcal{M}$, then the controller (9), (12), in which $X_i(\mu) = X_i^*(\mu)$, is optimal in the sense that the infimum of $\sup_{\Xi,\Pi,\Xi^u} J$ is achieved.

D. Design of Local Mode Dependent Controllers

Suppose a global mode dependent controller (9) is given. In this section we describe how a corresponding local mode dependent controller of the form (4) can be constructed. Also, we present a probabilistic interpretation of the proposed design.

Theorem 3: Given the controller (9), let

$$K_i(\upsilon_i) = \frac{\sum_{\mu=1}^{M}\{\tilde{K}_i(\mu)\pi_{\infty\mu}\mathbb{I}_i(\mu, \upsilon_i)\}}{\sum_{\mu=1}^{M}\{\pi_{\infty\mu}\mathbb{I}_i(\mu, \upsilon_i)\}} \quad (13)$$

for all $\upsilon_i \in \mathcal{M}_i$, $i \in \mathcal{N}$, where

$$\pi_\infty = \mathbf{e}(\mathbf{Q} + \mathbf{E})^{-1}$$

$$\mathbb{I}_i(\mu, \upsilon_i) = \begin{cases} 1 & \text{if } \upsilon_i = \Psi_i^{-1}(\mu) \\ 0 & \text{otherwise} \end{cases}$$

and $\mathbf{e} = [1\ 1\ \cdots\ 1] \in \mathbb{R}^{1\times M}$, $\mathbf{E} = [\mathbf{e}^T\ \mathbf{e}^T\ \cdots\ \mathbf{e}^T]^T \in \mathbb{R}^{M\times M}$. Then

$$K_i(\upsilon_i) = \lim_{t\to\infty} \mathrm{E}\left(\tilde{K}_i(\eta(t)) \mid \eta_i(t) = \upsilon_i\right).$$

Moreover,

$$\Delta_i(\mu) = \frac{\sum_{\nu=1,\nu\neq\mu}^{M}\{\mathbb{I}_i(\nu,\mu_i)\pi_{\infty\nu}\left[\tilde{K}_i(\mu) - \tilde{K}_i(\nu)\right]\}}{\sum_{\nu=1}^{M}\{\mathbb{I}_i(\nu,\mu_i)\pi_{\infty\nu}\}} \tag{14}$$

where $\mu_i = \Psi_i^{-1}(\mu)$.

Proof: First we observe that π_∞ is the steady state distribution of $\eta(t)$. The ergodic property of Markov process $\eta(t)$ implies that

$$\lim_{t\to\infty} e^{\mathbf{Q}t} = \begin{bmatrix}\pi_\infty\\ \vdots \\ \pi_\infty\end{bmatrix}.$$

Then the probability distribution of the control gain of the controller (9) has a limit as $t\to\infty$ given by

$$\lim_{t\to\infty} \Pr(\tilde{K}_i(\eta(t)) = \tilde{K}_i(\mu)) = \pi_{\infty\mu}$$

for all $\mu \in \mathcal{M}$. So the expected value of the control gain conditioned on the subsystem operation modes as time approaches infinity is

$$\begin{aligned}
&\lim_{t\to\infty} \mathrm{E}\left(\tilde{K}_i(\eta(t)) \mid \eta_i(t) = \upsilon_i\right)\\
&= \sum_{\mu=1}^{M}\{\tilde{K}_i(\mu)\lim_{t\to\infty}\Pr(\eta(t)=\mu \mid \eta_i(t)=\upsilon_i)\}\\
&= \sum_{\mu=1}^{M}\{\tilde{K}_i(\mu)\lim_{t\to\infty}\frac{\Pr(\eta(t)=\mu,\eta_i(t)=\upsilon_i)}{\Pr(\eta_i(t)=\upsilon_i)}\}\\
&= \frac{\sum_{\mu=1}^{M}\{\tilde{K}_i(\mu)\pi_{\infty\mu}\mathbb{I}_i(\mu,\upsilon_i)\}}{\sum_{\mu=1}^{M}\{\pi_{\infty\mu}\mathbb{I}_i(\mu,\upsilon_i)\}}\\
&= K_i(\upsilon_i).
\end{aligned}$$

Moreover, let $\mu_i = \Psi_i^{-1}(\mu)$. We have

$$\begin{aligned}
&\Delta_i(\mu) = \tilde{K}_i(\mu) - K_i(\mu_i)\\
&= \tilde{K}_i(\mu) - \frac{\sum_{\nu=1}^{M}\{\tilde{K}_i(\nu)\pi_{\infty\nu}\mathbb{I}_i(\nu,\mu_i)\}}{\sum_{\nu=1}^{M}\{\pi_{\infty\nu}\mathbb{I}_i(\nu,\mu_i)\}}\\
&= \frac{\sum_{\nu=1}^{M}\{\tilde{K}_i(\mu)\pi_{\infty\nu}\mathbb{I}_i(\nu,\mu_i)\} - \sum_{\nu=1}^{M}\{\tilde{K}_i(\nu)\pi_{\infty\nu}\mathbb{I}_i(\nu,\mu_i)\}}{\sum_{\nu=1}^{M}\{\pi_{\infty\nu}\mathbb{I}_i(\nu,\mu_i)\}}\\
&= \frac{\sum_{\nu=1}^{M}\{\mathbb{I}_i(\nu,\mu_i)\pi_{\infty\nu}\left[\tilde{K}_i(\mu)-\tilde{K}_i(\nu)\right]\}}{\sum_{\nu=1}^{M}\{\mathbb{I}_i(\nu,\mu_i)\pi_{\infty\nu}\}}\\
&= \frac{\sum_{\nu=1,\nu\neq\mu}^{M}\{\mathbb{I}_i(\nu,\mu_i)\pi_{\infty\nu}\left[\tilde{K}_i(\mu)-\tilde{K}_i(\nu)\right]\}}{\sum_{\nu=1}^{M}\{\mathbb{I}_i(\nu,\mu_i)\pi_{\infty\nu}\}}.
\end{aligned}$$

This completes the proof. ■

Remark 5: Controller (13) is the expected value of controller (9) conditioned on the subsystem operation modes as time approaches infinity, and also the minimum variance estimate of controller (9), in the sense that $\lim_{t\to\infty}\mathrm{E}\left(\left\|\tilde{K}_i(\eta(t)) - K_i(\eta_i(t))\right\|_{\mathrm{F}}^2 \mid \eta_i(t)\right)$ is minimal [9, Theorem 3.1]. Here $\|\cdot\|_{\mathrm{F}}$ denotes Frobenius norm.

Let us use an example to illustrate the design method.

Example 2: Suppose $N = 2$, $M_1 = 2$, $M_2 = 3$, $M = 6$, and

$$\pi_\infty = \begin{bmatrix}\pi_{\infty1} & \pi_{\infty2} & \cdots & \pi_{\infty6}\end{bmatrix}.$$

It follows from (13) that the local mode dependent control gain for subsystem $\mathcal{S}_1$ at mode 1 is

$$K_1(1) = \frac{\pi_{\infty1}\tilde{K}_1(1) + \pi_{\infty2}\tilde{K}_1(2) + \pi_{\infty3}\tilde{K}_1(3)}{\pi_{\infty1}+\pi_{\infty2}+\pi_{\infty3}}.$$

In addition, mismatches between global and local mode control gains can be computed. For example, $\Delta_1(1)$ can be computed as follows:

$$\Delta_1(1) = \frac{\pi_{\infty2}[\tilde{K}_1(1)-\tilde{K}_1(2)] + \pi_{\infty3}[\tilde{K}_1(1)-\tilde{K}_1(3)]}{\pi_{\infty1}+\pi_{\infty2}+\pi_{\infty3}}.$$

E. Computational Method

A computational method for the design of the controller (4) is presented in this subsection. The method is based upon Theorems 1, 2, and 3, and is formulated as a feasibility problem for a set of rank constrained linear matrix inequalities.

Theorem 4: Suppose there exist matrices $X_i(\mu) \in \mathbb{S}^+$, $\bar{X}_i(\mu) \in \mathbb{S}^+$, scalars $\tilde{\beta}_i(\mu) \in \mathbb{R}^+$, $\bar{\beta}_i(\mu) \in \mathbb{R}^+$, $\bar{\tau}_i^u \in \mathbb{R}^+$, $\bar{\tau}_i \in \mathbb{R}^+$, $\bar{\theta}_i \in \mathbb{R}^+$, $i \in \mathcal{N}$, $\mu \in \mathcal{M}$, such that the coupled linear matrix inequalities

$$\begin{bmatrix}\Upsilon_{i11}(\mu) & \Upsilon_{i12}(\mu) & \Upsilon_{i13}(\mu)\\ \Upsilon_{i12}^T(\mu) & \Upsilon_{i22}(\mu) & 0\\ \Upsilon_{i13}^T(\mu) & 0 & \Upsilon_{i33}(\mu)\end{bmatrix} < 0 \tag{15}$$

$$\begin{bmatrix}\bar{\tau}_i^u I & \Delta_i(\mu)\\ \Delta_i^T(\mu) & \tilde{\beta}_i(\mu)I\end{bmatrix} \geq 0 \tag{16}$$

$$\begin{bmatrix}\bar{X}_i(\mu) & I\\ I & X_i(\mu)\end{bmatrix} \geq 0 \tag{17}$$

$$\begin{bmatrix}\bar{\beta}_i(\mu) & 1\\ 1 & \tilde{\beta}_i(\mu)\end{bmatrix} \geq 0 \tag{18}$$

with rank constraints

$$\operatorname{rank}\left(\begin{bmatrix}\bar{X}_i(\mu) & I\\ I & X_i(\mu)\end{bmatrix}\right) \leq n \tag{19}$$

$$\operatorname{rank}\left(\begin{bmatrix}\bar{\beta}_i(\mu) & 1\\ 1 & \tilde{\beta}_i(\mu)\end{bmatrix}\right) \leq 1 \tag{20}$$

hold for all $i \in \mathcal{N}$, $\mu \in \mathcal{M}$, where

$$\begin{aligned}
\Upsilon_{i11}(\mu) &= \bar{X}_i(\mu)\tilde{A}_i^T(\mu) + \tilde{A}_i(\mu)\bar{X}_i(\mu) + q_{\mu\mu}\bar{X}_i(\mu)\\
&\quad - \tilde{B}_i(\mu)R_i^{-1}(\mu)\tilde{B}_i^T(\mu) + \bar{\tau}_i^u\tilde{B}_i(\mu)\tilde{B}_i^T(\mu)\\
&\quad + \bar{\tau}_i\tilde{E}_i(\mu)\tilde{E}_i^T(\mu) + \bar{\theta}_i\tilde{L}_i(\mu)\tilde{L}_i^T(\mu)\\
\Upsilon_{i12}(\mu) &= \bar{X}_i(\mu)\begin{bmatrix}I & I & \tilde{H}_i^T(\mu) & \tilde{H}_i^T(\mu) & \cdots\end{bmatrix.\\
&\qquad \begin{matrix}\tilde{H}_i^T(\mu) & \tilde{H}_i^T(\mu) & \cdots & \tilde{H}_i^T(\mu)\end{matrix}\Big]\\
\Upsilon_{i22}(\mu) &= -\operatorname{diag}(Q_i^{-1}(\mu), \bar{\beta}_i(\mu)I, \bar{\tau}_iI, \bar{\theta}_1I, \ldots,\\
&\qquad \bar{\theta}_{i-1}I, \bar{\theta}_{i+1}I, \ldots, \bar{\theta}_NI)
\end{aligned}$$

$$\Upsilon_{i13}(\mu) = \bar{X}_i(\mu)\left[\sqrt{q_{\mu,1}}I \quad \cdots \quad \sqrt{q_{\mu,\mu-1}}I \right.$$
$$\left. \sqrt{q_{\mu,\mu+1}}I \quad \cdots \quad \sqrt{q_{\mu,M}}I\right]$$
$$\Upsilon_{i33}(\mu) = -\operatorname{diag}(\bar{X}_i(1),\ldots,\bar{X}_i(\mu-1), \bar{X}_i(\mu+1),\ldots,\bar{X}_i(M))$$
$$\Delta_i(\mu) = \frac{1}{\sum_{\nu=1}^{M}\{\mathbb{I}_i(\nu,\mu_i)\pi_{\infty\nu}\}}\sum_{\nu=1,\nu\neq\mu}^{M}\{\mathbb{I}_i(\nu,\mu_i)\pi_{\infty\nu} \times \left[R_i^{-1}(\nu)\tilde{B}_i^T(\nu)X_i(\nu) - R_i^{-1}(\mu)\tilde{B}_i^T(\mu)X_i(\mu)\right]\Big\}$$

and μ_i in $\Delta_i(\mu)$ is given by $\mu_i = \Psi_i^{-1}(\mu)$. Then a stabilizing decentralized controller of the form (4) for the uncertain system (1) subject to the constraints (2), (3) is given by (13).

Proof: The idea of the proof follows that in [5], [7]. Define $\bar{X}_i(\mu) \triangleq (X_i(\mu))^{-1}$, $\bar{\tau}_i^u \triangleq (\tau_i^u)^{-1}$, $\bar{\tau}_i \triangleq (\tau_i)^{-1}$, $\bar{\theta}_i \triangleq (\theta_i)^{-1}$, $\tilde{\beta}_i(\mu) \triangleq \tau_i^u\beta_i^u(\mu)$, and $\bar{\beta}_i(\mu) \triangleq (\tilde{\beta}_i(\mu))^{-1}$. Consider the matrix inequalities in part 2) of Theorem 2 which were obtained from (11) by replacing "$=$" with "$<$". By pre- and post-multiplying both sides of the i-th inequality by $\bar{X}_i(\mu)$ we obtain the coupled algebraic Riccati inequalities

$$\begin{aligned}
&\bar{X}_i(\mu)\tilde{A}_i^T(\mu) + \tilde{A}_i(\mu)\bar{X}_i(\mu) + q_{\mu\mu}\bar{X}_i(\mu) \\
&+ \sum_{\nu=1,\nu\neq\mu}^{M} q_{\mu\nu}\bar{X}_i(\mu)(\bar{X}_i(\nu))^{-1}\bar{X}_i(\mu) + \bar{X}_i(\mu)\left[Q_i(\mu)\right. \\
&\left.+\tilde{\beta}_i(\mu)I + \left[\tau_i + \sum_{j=1,j\neq i}^{N}\theta_j\right]\tilde{H}_i^T(\mu)\tilde{H}_i(\mu)\right]\bar{X}_i(\mu) \\
&- \tilde{B}_i(\mu)R_i^{-1}(\mu)\tilde{B}_i^T(\mu) + \bar{\tau}_i^u\tilde{B}_i(\mu)\tilde{B}_i^T(\mu) + \bar{\tau}_i\tilde{E}_i(\mu)\tilde{E}_i^T(\mu) \\
&+ \bar{\theta}_i\tilde{L}_i(\mu)\tilde{L}_i^T(\mu) < 0. \qquad (21)
\end{aligned}$$

Furthermore, using Schur complement equivalence, (21) is equivalent to (15).

Since $\beta^u(\mu) = \tilde{\beta}_i(\mu)\bar{\tau}_i^u$, the norm condition in (10) becomes $\|\Delta_i(\mu)\|^2 \le \tilde{\beta}_i(\mu)\bar{\tau}_i^u$, which is further equivalent to (16) in view of (14) and (12).

In addition, equality constraints

$$X_i(\mu)\bar{X}_i(\mu) = I, \qquad \tilde{\beta}_i(\mu)\bar{\beta}_i(\mu) = 1 \qquad (22)$$

are also required because (15) is linear in $\bar{X}_i(\mu)$, $\bar{\beta}_i(\mu)$, $\bar{\tau}_i$, $\bar{\theta}_i$ and $\bar{\tau}_i^u$, while (16) linear in $X_i(\mu)$, $\bar{\tau}_i^u$ and $\tilde{\beta}_i(\mu)$. The equality constraints in (22) are equivalent to LMIs (17), (18) with rank constraints (19), (20).

Therefore, if LMIs (15), (16), (17), (18) with rank constraints (19), (20) have solutions, then both (10) and (21) hold. Therefore, controller (9) can be constructed by (12), and the stabilizing controller of form (4) is given by (13). ■

Remark 6: Due to the equality constraints in (22), the solution set to (15), (16) and (22) is non-convex, and it is generally difficult to solve such problems. Fortunately, several numerical algorithms have been proposed for this purpose, such as the cone complementarity linearization algorithm in [10], the tangent and lift method in [11]. These algorithms numerically perform very well although the convergence from an arbitrary initial condition to a feasible solution is not guaranteed. In this paper, we used the algorithm from [11].

F. Design Procedure

The proposed controller design procedure is summarized as follows.

1) Find a feasible solution to the rank constrained LMI problem in (15), (16), (17), (18), (19), (20) using the algorithm in [11];
2) Using this feasible solution, calculate the global mode dependent controller (9) according to equation (12);
3) Construct the local mode dependent controller (4) using equation (13).

IV. CONCLUSIONS

This paper has studied the decentralized stabilization problem for a class of uncertain Markovian jump large-scale systems. The controllers are entirely decentralized with respect to the subsystems. They use the local system states and the local operation modes of the subsystems to produce the local control inputs. A sufficient condition in terms of rank constrained linear matrix inequalities has been developed to construct the state feedback controllers.

A number of possible areas for future research are motivated by the results of this paper. These include an extension of the results to the output feedback case, consideration of the discrete-time case, and an extension to the problem of control synthesis with guaranteed H_∞ performance.

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