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Weight Selection for Gap Robustness with Degree-Constrained Controllers

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Abstract— **In modern robust control, control synthesis may be cast as an interpolation problem where the interpolant relates to robustness and performance criteria. In particular, robustness in the gap fits into this framework and the magnitude of the corresponding interpolant dictate the robustness to perturbations of the plant as a function of frequency. In this paper we consider the correspondence between weighted entropy functionals and minimizing interpolants in order to find appropriate interpolants for e.g. control synthesis. There are two basic issues that we address: we first characterize admissible shapes of minimizers by studying the corresponding inverse problem, and then we develop effective ways of shaping minimizers via suitable choices of weights. These results are used in order to systematize feedback control synthesis to obtain frequency dependent robustness bounds with a constraint on the controller degree.**

I. Introduction

The topic of this paper relates to the framework and the mathematics of modern robust control. The foundational work [33] of George Zames in the early 1980's cast the basic robust control problem as an analytic interpolation problem, where interpolation constraints ensure stability of the feedback scheme, and a norm bound guarantees performance and robustness. In this context, the analytic interpolant represents a particular transfer function of the feedback system. The work of Zames and the fact that the degree of the interpolant relates to the dimension of the closed-loop system motivated a program to investigate analytic interpolation with degree constraint (see [8], [9]). This led to an approach based on convex optimization, in which interpolants of a certain degree are obtained as minimizers of *weighted* entropy functionals. In this paper we study the correspondence between weights and such interpolants, and we develop a theory which allows for systematic shaping of interpolants to specification.

The basic issue of how the choice of weights and indices in optimization problems affects the final design is by no means new. It was R.E. Kalman [20] who, in the context of quadratic optimal control, first raised the question of what it is that characterizes optimal designs and, further, how to describe all performance criteria for which a certain design is optimal. Following Kalman's example we study here the analogous inverse problem for analytic interpolation with complexity constraint.

The analysis of the inverse problem leads to a new procedure for feedback control synthesis. More specifically, the quality of control depends on the frequency characteristics of the interpolant, which in turn dictates the loop shape of the feedback control system. The theory of [8], [9] provides a parametrization of all interpolants, having degree less than the number of interpolation constraints, in terms of weights in a suitable class. The choice of weights for feedback control design via this procedure has been the subject of several papers (see e.g., [27], [28]). The challenge stems from the fact that the correspondence between weights and the shape of interpolants is nonlinear. One of the contributions of this paper is to develop a systematic procedure for the selection of weights based on the inverse problem.

The synthesis proceeds in two steps. We first obtain an interpolant with the required shape, but without any restriction on the degree. Then, via the inverse problem, we identify all weights for which the given interpolant is a minimizer of the corresponding entropy functional. The problem of approximating the interpolant by one of lower degree is then replaced by approximating weights in a suitable class. This approximation problem is quasi-convex and can be solved by standard methods. Hence we have replaced a non-convex problem by one that is tractable.

This paper is a short version of [22] which is a considerable extension of [21]. If nothing else is stated, proofs of theorems, propositions, and lemmas are found in [22]. In Section II we establish notation and review basic facts on bounded analytic interpolation and complexity-constrained interpolation. We only discuss interpolation in the unit disc $\mathbb{D} = \{z : |z| < 1\}$, but the theory applies equally well to interpolation in the half plane. In Section III we consider a robust design example where the robustness margin is frequency-dependent. In Section IV we provide the characterization of minimizers of weighted entropy functionals and describe the set of weights which give interpolants of a prespecified bounded degree. Here we also formulate and solve the inverse problem which is one of the key tools needed in the paper. In Section V we study continuity properties of the mapping from weights to minimizers, and in Section VI we develop a method for degree reduction of interpolants via the corresponding weights. Finally, in Section VII, we revisit the motivating example of Section III and apply the procedure of Section VI.

II. Background

Given complex numbers $z_0, z_1, \ldots, z_n$ in $\mathbb{D}$ which we assume to be distinct for simplicity, and given complex numbers $w_0, w_1, \ldots, w_n$, the classical Pick interpolation problem asks for a function f in the *Schur class*

$$\mathcal{S} = \{f \in H_\infty(\mathbb{D}) : \|f\|_\infty \leq 1\}$$

which satisfies the interpolation condition

$$f(z_k) = w_k, \quad k = 0, 1, \ldots, n, \tag{1}$$

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where $H_\infty(\mathbb{D})$ (or simply H_∞) is the Hardy space of bounded analytic functions on $\mathbb{D}$. It is well-known (see, e.g., [12]) that such a function exists if and only if the Pick matrix

$$P = \left[\frac{1 - w_k \bar{w}_\ell}{1 - z_k \bar{z}_\ell}\right]_{k,\ell=0}^{n} \tag{2}$$

is positive semi-definite. The solution is unique if and only if P is singular, in which case f is a Blaschke product of degree equal to the rank of P. In this paper, throughout, we assume that P is positive definite and hence that there are infinitely many solutions to the Pick problem. A complete parameterization of all solutions was given by Nevanlinna (see e.g. [1]), and for this reason the subject is often referred to as Nevanlinna-Pick interpolation.

In engineering applications f usually represents the transfer function of a feedback control system or of a filter, and therefore the McMillan degree of f is of significant interest. Thus, it is natural to require that f be rational and of bounded degree. Such a constraint completely changes the nature of the underlying mathematical problem.

Following [9], [17], we consider the generalized entropy functional $\mathbb{K}_\Psi : \mathcal{S} \to \mathbb{R} \cup \infty$, defined by

$$\mathbb{K}_\Psi(f) = -\int_{\mathbb{T}} \Psi \log(1 - |f|^2) dm, \tag{3}$$

where Ψ is a non-negative log-integrable function on $\mathbb{T} = \{z = e^{i\theta} : \theta \in (-\pi, \pi]\}$ and $dm := d\theta/2\pi$ is the (normalized) Lebesgue measure on $\mathbb{T}$. We study how the minimizer of

$$\min\{\mathbb{K}_\Psi(f) \ : \ f \in \mathcal{S},\ f(z_k) = w_k,\ k = 0, \ldots, n\} \tag{4}$$

depends on the weighting function Ψ and then determine when an interpolant f is attainable as a minimizer of (4) for a suitable choice of Ψ. One particularly interesting case, as we will see below, is when $\Psi = |\sigma|^2$ and σ belongs to the class of rational functions with poles at the conjugate inverses of the interpolation points.

Let ϕ be the Blaschke product

$$\phi(z) = \prod_{k=0}^{n} \frac{z_k - z}{1 - \bar{z}_k z} \tag{5}$$

and let $U : f(z) \to z f(z)$ denote the standard shift operator on H_2. Then ϕH_2 is a shift invariant subspace, i.e. $f \in \phi H_2$ implies that $U(f) = zf \in \phi H_2$. Denote by $\mathcal{K}$ the co-invariant subspace $H_2 \ominus \phi H_2$. Then $\mathcal{K}$ is invariant under U^*, where U^* denotes the adjoint of U. Let $\mathcal{K}_0$ denote the set of outer functions in $\mathcal{K}$ that are positive at the origin. The following result is taken from [9].

Theorem 1: Suppose that the Pick matrix (2) *is positive definite, and let σ be an arbitrary function in $\mathcal{K}_0$. Then there exists a unique pair of elements $(a, b) \in \mathcal{K}_0 \times \mathcal{K}$ such that*

(i) $f = b/a \in H^\infty$ *with* $\|f\|_\infty \le 1$
(ii) $f(z_k) = w_k, \quad k = 0, 1, \ldots, n$, *and*
(iii) $|a|^2 - |b|^2 = |\sigma|^2$ *a.e. on* $\mathbb{T}$.

Conversely, any pair $(a, b) \in \mathcal{K}_0 \times \mathcal{K}$ satisfying (i) and (ii) determines, via (iii), a unique $\sigma \in \mathcal{K}_0$. Moreover, setting $\Psi = |\sigma|^2$, the optimization problem

$$\min \mathbb{K}_\Psi(f) \ \text{ s.t. } \ f(z_k) = w_k,\ k = 0, \ldots, n$$

has a unique solution f that is precisely the unique $f \in \mathcal{S}$ satisfying conditions (i), (ii) and (iii).

We define

$$\tau(z) := \prod_{k=0}^{n} (1 - \bar{z}_k z),$$

where $\{z_k\}_{k=0}^n$ are the interpolation points. Then, it is easy to see that

$$\mathcal{K} = \left\{\frac{p(z)}{\tau(z)} \ : \ p \in \mathrm{Pol}(n)\right\}, \mathcal{K}_0 = \left\{\frac{p(z)}{\tau(z)} \ : \ p \in \mathrm{Pol}_+(n)\right\},$$

where $\mathrm{Pol}(n)$ denotes the set of polynomials of degree at most n, and where $\mathrm{Pol}_+(n)$ denotes the subset of polynomials $p \in \mathrm{Pol}(n)$ such that $p(z) \neq 0$ in $\mathbb{D}$ and $p(0) > 0$.

The function $\sigma = p/\tau \in \mathcal{K}_0$ with $p \in \mathrm{Pol}_+(n)$, as in the theorem, represents a spectral factor of the nonnegative function $1 - |f|^2$ on the unit circle. Thus, the n roots of the polynomial $z^n \bar{p}(z^{-1})$ are often referred to as *spectral zeros*. We also note that the degree of f may be less than n, which happens when a, b, and σ, have common roots.

The theorem, stated in [9], allows for considerably more general interpolation conditions than (ii). In the case where the points $\{z_0, \ldots, z_n\}$ are not necessarily distinct, condition (ii) needs to be replaced by

$$f = f_0 + \phi q \text{ with } q \in H_\infty(\mathbb{D}),$$

which encapsulates interpolation of derivatives as well. The theorem is also valid when ϕ is an arbitrary inner function. The background to the derivation of Theorem 1 has a long history. The existence part of the parameterization was first proved in the covariance extension case in [13], [14] and in the Nevanlinna-Pick case in [15]. The uniqueness part (as well as well-posedness) in [7]. The optimization approach was initiated in [5] (also, see the extended version [6]) and further developed in, e.g., [8], [4], [16].

III. Frequency-dependent robustness margin

To motivate our procedure we consider a problem related to H_∞ loop-shaping and robustness in the gap metric.

Let P_0 denote the transfer function of a single-input, single-output finite-dimensional linear system

$$P_0 = \frac{N_0}{M_0},$$

with stable coprime factors $M_0, N_0 \in H_\infty$, normalized to satisfy

$$M_0^* M_0 + N_0^* N_0 = 1, \text{ on } \mathbb{T}, \tag{6}$$

where $f(z)^* := \overline{f(\bar{z}^{-1})}$. Then, as is well-known, all stabilizing controllers for P_0 are parameterized by $q \in H_\infty$ via

$$C = \frac{U_0 + M_0 q}{V_0 + N_0 q}, \tag{7}$$

where $U_0, V_0 \in H_\infty$ satisfy $V_0M_0 - U_0N_0 = 1$, see, e.g., [10], [30]. To model perturbations of the coprime factors for frequency-dependent uncertainty, we consider plants $P = N/M$ such that

$$\left\| \begin{pmatrix} M(z) - M_0(z) \\ N(z) - N_0(z) \end{pmatrix} \right\| < \alpha|w(z)| \ \text{ for } z \in \mathbb{T}, \tag{8}$$

where $\|\cdot\|$ denotes Euclidean vector norm and w is an outer function shaping the radius. Moreover, the size of the radius is controlled by a separate scaling parameter $\alpha \in \mathbb{R}_+$. Thus, we consider the problem of robust stabilization of the ball of plants

$$\mathcal{B}(P_0, \alpha w) := \left\{ P = \frac{N}{M} \ : \ (8) \text{ holds} \right\},$$

around the center P_0.

As shown in [31], a controller specified by q stabilizes $\mathcal{B}(P_0, \alpha w)$ provided

$$\left\| \begin{pmatrix} U_0 + M_0 q \\ V_0 + N_0 q \end{pmatrix} \alpha w \right\|_\infty \leq 1. \tag{9}$$

This condition can be expressed as a Nevanlinna-Pick problem. Indeed, taking advantage of the normalization of the coprime factors as in [24] (see also [18]), we define the transformation

$$Z := \begin{pmatrix} M_0^* & N_0^* \\ -N_0 & M_0 \end{pmatrix}$$

which is unitary, i.e., $ZZ^* = Z^*Z = I$. We also denote by ϕ the Blaschke product that vanishes at the conjugate inverse of the poles of M_0, N_0. Hence, ϕM_0^*, $\phi N_0^* \in H_\infty$. Then, the left hand side of (9) is

$$\left\| \begin{pmatrix} \phi & 0 \\ 0 & 1 \end{pmatrix} Z \begin{pmatrix} U_0 + M_0 q \\ V_0 + N_0 q \end{pmatrix} \alpha w \right\|_\infty = \left\| \begin{pmatrix} F \\ 1 \end{pmatrix} \alpha w \right\|_\infty, \tag{10}$$

where $F_0 = \phi M_0^* U_0 + \phi N_0^* V_0 \in H_\infty$ and

$$F = F_0 + \phi q. \tag{11}$$

It can be seen that the values of F at the roots of ϕ are independent of q and are specified by the plant. Moreover, as seen from (10), condition (9) holds provided $F \in H_\infty$ satisfies (11) and

$$\sqrt{|F|^2 + 1} \leq \frac{1}{\alpha|w|}, \ \text{ on } \mathbb{T}. \tag{12}$$

Conversely, for any $F \in H_\infty$ satisfying (11), there corresponds a unique parameter q and a controller C, where C stabilizes the ball of plants $\mathcal{B}(P_0, \alpha w)$ with radius

$$\alpha|w| = (|F|^2 + 1)^{-\frac{1}{2}}.$$

Furthermore, if the degree of F is small, so is the degree of the controller C. This is stated in the following proposition.

Proposition 2: Let $F \in H_\infty$ satisfy (11) and C be the controller specified via (11) and (7). Then

$$\deg C \leq \deg F.$$

We consider

$$\Pi_{P//C} := \begin{pmatrix} \frac{P_0}{1-P_0C} & \frac{-P_0C}{1-P_0C} \\ \frac{1}{1-P_0C} & \frac{-C}{1-P_0C} \end{pmatrix} = Z^* \begin{pmatrix} 1 & -\phi^* F \\ 0 & 0 \end{pmatrix} Z,$$

which is the matrix of transfer functions from disturbances at the input and output ports of the plant to the plant input and output. This is a rank-one matrix function (see [18]) with singular value $\sqrt{|F|^2 + 1}$. Thus, the shape of $|F|$ relates directly to amplification of external disturbances in the loop, and it also dictates how robust the control system is to plant uncertainty in the coprime factor (or, in the gap metric; cf. [24], [18]). In fact,

$$b_{\text{opt}}(P) := \max_{C \text{ stabilizing}} \|\Pi_{P//C}\|_\infty^{-1}$$

is precisely the optimal robustness radius for gap-ball uncertainty (see [18]) and coincides with $1/\sqrt{|F|^2 + 1}$ for the smallest $\|F\|_\infty$ consistent with (11).

The use of a frequency-dependent weight w allows shaping the loop-gain [24] as well as the performance and the robustness of the closed-loop system over different frequency bands [31], [3], [32]. By scaling α in (12) one can maximize the radius of $\mathcal{B}(P_0, \alpha w)$ for which a stabilizing controller exists (as in [31], [24], [18]). The maximal value $\alpha_{\max}$ and the optimal interpolant F, consistent with (11) and (12), satisfy

$$|F|^2 = \frac{1 - \alpha_{\max}^2 |w|^2}{\alpha_{\max}^2 |w|^2}.$$

Thus the use of a nontrivial weight w forces the interpolant to have a nontrivial outer factor. This causes a corresponding increase in the degree of the closed-loop system and of the controller. In this paper we shall develop techniques for reducing the degree of the control system while relaxing design requirements in a controlled fashion, and to illustrate this we shall consider a design example in Section VII.

IV. Characterization of $\mathbb{K}_\Psi$-minimizers and the Inverse problem

Theorem 1 provides a (complete) parametrization of Nevanlinna-Pick interpolants of degree $\leq n$. It states that such interpolants are in correspondence with $\Psi = |\sigma|^2$ for $\sigma \in \mathcal{K}_0$. Furthermore, it states that such interpolants originate as minimizers of the generalized entropy integral $\mathbb{K}_\Psi$ specified by such a weight Ψ. In this paper we are also interested in interpolants of higher degree. Thus, we are led to consider $\mathbb{K}_\Psi$-entropy minimizers for more general choices of Ψ. Indeed, the entropy functional $\mathbb{K}_\Psi$ can be defined for arbitrary nonnegative functions Ψ and since the minimizer relates algebraically to Ψ, choices of a rational Ψ generate minimizers of a suitable degree. Thus, he following theorem is a generalization of Theorem 1 which allows for the use of general weights Ψ. This is one of our main results.

Theorem 3: Suppose that the Pick matrix (2) *is positive definite and Ψ is a log-integrable nonnegative function on the unit circle. A function f is a minimizer of* (4) *if and only if the following three conditions hold:*

(i) $f(z_k) = w_k$ *for* $k = 0, \ldots, n$,

(ii) $f = \frac{b}{a} \in \mathcal{S}$ *where* $b \in \mathcal{K}$ *and* a *is outer,*
(iii) $\Psi = |a|^2 - |b|^2$.

Any such minimizer is necessarily unique.

As seen from Theorem 1, any choice of $\Psi = |\sigma|^2$ with $\sigma \in \mathcal{K}_0$ gives rise to $a, b \in \mathcal{K}$, and hence to an interpolant f with a degree bounded by n. Even if Ψ is rational of an arbitrarily high degree or is irrational, b still belongs to $\mathcal{K}$. In fact, the additional "complexity" is absorbed in a. Naturally, in such a case, the interpolant f will also be rational of a high degree or irrational, respectively. This observation allows us to characterize all minimizers of $\mathbb{K}_\Psi$ of degree at most $n+m$ for any given $m \in \mathbb{N}_+$. More specifically, let

$$\begin{aligned}\mathcal{K}_m &:= \big\{\sigma = \sigma_0 p : \sigma_0 \in \mathcal{K}_0, \deg p \le m, \\ &\qquad\qquad\qquad p \text{ outer}, \ p(0) > 0\big\} \\ &= \left\{ \frac{q}{\tau\pi} \ : \ \pi \in \mathrm{Pol}_+(m), q \in \mathrm{Pol}_+(n+m)\right\} \end{aligned} \tag{13}$$

The following statement gives the sought characterization.

Proposition 4: Let $\Psi = |\sigma|^2$ *with* $\sigma \in \mathcal{K}_m$. *Then the minimizing function* f *in* (4) *satisfies*

(i) $f(z_k) = w_k$ *for* $k = 0, \dots, n$,
(ii) f *has at most* n *zeros in* $\mathbb{D}$,
(iii) *the degree of* f *is at most* $n + m$.

Conversely, for any $f \in \mathcal{S}$ *which satisfies (i), (ii), and (iii), there exists a corresponding choice of* $\sigma \in \mathcal{K}_m$ *so that* f *is the minimizer of* (4).

Corollary 5: If $f \in \mathcal{S}$ *is a minimizer of* $\mathbb{K}_\Psi$ *for some choice of a log-integrable non-negative function* Ψ, *then* f *has at most* n *zeros in* $\mathbb{D}$.

This corollary underscores the significance of Theorem 3 for understanding the structure of minimizers.

We now consider the inverse problem of ananlytic interpolation with a degree constraint, namely the problem to decide when a particular interpolant is a minimizer of some weighted entropy functional, and if so, to determine the set of all admissible weights. It turns out that the number of roots in $\mathbb{D}$ determine whether an interpolant f is a minimizers of $\mathbb{K}_\Psi$ for some choice of Ψ. Furthermore, it is possible to characterize all such Ψ. This is stated below.

Proposition 6: Any function $f \in \mathcal{S}$ *that satisfies*

(i) $f(z_k) = w_k$ *for* $k = 0, \dots, n$,
(ii) f *has at most* n *zeros in* $\mathbb{D}$,
(iii) $\log(1 - |f|^2) \in L_1(\mathbb{T})$,

is the unique minimizer of (4) *for*

$$\Psi = (|f|^{-2} - 1)|b|^2 \tag{14}$$

with $b \in \mathcal{K}$ *chosen so that* bf^{-1} *is outer. Conversely, a (nonzero) function* f *having more than* n *zeros in* $\mathbb{D}$ *cannot arise as the minimizer of* (4) *for any choice of* Ψ.

The choice of $b \in \mathcal{K}$ in Theorem 6 is not unique, in general. The selection of b must prevent bf^{-1} from having poles in $\mathbb{D}$, and hence any zero of f must also be a zero of b. If f has more than n zeros in $\mathbb{D}$, there is no such b, whereas if f has exactly n zeros in $\mathbb{D}$, then b is uniquely defined. In all other cases, when f has $n_f < n$ zeros in $\mathbb{D}$, the family of possible choices of b, and hence the family of possible weights

$$\{\Psi \ : \ \Psi = (|f|^{-2} - 1)|b|^2, b \in \mathcal{K}, \ bf^{-1}\text{outer}\}$$

has dimension $n - n_f$. The design freedom offered by this nonuniqueness will be exploited in Section VI for finding a weight corresponding to f that is close to a low-degree weight.

It is not unusual that it is the shape of the interpolant $|f|$, instead of the interpolant f itself, that is of interest. Suppose that $g \in \mathcal{S}$ is a given outer function. When does there exist a minimizer of (4), specified by a suitable choice of Ψ, which satisfies

$$|f(e^{i\theta})| = |g(e^{i\theta})|, \quad \theta \in (-\pi, \pi]?$$

This inverse problem is closely connected to the the Pick-matrix

$$\mathbf{Pick}(g) := \left[\frac{1 - w_k g(z_k)^{-1}\overline{w_l g(z_l)^{-1}}}{1 - z_k \overline{z_l}}\right]_{k,l=0}^{n} \tag{15}$$

according to the folowing proposition.

Proposition 7: Let $g \in \mathcal{S}$ *be outer and such that* $\log(1 - |g|^2) \in L_1$. *Then there exists a pair* (Ψ, f) *of functions on* $\mathbb{T}$ *such that*

(i) $\log \Psi \in L_1$,
(ii) f *is the solution of (4), and*
(iii) $|f| = |g|$ *on* $\mathbb{T}$

if and only if $\mathbf{Pick}(g)$ *is positive semidefinite and singular. Furthermore,* f *is uniquely determined.*

V. Continuity properties of the map from weights to minimizers

Assume that f is the minimizer of the entropy functional, as in (4), for a suitable weight selected without regard to the degree. We begin by studying the properties of the nonlinear transformation

$$\varphi \ : \ \Psi \mapsto f \tag{16}$$

which maps a space of weights

$$\Psi \in \mathcal{M} := \{\Psi \mid \log \Psi \in L_\infty(\mathbb{T})\}$$

to the corresponding minimizers f of (4). We define the metric on $\mathcal{M}$ as $d(\Psi, \Psi_r) := \|\log(\Psi) - \log(\Psi_r)\|_\infty$, and, it turns out that, φ is continuous when the range is taken to be H_2, but not with range H_∞. On the other hand, the map $\Psi \mapsto |f| \in L_\infty$ is again continuous.

Remark 1: Here we only study the case where $\Psi \in \mathcal{M}$. However, it is easy to show that these continuity properties also holds for arbitrary L_∞ perturbations of $\log \Psi$ when Ψ is non-negative and log-integrable.

Lemma 8: Let Ψ *and* Ψ_r *be nonnegative log-integrable functions on* $\mathbb{T}$ *that satisfy*

$$\|\log(\Psi) - \log(\Psi_r)\|_\infty = \epsilon, \tag{17}$$

and set $f := \varphi(\Psi)$ *and* $f_r := \varphi(\Psi_r)$. *Then the inequalities*

$$\int_{\mathbb{T}} \Psi \log\left(1 + \frac{|f - f_r|^2}{8}\right) dm \le (e^{2\epsilon} - 1)\mathbb{K}_\Psi(f) \tag{18}$$

and

$$\|\sigma(f-f_r)\|_2^2 \le 10(e^{2\epsilon}-1)\mathbb{K}_\Psi(f) \tag{19}$$

hold.

A direct consequence of this is the sought continuity of φ.

Proposition 9: The map φ in (16) with range H_2, is continuous.

The mapping φ from $\Psi \in \mathcal{M}$ to $f \in H_\infty$ is not continuous. A counterexample that shows this is presented in [22].

From an engineering viewpoint, ∞-norm bounds on the approximation error are important in order to guarantee performance and robustness. To this end, consider the mapping

$$\psi : \Psi \mapsto |f| \in L_\infty, \tag{20}$$

which maps a choice of weight $\Psi \in \mathcal{M}$ to the magnitude $|f|$ of the corresponding minimizer f of (4). The lack of H_∞ continuity of interpolants is due to the fact that spectral factorization is not continuous. This problem does not ocur with ψ, which is continuous, and small approximation error on the weight Ψ will correspond to small L_∞ error in the shape of the interpolant.

Proposition 10: The map ψ in (20) is continuous.

VI. Approximation of interpolants

The continuity properties described in the previous section suggests a new approach for approximating interpolants that exploits the correspondence between minimizers and weights. Given an interpolant f we would like to find a degree-r approximating interpolant f_r of f, where $r \ge n$. From the inverse problem there is a set $\varphi^{-1}(f)$ of admissible weights Ψ for which a given f is the minimizer of (4). Our first task is to find a pair (Ψ, Ψ_r) for which $\Psi \in \varphi^{-1}(f)$ and $\Psi_r = |\sigma_r|^2$, with $\sigma_r \in \mathcal{K}_{r-n}$, so that their logarithmic distance is minimal. That is, we solve the following optimization problem

$$\begin{aligned} &\min \|\log(\Psi) - \log(\Psi_r)\|_\infty \\ &\text{subject to} \\ &\Psi \in \varphi^{-1}(f) \text{ and } \Psi_r = |\sigma_r|^2 \text{ with } \sigma_r \in \mathcal{K}_{r-n}. \end{aligned} \tag{21}$$

This optimization problem may be reformulated as a quasi-convex optimization and solved efficiently. By Proposition 4 the degree of the interpolant $f_r = \varphi(\Psi_r)$ is bounded by r, and by Lemma 8, a bound on the approximation error $f - f_r$ is obtained based on the quality of approximation obtained via the quasi-convex optimization.

By Proposition 4, the function φ, defined in (16), maps $\mathcal{K}_{r-n}$ into the set of interpolants of degree at most r. Thus, the basic idea is to replace the hard nonconvex problem of approximating f by another interpolating function f_r of degree at most r, by the simpler quasi-convex problem to approximate a $\Psi \in \varphi^{-1}(f)$ by a $\Psi_r = |\sigma_r|^2$ with $\sigma_r \in \mathcal{K}_{r-n}$.

The theory presented so far suggests a computational procedure in several steps, which we now summarize. In general, the required bound on the norm of the interpolant may differ from one, and therefore we consider the more general problem to find a function F, of a desired shape, which satisfies $\|F\|_\infty \le \gamma$ and the interpolation conditions

$$F(z_k) = W_k \text{ for } k = 0, 1 \ldots, n. \tag{22}$$

Step 1. Find an interpolant having the desired shape, but without restricting its degree. To this end, we begin with a family of functions $\{g_\alpha\}$ having desired shape, and we select one function g in this class for which the Pick condition in Proposition 7 is satisfied. Then, by Proposition 7, there is a Ψ such that $f := \varphi(\Psi)$ satisfies $|f(z)| = |g(z)|$ for all $z \in \mathbb{T}$.

Typically we choose this family of functions in such a way that $|g_\alpha|$ is monotonically decreasing in α. In our motivating example we take

$$|g_\alpha|^2 = \frac{1 - \alpha^2|w|^2}{\alpha^2|w|^2}$$

which leads to a typical 2-block problem. Another typical choice would be $g_\alpha = w/\alpha$, which leads to a standard H_∞ optimization problem.

We then seek a solution to the optimization problem

$$\begin{aligned} &\max \alpha \\ &\text{subject to} \\ &|F| \le |g_\alpha| \text{ and } F(z_k) = W_k,\ k = 0, 1 \ldots, n. \end{aligned}$$

The optimum is attained when $\mathbf{Pick}(g_\alpha)$ is positive semidefinite and singular (Proposition 7). Here the bound γ must satisfy

$$\|F\|_\infty \le \gamma \text{ and } \log(\gamma^2 - |F|^2) \in L_1,$$

or else the bound γ needs to be relaxed. Then the normalized interpolant

$$f := \frac{1}{\gamma} F$$

satisfies the interpolation conditions

$$f(z_k) = w_k := \frac{1}{\gamma} W_k,\ k = 0, 1, \ldots, n,$$

as well as the log-integrability condition of $1 - |f|^2$. Hence we have constructed an interpolant with the required shape, but which in general does not satisfy the desired degree constraint.

Step 2. For some $r \ge n$, find an approximation f_r of f of degree at most r which satisfies the same interpolation conditions. To this end, find functions Ψ and Ψ_r that solve the optimization problem (21), where

$$\varphi^{-1}(f) = \{\Psi : \Psi = (|f|^{-2} - 1)|b|^2 \mid b \in \mathcal{K},\ bf^{-1}\text{outer}\}.$$

This is a quasi-convex optimization problem. In fact, $\|\log(\Psi) - \log(\Psi_r)\| \le \epsilon$ if and only if

$$e^{-\epsilon} \le \frac{\Psi_r(z)}{\Psi(z)} \le e^{\epsilon} \text{ for all } z \in \mathbb{T}. \tag{23}$$

The constraints (23) define an infinite set of linear constraints on the pseudo-polynomials representing the nominator and denominator, respectively, of Ψ_r/Ψ. Since the sublevel set of nominators and denominators solving (23) is convex for each $\epsilon > 0$, the problem is quasiconvex. The reader is referred to [23] for a detailed description how to solve this problem.

Step 3. Next we solve the optimization problem

$$\min\{\mathbb{K}_{\Psi_r}(f_r) \ : \ f_r \in \mathcal{S},\ f_r(z_k) = w_k,\ k = 0, \ldots, n\}$$

for the unique solution f_r. In Step 2 we have determined the weight Ψ_r as an approximation of Ψ, and therefore f_r will also be an approximation of to f, for which the bounds (18) and (19) hold. Furthermore, since $\Psi_r = |\sigma_r|^2$ with $\sigma_r \in \mathcal{K}_{r-n}$, the degree of f_r is bounded by r (Proposition 4). Finally, we renormalize the interpolant

$$F_r := \gamma f_r$$

to obtain the approximant which solves the original interpolation problem.

VII. Example

We now return to the example in Section III. The underlying mathematical problem is an analytic interpolation problem where a desired shape is sought for the interpolant. This problem is addressed using the procedure outlined in Section VI.

We consider a continuous-time plant having one integrator, a slow unstable pole, and a time-lag, modeled via $\frac{s-1}{s+1}\frac{1}{s(s-0.1)}$. We base our design on its discrete-time counterpart

$$P_0(z) = \frac{0.08772z^3 - 0.08772z^2 - 0.4386z - 0.2632}{z^3 - 2.439z^2 + 1.807z - 0.3684}$$

obtained via the Möbius transform $s \mapsto z = (2+s)/(2-s)$ and restrict our analysis to the discrete-time domain.

The design objective is encapsulated in the choice of a weight w, chosen as in Section III to increase robustness to high-frequency modeling uncertainty. The selected "nominal weight" w is shown in Figure 1. The maximal scaling parameter $\alpha_{\max}$ can be readily computed by first calculating the outer factor g_α of

$$|F|^2 = \frac{1-\alpha^2|w|^2}{\alpha^2|w|^2} = |g_\alpha|^2 \text{ on } \mathbb{T},$$

and then, finding the maximal value $\alpha_{\max}$ for which the Pick matrix $\mathbf{Pick}(g_\alpha)$ is positive semidefinite (cf. Proposition 7). The Pick matrix $\mathbf{Pick}(g_\alpha)$ is defined in (15) and requires the interpolation data that can be obtained by evaluating F_0 at the roots of ϕ in (11). Denote by F_{ideal} the unique interpolant which satisfies $|F_{\text{ideal}}| = |g_{\alpha_{\max}}|$, and denote the corresponding controller by C_{ideal}. Since w is not rational, neither are F_{ideal} and C_{ideal}. Next we describe how to approximate F_{ideal} with an admissible interpolant of low degree. Using the corresponding controller leads to closed-loop transfer functions of low degree.

The uniform robustness margin corresponding to the controller C_{ideal} is determined by the value of $\|F_{\text{ideal}}\|_\infty$ via (12). For the above choices $\|F_{\text{ideal}}\|_\infty = 10.87$. In order to achieve the desired characteristic for the frequency-dependent robustness margin, we relax the H_∞ bound on F.

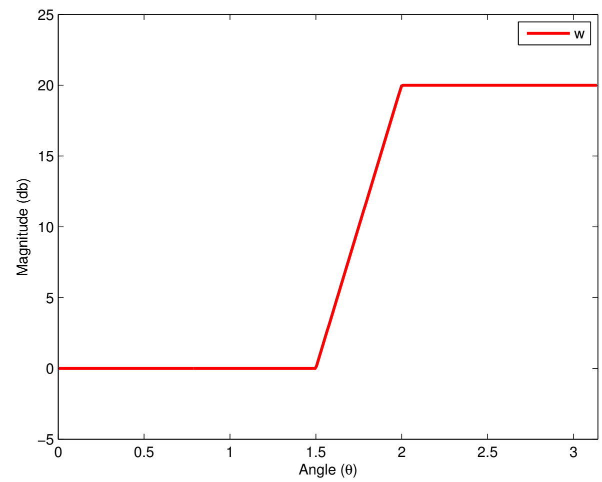

Fig. 1. The frequency-dependent robustness shape w

For the particular example, it is deemed appropriate to allow $\|F\|_\infty \le \gamma = 20$. We normalize F by defining $f = \frac{1}{\gamma}F$, which is then required to satisfy

$$f(z) = \frac{1}{\gamma}F_0(z) \text{ whenever } \phi(z) = 0, \tag{24}$$

in view of the interpolation condition (11). Then we follow the steps of Section VI to obtain approximants to $f_{\text{ideal}} = \frac{1}{\gamma}F_{\text{ideal}}$.

For a given $r \ge n$ we determine a degree-r approximant f_r of f_{ideal} as follows. We first compute a minimizer σ_r of the quasi-convex optimization problem to find a $\sigma_r \in \mathcal{K}_{r-n}$ and a $\Psi \in \varphi^{-1}(f_{\text{ideal}})$ which minimize

$$\left\|\log(|\sigma_r|^2) - \log(\Psi)\right\|_\infty .$$

Next we determine f_r as the minimizer of the convex optimization problem

$$f_r = \arg\min\{\mathbb{K}_{|\sigma_r|^2}(f_r) \big| f \in \mathcal{S} \text{ and } (24)\}.$$

A corresponding controller C_r can now be determined via q from (11) and (7).

The uniform robustness radius for gap-metric uncertainty is maximal for an optimal choice of the controller C_{opt} and equals $b_{\text{opt}}(P)$ ([31], [18]). This is the inverse of the H_∞-norm of the "parallel projection" operator $\Pi_{P//C_{\text{opt}}}$, and this value is shown in Figure 2 with a dash-dotted line. On the other hand, the inverse of the maximal singular value of $\Pi_{P//C_{\text{ideal}}}$, plotted as a function of frequency with solid line, represents a frequency-dependent robustness radius [31]. Both are now compared with a degree-four approximant

$$C_4 = \frac{0.876z^4 - 0.190z^3 - 0.0669z^2 - 0.460z + 0.157}{z^4 + 0.1205z^3 + 1.389z^2 + 0.07538z + 0.2214},$$

and it is seen that there is substantial improvement of robustness as compared to $b_{\text{opt}}(P)$ in the high-frequency range. Figure 3 compares the gains of C_4 and C_{opt}. Similarly, Figure 4 and Figure 5 compare the loop-gains and the

Nyquist plots, respectively, for the two cases. It is seen that some form of phase compensation is effected by C_4 around 1.6 rad/sec, as compared to $C_{\rm opt}$ so as to gain the sought advantage. Figure 6 compares the gains of the four entries of the closed-loop transfer matrix $\Pi_{P//C}$. The improvement in the sensitivity function at middle range becomes evident.

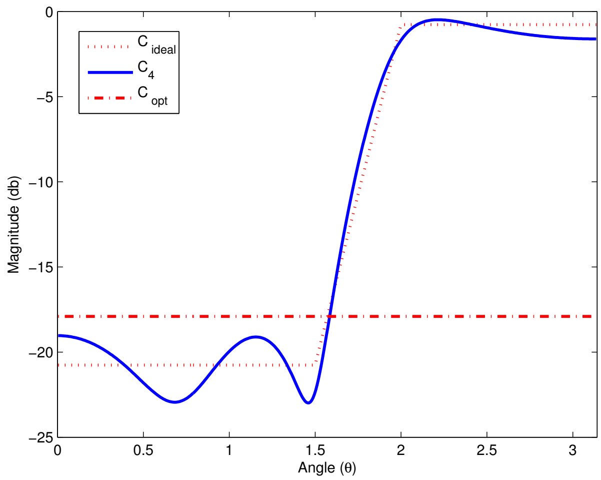

Fig. 2. The robustness radius obtained for the controllers $C_{\rm ideal}$, C_4, and $C_{\rm opt}$.

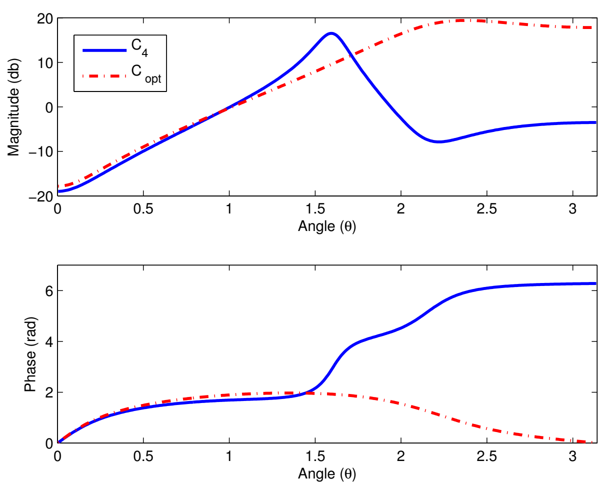

Fig. 3. Bode plots of controllers C_4, and $C_{\rm opt}$

VIII. Concluding remarks

The formulation of feedback control synthesis as an analytic interpolation problem has been at the heart of modern developments in robust control. Yet, many of the standard approaches often lead to designs of a large degree, due to degree inflation when introducing and absorbing "weights" into the controller. At various stages, alternative methodologies for dealing with control design under structural and

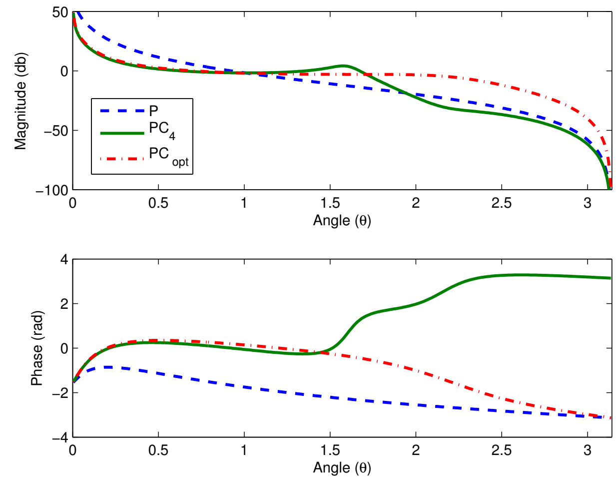

Fig. 4. Bode plots of P and of loop gains PC_4, and $PC_{\rm opt}$

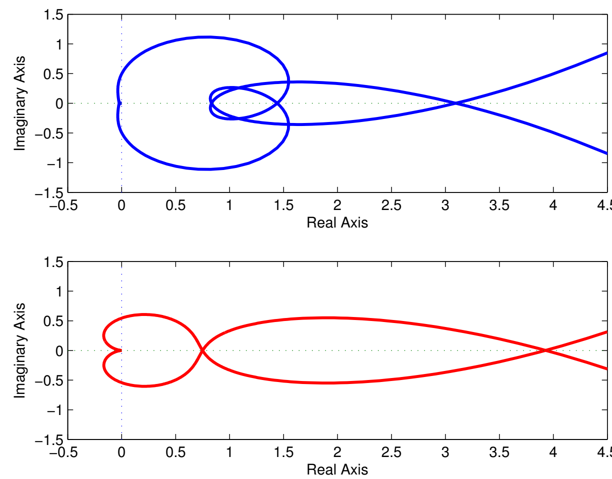

Fig. 5. Nyquist plots of the loop gains PC_4 (above) and $PC_{\rm opt}$ (below), respectively

dimensionality constraints were developed by several authors based primarily on suitable approximations and a linear matrix inequality formalism (see [11], [29], [19], [2]). In particular, a comparison between the viewpoint in Gahinet and Apkarian [11] and Skelton, Iwasaki, and Grigoriadis [29] and the viewpoint advocated in our work is provided in [17].

Our approach builds on the original H_∞-formulation of control synthesis as an analytic interpolation problem and on the recently discovered fact that, in contrast to H_∞-minimization, dimensionality and performance are inherited by the weighted-entropy minimization. In this setting, "weights" provide the means of shaping interpolants in a manner akin to H^∞ design. Thus, the advantage of the new methodology which involves entropy functionals stems from the fact that selection of weights within a specific class does not unduly penalize the degree of the design. However, the

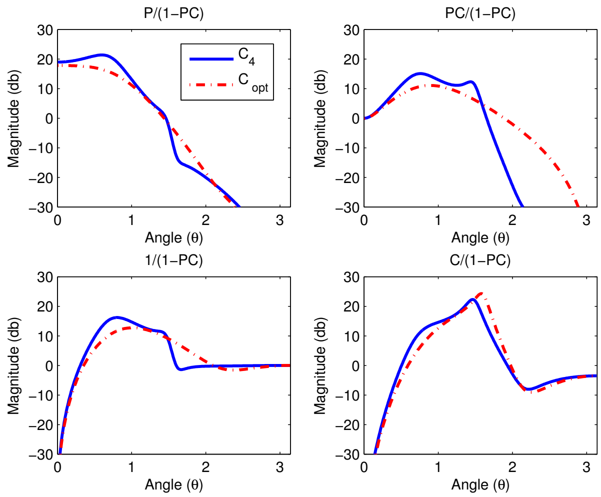

Fig. 6. The four closed loop transfer functions of $\Pi_{P//C_4}$ and $\Pi_{P//C_{\rm opt}}$

choice of weights is not immediate, as it is in the standard H_∞ paradigm [10]. The choice of weights that lead to acceptable controllers is, in itself, a non-convex optimization problem. Thus, one of the contributions of this paper is a relaxation of this non-convex problem into one which is quasi-convex, and thus solvable by standard methods. The methodology builds on a more fundamental question which forms a main theme of the paper, namely the characterization of all possible minimizers of weighted entropy functionals. The inverse problem of constructing weights for permissible minimizers is the basis for our new design theory. In a more general context, the results of this paper provide a solution to the longstanding open problem of determining which spectral zeros correspond to a certain desired shape of the interpolant.

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