Proceedings of the
47th IEEE Conference on Decision and Control
Cancun, Mexico, Dec. 9-11, 2008

Constrained $\mathcal{H}_\infty$ control for discrete-time LPV systems using interpolation

Tamas Péni, Balázs Kulcsár, József Bokor, Michel Verhaegen

Abstract— **The paper proposes an interpolation based control method as a possible solution to the constrained $\mathcal{H}_\infty$ control of discrete-time, linear parameter varying (LPV) systems. The control policy is constructed by interpolating among a priori designed, unconstrained state feedback controllers. The predefined $\mathcal{H}_\infty$ performance level remains guaranteed under hard state and input constrains. By applying invariant set theory it is also shown that the domain of applicability of the proposed control method is significantly larger than that can be achieved by any, single state feedback.**

I. INTRODUCTION

In recent years the constrained $\mathcal{H}_\infty$ methodology became one of the most attractive control field to assure robustness under state or control input constraints. Alternate and efficient solutions has only been elaborated for linear time invariant systems [7], [1], [12], [2]. For nonlinear systems the receding horizon control is one of the most suitable, and thus, preferred approach [5], [8]. Unfortunately, it involves complex optimization problems that have to be solved online. For the existence of an off-line designed, static, state feedback solution, [9] derives necessary and sufficient conditions, but it restricts only on Euclidean norm constraints. The presented method requires to solve nonlinear matrix inequalities, which is computationally demanding.

Rewriting a nonlinear dynamics in linear parameter-varying (LPV) [19] form may allow to extend Linear Time Invariant (LTI) control techniques to the reformulated nonlinear systems. In [3], [4] the design of the model predictive constrained $\mathcal{H}_\infty$ controller is traced back to a convex optimization problem involving linear matrix inequalities. Unfortunately, the constraint handling is based on a conservative estimation of the invariant set, therefore the controller can only be applied over a small subset of states, close to the origin.

To overcome the difficulties of the existing control methods, such as computational complexity and conservatism, in the paper an interpolation based control structure is proposed. The control policy is constructed by interpolating among unconstrained, state feedback controllers so that the constraint satisfaction and the prescribed performance level remain guaranteed. The domain of applicability of the controller is determined from the polyhedral disturbance invariant set of the closed-loop system. The disturbance invariant set is computed directly from the system dynamics, which significantly reduces the conservativeness of the solution.

The algorithms presented here are extensions of the results published in [15], [17], [13], [14]. These papers propose efficient methods for the invariance set computation and for the construction of the interpolation based controller, but they focus only on the disturbance-free case and solve an LQ-like, constrained optimal control problem. In this paper the LPV system is completed with additive disturbance and the original methods are extended to the constrained $\mathcal{H}_\infty$ control problem.

The paper is organized as follows. First, the problem is formulated and the solution is outlined. In section III an efficient algorithm is proposed for the computation of the maximal d-invariant set. The interpolation based control scheme is presented in section IV. The algorithm is tested by a numerical simulation example of an LPV system. In the section VI the results are summarized and the conclusions are drawn.

II. PROBLEM FORMULATION

Let a discrete-time, linear parameter-varying (LPV) system be given in polytopic form by

$$\begin{aligned} x_+ &= A(\delta)x + B_1(\delta)w + B_2(\delta)u \\ z &= C(\delta)x + D_1(\delta)w \end{aligned} \qquad (1)$$

where $x \in \mathbb{R}^{n_x}$, $u \in \mathbb{R}^{n_u}$, $z \in \mathbb{R}^{n_z}$ and $w \in \mathbb{R}^{n_w}$ are the states, control input, performance output and the disturbance signals respectively. (x_+ denotes the value of x at the *next* time instant). The polytopic representation of the model is written as

$$\delta \in \Delta, \quad \Delta = \{\delta = [\delta^1, \ldots, \delta^L] \mid \delta^i \in \mathbb{R}^+, \sum_{i=1}^{L} \delta^i = 1\}$$
$$[A(\delta), B_1(\delta), B_2(\delta), C(\delta), D_1(\delta)] = \sum_{i=1}^{L} \delta^i \cdot [A_i, B_{1i}, B_{2i}, C_i, D_{1i}] \qquad (2)$$

with known matrices $A_i, B_{2i}, C_i, D_{1i}, D_{2i}$ of appropriate dimension. It is assumed that the state x and the parameter vector δ are measured at each time instant. Assume $w \in W$ and that the state and control input are subject to constraints: $x \in X$, $u \in U$. The sets X, U, W are closed, convex and contain the origin in their interior. In order to formulate the control problem itself the notion of the disturbance invariant (d-invariant) set has to be recalled [11].

T. Péni and J. Bokor are with Systems and Control Laboratory, Computer Automation Research Institute, Hungarian Academy of Sciences, H-1518, Budapest, P.O.Box 63., pt@scl.sztaki.hu; bokor@sztaki.hu

B. Kulcsár and M. Verhaegen are with Delft Center for System and Control, Faculty of Mech, Maritime and Materials Eng., Technical University of Delft, Mekelweg 2, 2628CD Delft, Netherland, b.a.kulcsar@tudelft.nl, M.Verhaegen@moesp.org

978-1-4244-3124-3/08/$25.00 ©2008 IEEE

Definition 1. (d-invarinat set, maximal) The set $S \subseteq X$ is a d-invariant set generated by the stabilizing controller $u(x)$ if S is the d-invariance set of the closed loop system $A(\delta)x + B_2(\delta)u(x) + B_1(\delta)w$, i.e. $x \in S$ implies $u(x) \in U$ and $A(\delta)x + B_2(\delta)u(x) + B_1(\delta)w \in S$ for all $w \in W$. The largest (maximal) d-invariant set is denoted by $\overline{S}$. By definition $S \subseteq \overline{S}$ holds for all d-invariant sets S.
The control problem can now be formulated as follows:

Problem 1. (constrained $\mathcal{H}_\infty$) Let $0 < \gamma^*$ be a given predefined performance level. Find a stabilizing state feedback control policy $u = u(x)$ for system (1) so that

(i) there exists a positive definite storage function $V(x, \delta) : \mathbb{R}^{n_x} \times \mathbb{R}^{n_\delta} \to \mathbb{R}^+$, $V(0, \delta) = 0$ and a positive scalar $\gamma < \gamma^*$ s.t. the state x and output z of the closed-loop system $x_+ = A(\delta)x + B_2(\delta)u(x) + B_1(\delta)w$, $z = C(\delta)x + D_1(\delta)w$ satisfy the dissipation inequality
$$z^T z - \gamma^2 w^T w + V(x_+, \delta_+) \le V(x, \delta) \tag{3}$$
for all possible disturbance sequence $[w_0, w_1, \ldots] : \sum_{i=0}^{\infty} w_i^T w_i < \infty$. Moreover

(ii) $u(x)$ generates a nonempty (or practically 'large') d-invariant set S.

The Part (i) is the generic formulation of the the unconstrained $\mathcal{H}_\infty$ control problem. Satisfying (3) assures that the induced $\mathcal{L}_2$ norm between w and z is less than γ (respectively γ^*). The set S defined by Part (ii) has to contain the initial state x_0 for the constraint to be satisfied along the entire future state trajectory. Therefore, S can be considered as the *region of applicability* of the controller $u(x)$. S is obviously required to be as large as possible.

The control input $u(x)$ is computed by interpolation among a set of unconstrained and constant state feedback controllers $u_i(x) = K_i x$. The storage function is chosen to be parameter independent and quadratic $V(x) = x^T P x, P > 0$. The selection of the linear feedbacks may seem to be too conservative, but they have considerable advantages compared to nonlinear control policies. First, they can be computed efficiently by rewriting (3) to linear matrix inequalities [10] and solving them by convex programming [18]. Second, there are powerful algorithms available for the approximation of the maximal d-invariant set (see section III) they generate.

On the other hand, a single state feedback can not provide in general satisfactory result. There is namely a trade-off between the performance and the region of applicability. The controller providing low γ has large feedback gain, thus – due to the input constraints – generates small, or even empty, d-invariant set.

To overcome this trade-off, the set of acceptable controllers (that have γ smaller than γ^*) is augmented with an unacceptable one, which has 'large' maximal d-invariant set. Interpolation is performed then, to generate $u(x)$ so that the overall performance stays acceptable (smaller than than the predefined level), while the d-invariant set generated by the controller is *much* larger than the one could be achieved by a single feedback.

III. MAXIMAL DISTURBANCE INVARIANT SET OF LPV SYSTEMS

As it was previously mentioned, the d-invariant sets of the closed-loop system play a key role in the determination of the region of applicability of the applied control policy. At the same time the exact computation of these sets is generally a complex problem, even if the system is linear time invariant [11][6],[16]. For the construction of the maximal invariant set of a polytopic system [13] provides a computationally efficient method, but it focuses only on the disturbance free case. Revising and extending the earlier results this section proposes an efficient algorithm for the approximation of the maximal d-invariant set of system (1), generated by a parameter-independent state feedback controller $u(x) = Kx$.

The invariant set algorithm proposed by [13] is adjusted to this problem. For this, assume that the constraints $x \in X$ and $Kx \in U$ are expressed by one linear constraint set $A_S x \le b_S$, where each row of $[A_S, b_S]$ corresponds to a linear constraint of the form $a^T x \le b$. Consider now the the sequence of sets $S_0, S_1, S_2, \ldots$ defined as follows: $S_0 = \{x \,|\, A_S x \le b_S\}$, $S_t = \{x \in S_{t-1} \,|\, (A(\delta) + B_2(\delta)K)x + B_1(\delta)w \in S_{t-1},\ \forall w \in W, \delta \in \Delta\}$, $t > 0$. If S_0 is a polytope then S_t and $S_i \subseteq S_{i-1}$ are polytopes as well. The following lemma proves the convergence of the approximation towards $\overline{S}$.

Lemma 1. (convergence $\overline{S}$) *If the sequence of sets* $S_0 = \{x \,|\, A_S x \le b_S\}$, $S_t = \{x \in S_{t-1} \,|\, (A(\delta) + B(\delta)K)x \in S_{t-1},\ \forall w \in W, \delta \in \Delta\}$ *is convergent*[1] *then* $\lim_{t\to\infty} S_t = \overline{S}$

Proof. (The proof is similar to the proof in [13].) It is proved that $S^* = \lim_{t\to\infty} S_t$ is a d-invariant set and it contains $\overline{S}$. Since $S_t \subseteq S_{t-1}$ $\forall t$ and $S^* = \lim S_t$ thus the set $S' = \{x \in S^* \mid (A(\rho) + B_2(\rho))x + B_1(\delta)w \in S^*,\ \forall w \in W, \forall \delta \in \Delta\}$ can not be a real subset of S^*. Consequently $S' = S^*$, i.e. S^* is d-invariant. Suppose now $\overline{S} \nsubseteq S^*$. This means that there exists j s.t. $\overline{S} \subseteq S_{j-1}$, but $\overline{S} \nsubseteq S_j$. Since $\overline{S}$ is d-invariant, for all $x \in \overline{S}$ $(A(\rho) + B_2(\rho))x + B_1(\delta)w \in S_{j-1}$. This implies that $\overline{S} \subseteq S_j$, which contradicts the assumption. ■

The algorithm constructing the outer approximation of the maximal d-invariant set can be given as follows:

Algorithm 1. (Outer approximation of the maximal d-invariant set)

1) *Initialize* $c_{max} = 0$, $S_0 = \{x | A_{S_0} x \le b_{S_0}\}$, $A_{S_0} = A_S, b_{S_0} = b_S$, $t = 1$
2) *Set* $M = [], m = []$
3) *Perform the following steps while* j *is not larger than the number of rows in* $A_{S_{t-1}}$
   a) *Take the* j*th inequality* $a^T x \le b$
   b) *Check whether there exists* $x \in S_{t-1}, w \in W$ *and* δ *s.t.* $(A(\delta) + B(\delta)K)x + B_1(\delta)w \notin S_{t-1}$. *For this,*

[1] Set convergence is defined according to [11]/Theorem 4.1, i.e. $F_t \to F$ if for every $\epsilon > 0$ there exists t s.t. $F \subset F_t + \epsilon B$, where B is a unit ball.

compute for each triplet (A_i, B_{1i}, B_{2i}), $i = 1 \dots L$

$$c_i = \max_{w\in W, x\in S_{t-1}} a^T[(A_i + B_{2i}K)x + B_{1i}w] - b$$

$$(x_i^*, w_i^*) = \arg\max_{w\in W,\, x\in S_{t-1}} a^T[(A_i + B_{2i}K)x + B_{1i}w] - b \quad (4)$$

If for any i $c_i > 0$ *save the inequality* $a^T(A_i + B_{2i}K)x \le b - a^T B_{1i}w_i^*$, *i.e. let*

$$M = \begin{bmatrix} M \\ a^T(A_i + B_{2i}K)\end{bmatrix} \qquad m = \begin{bmatrix} m \\ b - a^T B_{1i}w_i^*\end{bmatrix}$$

c) *let* $c_{\max} = \max(c_{\max}, c_1, \dots, c_L)$

4) *let* $A_{S_t} = \begin{bmatrix} A_{S_{t-1}} \\ M\end{bmatrix}$, $b_{S_t} = \begin{bmatrix} b_{S_{t-1}} \\ m\end{bmatrix}$ *and* $S_t = \{x \,|\, A_{S_t}x \le b_{S_t}\}$.

5) $[A_{S_t}, b_{S_t}] = \text{reduce}(A_{S_t}, b_{S_t})$

6) *if* $c_{\max} < \epsilon$ *then let* $\hat{\overline{S}} = S_t$ *and stop, else* $t := t + 1$ *and go to step 2*

The algorithm assumes first, that the set $\{x \,|\, A_S x \le b_S\}$ generated from the constraints is equal to a d-invariant set. Afterwards, by going through each constraint it checks whether there exist an $x \in S$ that violates the constraint, i.e. whether there exists an $x \in S$ and w s.t. $(A(\delta)+B_2(\delta)K)x+B_1(\delta)w \notin S$. In this case a new constraint is added to the existing set of rules. Involving more and more constraints, an outer approximation is given for the maximal d-invariant set. By increasing the number of the rules, the invariant set becomes smaller and smaller till it covers the maximal set with the predefined precision ϵ. Since the algorithm only adds and never removes constraints it is worth revising occasionally the constraint set and removing the redundant constraints. This can be performed in a straightforward way by linear programming. The details can be found in [13]. The constraint set reduction is indicated in the algorithm above by calling the *reduce()* function in step 5. The presented algorithm can not construct the maximal d-invariant set in finite steps. Therefore the procedure is completed with a terminal condition $c_{max} < \epsilon$, where the variable $c_{\max}$ measures the 'difference' between two consecutive sets and if it is acceptable small the computation stops.

IV. INTERPOLATION BASED CONTROLLER

In this section the interpolation based control structure is presented and its main properties are derived. Suppose m different unconstrained $\mathcal{H}_\infty$ controllers have already been designed for the system (1). Let they be given by the following ordered pairs:

$$(K_1, \gamma_1), (K_2, \gamma_2), \dots, (K_m, \gamma_m) \quad (5)$$

where γ_i is the induced $\mathcal{L}_2$ gain provided by the state feedback control $u = K_i x$. The ordering is according to γ, i.e. $0 < \gamma_1 \le \gamma_2 \le \dots \le \gamma_{m-1} \le \gamma^* \ll \gamma_m < \infty$. (Note that the last controller can not be used in itself since the performance it provides is worse than the acceptable level.)

Consider now the extended system constructed from the m closed loop dynamics formed by the m controllers:

$$\Sigma: \begin{bmatrix} \hat{x}_+^1 \\ \vdots \\ \hat{x}_+^m \end{bmatrix} = \begin{bmatrix} \Phi_1(\delta) & & \\ & \ddots & \\ & & \Phi_m(\delta)\end{bmatrix} \begin{bmatrix} \hat{x}_k^1 \\ \vdots \\ \hat{x}_k^m \end{bmatrix} + \begin{bmatrix} B_1(\delta)/m \\ \vdots \\ B_1(\delta)/m \end{bmatrix} \hat{w}$$

$$\Phi_i(\delta) = A(\delta) + B_2(\delta)K_i$$

$$\hat{z} = [C(\delta), \dots, C(\delta)] \begin{bmatrix} \hat{x}_k^1 \\ \vdots \\ \hat{x}_k^m \end{bmatrix} + D_1(\delta)\hat{w}$$

where $\hat{x}^i \in \mathbb{R}^{n_x}$. The upper index i denotes that the vector is the i-th partition in the state vector of system Σ_m. The following lemma can be easily checked:

Lemma 2. (I/O equivalence) *The original* (1) *and the extended* Σ *system are input-output equivalent, i.e.* $[\hat{w}_0, \hat{w}_1, \dots, \hat{w}_{k-1}] \equiv [w_0, w_1, \dots, w_{k-1}] \Rightarrow [\hat{z}_1, \hat{z}_2, \dots, \hat{z}_k] \equiv [z_1, z_2, \dots, z_k]$ *if* $x_0 = \sum_{i=1}^m \hat{x}_0^i$ *and* $u = \Sigma_{i=1}^m K_i\hat{x}^i$. *Moreover, if* $u_i = K_i x$ *is stabilizing, the control policy* $u = \Sigma_{i=1}^m K_i\hat{x}^i$ *is stabilizing as well.*

Proof. Applying the control input $u_0 = \sum_{i=1}^m K_i\hat{x}_0^i$ to (1) and using $x_0 = \sum_{i=1}^m \hat{x}_0^i$ and $w_0 = \hat{w}_0$ we get

$$\begin{aligned} x_1 &= A(\delta_0)x_0 + B_2(\delta_0)\sum_{i=1}^m K_i\hat{x}_0^i + B_1(\delta)w_0 \\ &= \sum_{i=1}^m \left((A(\delta_0) + B_2(\delta_0)K_i)\hat{x}_0^i + \frac{B_1(\delta)}{m}\hat{w}_0 \right) \\ &= \sum_{i=1}^m \hat{x}_1^i \\ z_0 &= C(\delta_0)x_0 + D_1(\delta_0)w_0 \\ &= \sum_{i=1}^m C(\delta_0)\hat{x}_0^i + D_1(\delta_0)\hat{w}_0 = \hat{z}_0 \end{aligned} \quad (6)$$

Repeating the computation above for time instants $k > 0$ completes the proof. The stabilizing property of the interpolating controller follows from the stability of subsystems $A(\delta) + B_2(\delta)K_i$. ■

Lemma 2. asserts that if system (1) is controlled by

$$u = \sum_{i=1}^m K_i\hat{x}^i \quad (7)$$

then its output is equal to the output of the extended system, i.e. the input-output properties (e.g. the induced $\mathcal{L}_2$ gain) of the closed loop system can be determined from the behavior of the extended system. By exploiting the equivalence, it can be shown in the rest of the section that the partitioning $(\hat{x}_0^1, \dots, \hat{x}_0^m)$ of the initial state x_0 can be chosen so that the control policy (7) solves Problem 1. First, the following lemma is proved:

Lemma 3. ($\mathcal{L}_2$ gain of Σ) *The system* Σ *has finite* $\mathcal{L}_2$ *gain* $\hat{\gamma} \le \sqrt{\sum_{i=1}^m \frac{\gamma_i^2}{m}}$.

Proof. It is proved that there exists a positive definite function $\hat{V} : \mathbb{R}^{m \cdot nx} \to \mathbb{R}^+$ and a positive constant $\hat{\gamma}$ for system Σ s.t. the dissipation inequality

$$\hat{z}^T \hat{z} - \hat{\gamma}_m^2 \hat{w}^T \hat{w} + \hat{V}(\hat{x}_+^1, \dots, \hat{x}_+^m) \leq \hat{V}(\hat{x}^1, \dots, \hat{x}^m) \tag{8}$$

is satisfied. For this, consider the subsystems:

$$\begin{aligned} \hat{x}_+^i &= (A(\delta) + B_2(\delta)K_i)\hat{x}^i + B_1(\delta)\frac{\hat{w}}{m} \\ \hat{z}^i &= C(\delta)\hat{x}^i + D_1(\delta)\frac{\hat{w}}{m} \end{aligned} \tag{9}$$

which have the property $\hat{z} = \sum_{i=1}^m \hat{z}^i$. Since K_i is a $\mathcal{H}_\infty$ controller, the subsystem above has $\mathcal{L}_2$ gain γ_i, therefore the inequality

$$(\hat{z}^i)^T \hat{z}^i - \frac{\gamma_i^2}{m^2} \hat{w}^T \hat{w} + V_i(\hat{x}_+^i) \leq V_i(\hat{x}^i) \tag{10}$$

holds for all $\hat{w} \in W$, with appropriate storage function V_i. Summing up the inequalities above and multiplying the result by m the following relation is obtained

$$m\sum_{i=1}^m (\hat{z}^i)^T \hat{z}^i - \sum_{i=1}^m \frac{\gamma_i^2}{m} \hat{w}^T \hat{w} + m\sum_{i=1}^m V_i(\hat{x}_+^i) \leq m\sum_{i=1}^m V_i(\hat{x}^i) \tag{11}$$

Consider now the following inequality defined over arbitrary vectors $v_1, v_2, \dots, v_m$

$$(v_1 + \dots + v_m)^T (v_1 + \dots + v_m) \leq m(v_1' v_1 + \dots + v_m' v_m) \tag{12}$$

(the proof follows from the inequality $0 \leq \Sigma_{i<j}(v_i - v_j)^T(v_i - v_j)$.) Substituting $v_i = \hat{z}^i$ into (12) the following lower bound can be calculated for (11):

$$\begin{aligned} &\hat{z}^T \hat{z} - \sum_{i=1}^m \frac{\gamma_i^2}{m} \hat{w}^T \hat{w} + m\sum_{i=1}^m V_i(\hat{x}_+^i) \\ &\leq m\sum_{i=1}^m \left(\hat{z}^i\right)^T \hat{z}^i - \sum_{i=1}^m \frac{\gamma_i^2}{m} \hat{w}^T \hat{w} + m\sum_{i=1}^m V_i(\hat{x}_+^i) \\ &\leq m\sum_{i=1}^m V_i(\hat{x}^i) \end{aligned} \tag{13}$$

Note that (13) is the same as (8) with $\hat{\gamma}^2 = \sum_{i=1}^m \frac{\gamma_i^2}{m}$ and $\hat{V}(\hat{x}^1, \dots, \hat{x}^m) = m\sum_{i=1}^m V_i(\hat{x}^i)$. ■

Remark. 1. The performance value $\sqrt{\sum_{i=1}^m \frac{\gamma_i^2}{m}}$ is only an upper bound in general for the real $\mathcal{L}_2$ gain. A better approximation can be found by setting $\hat{V}(\hat{x}) = \hat{x}^T \hat{P} \hat{x}$, substituting the system dynamics Σ into (8) and rewriting the inequality obtained into an equivalent linear matrix inequality, which is then solved for variables $\hat{P}$ and $\hat{\gamma}$ by convex programming.

Remark. 2. It can be easily checked that the number m of the controllers and the controllers themselves in (5) can be chosen so that $\hat{\gamma} = \sqrt{\sum_{i=1}^m \frac{\gamma_i^2}{m}} \leq \gamma^*$ holds even if $\gamma_m \gg \gamma^*$. This means that a set of 'proper' controllers (having $\mathcal{L}_2$ gain $\leq \gamma^*$) can be completed with an 'auxiliary' controller, which does not satisfy the performance specification.

By Lemma 3. we have proved that the control policy (7) solves Part (i) of Problem 1. To satisfy Part (ii) a nonempty d-invariant set has to be computed for the controlled closed loop system

$$\begin{aligned} x_+ &= A(\delta)x + B_2(\delta)\sum_{i=1}^m K_i \hat{x}^i + B_1(\delta)w \\ z &= C(\delta)x + D_1(\delta)w \end{aligned} \tag{14}$$

The following lemma helps.

Lemma 4. (d-invariance under interpolation) *Let $\overline{\hat{S}}$ be the maximal d-invariant set of the extended system Σ, satisfying the following constraints*

$$\Pi_x \hat{x} \in X \qquad \Pi_u \hat{K} \hat{x} \in U \tag{15a}$$

$$\hat{x}^i \in X \qquad K_i \hat{x}^i \in U \tag{15b}$$

$$\text{for all } \hat{x} = [\hat{x}^1, \dots, \hat{x}^m] \in \overline{\hat{S}},\ \hat{x}^i \in \mathbb{R}^{n_x}$$

where

$$\begin{aligned} \Pi_x &= \begin{bmatrix} I_{n_x \times n_x} & \dots & I_{n_x \times n_x} \end{bmatrix} \in \mathbb{R}^{n_x \times mn_x} \\ \Pi_u &= \begin{bmatrix} I_{n_u \times n_u} & \dots & I_{n_u \times n_u} \end{bmatrix} \in \mathbb{R}^{n_u \times mn_u} \\ \hat{K} &= \text{diag}(K_1, \dots, K_m). \end{aligned} \tag{16}$$

Then the set $S = \{x \in \mathbb{R}^{n_x} \,|\, x = \Pi_x \hat{x},\, \hat{x} \in \overline{\hat{S}}\}$ is a d-invariant set of the closed loop system (14).

Proof. By the construction of the interpolating controller (7) the relation $x_k = \Pi_x \hat{x}_k$ holds between the trajectories of the extended and the controlled system (14), (provided that $x_0 = \sum_{i=1}^m \hat{x}^i$). Moreover, by definition $\sum_{i=1}^m \hat{x}_k^i = \Pi_u \hat{K} \hat{x}$. Therefore, if the trajectory of the extended system satisfies (15a), the corresponding trajectory of the controlled system (14) satisfies the original constraints $x \in X, u \in U$. Furthermore, by the construction of S the trajectory x_k stays in S if $\hat{x}_k$ runs inside $\overline{\hat{S}}$. Thus S is d-invariant. Constraint (15b) is rather technical, since (15a) in itself does not define close set over $\mathbb{R}^{mn_x}$. ■

To apply Lemma 4. one has to construct the maximal d-invariant set $\overline{\hat{S}}$. This is not difficult by using Algorithm 1. with system Σ and constraints (15a), (15b). Note that these constraints are linear, since X, U are convex. The size of the d-invariant set S is of course, influenced by the size of the d-invariant sets generated by the controllers $K_1, \dots, K_m$. It will be shown in the next section, if the 'auxiliary' K_m controller is appropriately chosen (with 'large' d-invariant set) the set S becomes significantly larger than the maximal d-invariant set generated by any single 'proper', linear state feedback controller.

The interpolation based control algorithm can now be summarized as follows:

Algorithm 2. (Interpolation based control) *The initial state x_0 is given and the positive definite matrix $\hat{P}$ defining the storage function $\hat{V}(\hat{x}^1, \dots, \hat{x}^m) = \hat{x}^T \hat{P} \hat{x}$ has already* *been computed either by Lemma 3. or by Remark 1. Let the polytopic d-invariant set $\overline{\hat{S}}$ be defined by linear inequalities $(A_{\overline{\hat{S}}}, b_{\overline{\hat{S}}})$, i.e. $\hat{x} \in \overline{\hat{S}} \Leftrightarrow A_{\overline{\hat{S}}}\hat{x} \le b_{\overline{\hat{S}}}$.*

1) *Let*

$$\begin{aligned} \hat{x}_0 \quad := \quad & \arg\min_{\hat{x}\in\mathbb{R}^{mn_x}} \hat{x}^T \hat{P}\hat{x} \\ & \textit{w.r.t. } A_{\overline{\hat{S}}}\hat{x} \le b_{\overline{\hat{S}}} \textit{ and } x_0 = \Pi_x \hat{x} \end{aligned} \tag{17}$$

If the optimization is not feasible, i.e. $x_0 \notin S$, our algorithm can not give feasible solution. Thus $\Rightarrow$ STOP.

2) *Let k=0*
3) *Measure x_k, δ_k. Perform* <u>either</u> *of the following steps:*

a)

$$\begin{aligned} v_{k-1} \quad &\triangleq \quad \frac{1}{m} B_1(\delta_{k-1}) w_{k-1} \\ &= \quad \frac{1}{m}[x_k - A(\delta_{k-1})x_{k-1} - B_2(\delta_{k-1})u_{k-1}] \\ \hat{x}_k^i \quad &= \quad (A(\delta_{k-1}) + B_2(\delta_{k-1})K_i)\hat{x}_{k-1}^i + v_{k-1} \end{aligned} \tag{18}$$

b)

$$\begin{aligned} \hat{x}_k \quad := \quad & \arg\min_{\hat{x}\in\mathbb{R}^{mn_x}} \hat{x}^T \hat{P}\hat{x} \\ & \textit{w.r.t. } A_{\overline{\hat{S}}}\hat{x} \le b_{\overline{\hat{S}}} \textit{ and } x_k = \Pi_x \hat{x} \\ \hat{x}_k := & [\hat{x}_k^1, \ldots, \hat{x}_k^m] \end{aligned} \tag{19}$$

4) $u_k = \sum_{i=1}^m K_i \hat{x}_k^i$
5) *$k := k+1$, go to step 3*

In step 1 the initial state $\hat{x}_0$ of the extended system is computed so that it satisfies $\hat{x}_0 \in \overline{\hat{S}}$ and $\sum_{i=1}^m \hat{x}_0^i = x_0$. From the infinitely many solutions that is chosen, which minimizes the storage function. Step 3 can be executed in two different ways. Step a) computes simply the next state of Σ by using the measurements x_k, δ_{k-1} and the previous state $\hat{x}_{k-1}$. Step b) repartitions the state x_k by minimizing again the storage function $\hat{V}$. With the new state the dissipation inequality remains valid, but the performance improves. If the algorithm is applied with Step a) no on-line computation is needed (except at time 0/step 1). With step b) one achieves better performance for the price of higher computation time.

V. NUMERICAL EXAMPLE

The system to be controlled is defined by the following system matrices:

$$\begin{aligned} & A_1 = \begin{bmatrix} 0.8 & 0.1 \\ 0.3 & 1.2 \end{bmatrix} \; A_2 = \begin{bmatrix} 1.1 & 0.2 \\ 0.1 & 0.8 \end{bmatrix} \; B_1 = \begin{bmatrix} 0.15 \\ 1 \end{bmatrix} \\ & B_{2_1} = \begin{bmatrix} 0.1 \\ 1 \end{bmatrix} \; B_{2_2} = \begin{bmatrix} 0.2 \\ 1.5 \end{bmatrix} \\ & C_1 = \begin{bmatrix} 0.6 & 0.3 \end{bmatrix} \; C_2 = \begin{bmatrix} 0.7 & 0.3 \end{bmatrix} \\ & D_{1_1} = -0.3, \;\; D_{1_2} = 0.3 \end{aligned} \tag{20}$$

Let the polytopes X, U, W be given as follows

$$\begin{aligned} X = \{x \mid \begin{bmatrix} +1 & 0 \\ -1 & 0 \\ 0 & +1 \\ 0 & -1 \end{bmatrix} x \le 5\}, \qquad U = \begin{bmatrix} -1 & 1 \end{bmatrix}, \\ W = \begin{bmatrix} -0.3 & 0.3 \end{bmatrix} \end{aligned} \tag{21}$$

The prescribed performance level is $\gamma^* = 1$.

Two alternate $\mathcal{H}_\infty$ controllers K_a and K_b have been designed independently with the following performance list:

$$\gamma_{a,b} = \begin{bmatrix} 0.6093 & 2.0646 \end{bmatrix} \tag{22}$$

The first controller satisfies the performance requirement $\gamma_a < \gamma^*$, while the second one does not. On the other hand, the maximal d-invariant set associated to K_b is much larger than the set that corresponds to K_a.

An extended system is constructed by repeating the K_a controller 7 times, i.e. $K_1 = K_2 = \ldots K_7 = K_a$ and using K_b as K_8. The $\mathcal{L}_2$ gain computed by Lemma 3. is $\sqrt{\sum_{i=1}^8 \frac{\gamma_i^2}{8}} = 0.9261$. The gain according to Remark 1. has been reevaluated and obtained $\hat{\gamma}_8 = 0.7721 (< \gamma^* = 1)$.

The invariant sets of interest are depicted in figure 1. The largest set $\overline{S}_b (= \overline{S}_8)$ is the maximal d-invariant set generated by controller $K_b (= K_8)$. $\overline{S}_a (= \overline{S}_i,\, i \le 7)$ is contained in $\overline{S}_b$ as well. The d-invariant set S (the region of applicability) of the interpolation based controller is drawn by bold line. The set has been determined by using Lemma 4.

Furthermore, a constant K_c controller has been designed to assure approximately the same performance as the interpolating controller $\gamma_c = 0.7763$ over the largest possible domain. K_c gives a maximal domain of applicability ($\overline{S}_c$) that can be achieved by a single constrained $\mathcal{H}_\infty$ controller. The set S is *much larger* than the set $\overline{S}_c$.

After constructing the invariant sets the simulation is started at $x_0 = [-4.3 \;;\; 3]$. The parameter δ^1 is given as a signal with an amplitude 0.5 and a time period 60, shifted between 0 and 1. $\delta^2 = 1 - \delta^1$. The control input, system output and the trajectory can be seen in figures 1 and 2. (Algorithm 2. was used with step 3/b.)

Other methods using conservative approximations for the invariant sets can not exploit, in general, the entire control input range U. They prescribe much less control input than that is allowed by the constraints. In contrast, our algorithm operates on the entire domain, e.g. at time 0 it prescribes a control input close to 1 ($u_0 = 0.93$).

VI. CONCLUSION

A novel constrained $\mathcal{H}_\infty$ control design method has been developed for discrete-time LPV systems. The method is based on interpolating among appropriately chosen, unconstrained, linear feedback controllers. It was shown that the domain of applicability of the new controller is much larger than of any single state feedback.

It was also shown, that the proposed controller is less conservative in constraint handling compared to other methods using approximation of the level sets of the storage function.

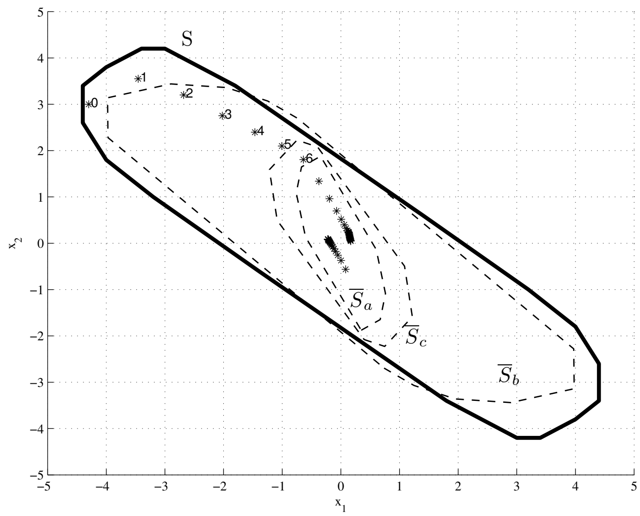

Fig. 1. The maximal d-invariant sets generated by the controllers K_a, K_b, K_c and the region of applicability S of the interpolating controller.

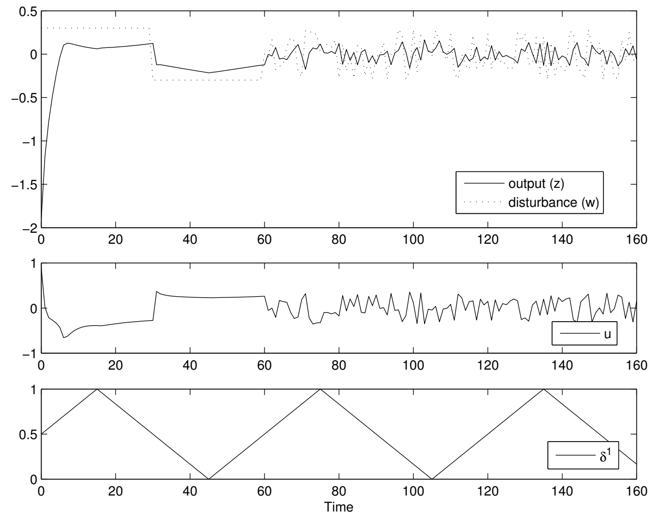

Fig. 2. The system output z, the disturbance w (dotted), the applied control input u and the parameter δ^1 ($\delta^2 = 1 - \delta^1$).

The applicability of the method is tested and demonstrated on numerical examples. In contrast to other nonlinear or LPV-MPC based approaches, the presented method does not necessarily require on-line optimization, therefore it can be used in real-time.

Although, the results are derived for constant state feedback controllers, part of the statements remain valid if parameter dependent $K_i(\rho)$ gains are applied in the interpolation. In the paper the polytopic LPV system is assumed to be perfectly known, no uncertainty is considered. Further research has to be carry on the robustness aspects of the proposed method and to extend the results for output feedback, dynamic controllers.

VII. ACKNOWLEDGEMENTS

This work has been supported by the Hungarian Science Fund (OTKA) through grant K60767 and the Hungarian National Office for Research and Technology through the project "Advanced Vehicles and Vehicle Control Knowledge Center" (No: OMFB-01418/2004) which is gratefully acknowledged.

REFERENCES

[1] D. Munoz de la Pena, D.R. Ramírez, E.F. Camacho, and T. Alamo. Explicit solution of min-max MPC with additive uncertainties and quadratic criterion. *Systems & Control Letters*, 55:266–274, 2006.

[2] F. Borrelli. *Constrained Optimal Control of Linear and Hybrid Systems*, volume 290 of *Lecture Notes in Control and Information Sciences*. Springer, 2003.

[3] H. Chen and C. W. Scherer. An lmi based model predictive control scheme with guaranteed $\mathcal{H}_\infty$ performance and its application to active suspension. In *American Control Conference*, 2004, Boston.

[4] H. Chen and C. W. Scherer. Moving horizon $\mathcal{H}_\infty$ control with performance adaptation for constrained linear systems. *Automatica*, 42:1033–1040, 2006.

[5] H. Chen, C. W. Scherer, and F. Allgöwer. A game theoretic approach to nonlinear robust receding horizon control of constrained systems. In *American Control Conference, Albuquerquie, New Mexico*, pages 3073–3077, 1997.

[6] E. G. Gilbert and K. T. Tan. Linear systems with state and control constraints: The theory and application of maximal output admissible sets. *IEEE Transactions on Automatic Control*, 36(9):1008–1020, 1991.

[7] P. J. Goulart and E. C. Kerrigan. A convex formulation for receding horizon control of constrained discrete-time systems with guaranteed ℓ_2 gain. In *45th IEEE Conference on Decision & Control*, 2006, San Diego.

[8] É. Gyurkovics. Receding horizon $\mathcal{H}_\infty$ control for nonlinear discrete-time systems. *IEE Proc-Control Theory Appl.*, 149/6:540–546, 2002.

[9] É. Gyurkovics and T. Takacs. Quadratic stabilisation with $\mathcal{H}_\infty$ norm bound of non-linear discrete-time uncertain systems with bounded control. *Systems & Control Letters*, 50:277–289, 2003.

[10] I. Kaminer, P. P. Khargonekar, and M. A. Rotea. Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control for discrete time systems via convex optimization. *Automatica*, 29:57–70, 1993.

[11] I. Kolmanovsky and E. G. Gilbert. Theory and computation of disturbance invariant sets for discrete-time linear systems. *Mathematical Problems in Engineering*, 4:317–367, 1998.

[12] D. Q. Mayne, S. V. Rakovic, R. B. Vinter, and E. C. Kerrigan. Characterization of the solution to a constrained $\mathcal{H}_\infty$ optimal control problem. *Automatica*, 42:371–382, 2006.

[13] B. Pluymers, J.A.Rossiter, J.A.K.Suykens, and B. De Moor. The efficient computation of polyhedral invariant sets for linear systems with polytopic uncertainty. In *American Control Conference*, 2005, Portland.

[14] B. Pluymers, J.A.Rossiter, J.A.K. Suykens, and B. De Moor. A simple algorithm for robust MPC. In *IFAC World Congress*, 2005.

[15] B. Pluymers, J.A.Rossiter, J.A.K. Suykens, and B. De Moor. Interpolation based MPC for LPV systems using polyhedral invarinat sets. In *American Control Conference*, 2005, Portland.

[16] S.V. Rakovic, E.C. Kerrigan, K.I. Kouramas, and D.Q. Mayne. Invariant approximations of the minimal robust positively invariant set. *IEEE Transactions on Automatic Control*, 50(3):406–410, 2005.

[17] J.A. Rossiter, B. Kouvaritakis, and M. Bacic. Interpolation based computationally efficient predictive control. *Interantional Journal of Control*, 77 (3):290–301, 2004.

[18] C. Scherer and S. Weiland. *Linear Matrix Inequalities in Control*. 2000. *http://www.dcsc.tudelft.nl/ cscherer/2416/lmi.html*.

[19] Fen Wu. *Control of Linear Parameter Varying Systems*. PhD thesis, University of California at Berkeley, 1995.