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A Polynomial Approach to the Synthesis of Observers for Nonlinear Systems

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Abstract— **This paper discusses the problem of transforming the single-input single-output nonlinear control system into the nonlinear observer (input-output injection) form using the notion of adjoint polynomials. Such a polynomial approach to the sysnthesis of observers results in the transparent and elegant way for the computation of the one-forms needed to solve the problem.**

I. INTRODUCTION

The paper addresses and revisits the problem of transforming the single-input single-output nonlinear control system into the nonlinear observer (input-output injection) form using the state coordinate transformation. This problem has been addressed within different approaches, see for example [7], [8], [12], [13], [14], [15], [16] and [19]. Our results are related to the solution in terms of the differential one-forms. In the continuous- and discrete-time settings the necessary and sufficient solvability conditions were given in [7] and [12], [13] respectively. Note that [12] considers only autonomous systems without input. Remark that in both cases certain one-forms, defined by the corresponding algorithms, have to be exact or closed for the problem to be solvable. The novelty of our approach lies in two aspects.

First, the results are stated for nonlinear control systems described by the pseudo-linear operator and results are obtained using the pseudo-linear algebra [1], [5], [11]. Such system description unifies the study of discrete- and continuous-time cases into a single framework and moreover, provides an extension. For the pseudo-linear operator covers not just the classical continuous- and discrete-time cases, but accommodates also difference, q-shift and q-difference operators [11]. Second, using the notion of adjoint polynomials [1], we suggest a very easy, transparent and mathematically elegant way to compute the necessary one-forms. Note that our approach simplifies the computations especially in the continuous-time case, for the procedures for handling Ore rings and skew polynomials, including adjoint polynomials as well, are already implemented for instance in the computer algebra system Maple. Finally, the paper demonstrates the applicability of pseudo-linear algebra to solve the problem of transforming the state equations into a special form. In a similar manner a pseudo-linear algebra may be applied

to many different control problems like system reduction, feedback linearization, accessibility and realization of the input-output equations in the state space form, to name a few possibilities.

II. ALGEBRAIC BACKGROUND

The algebraic framework which allows us to handle various nonlinear control systems from a unique standpoint was recently suggested in [11]. This formalism introduces the notion of a σ-differential field which represents a generalization of the notions of differential and difference fields usually employed to analyze the properties of continuous-and, respectively, discrete-time nonlinear systems.

A. Pseudo-linear algebra

Pseudo-linear algebra [1], [5] is the study of common properties of linear differential, shift, difference, q-shift, q-difference and other types of operators. Such operators are expressed in terms of the so-called skew polynomials. Note that while mainly the differential and shift operators play a key role in the control theory also applications employing other types of operators can be found [3], [9], [10]. For instance, the q-shift operator can be in some cases used to model discrete-time systems with the varying sampling period. The sampling period may be changed based on the resources availabilty [2].

We begin with introducing the notion of a pseudo-derivation.

Definition 1: Let K be a field and $\sigma : K \rightarrow K$ an automorphism of K. A map $\delta : K \rightarrow K$ which satisfies

$$\delta(a + b) = \delta(a) + \delta(b)$$
$$\delta(ab) = \sigma(a)\delta(b) + \delta(a)b \qquad (1)$$

is called a pseudo-derivation (or a σ-derivation).

The notion of a pseudo-derivation unifies the notion of a standard derivation and various difference operators [1], [5]. Clearly, if $\sigma = 1_K$ then (1) is just the rule for a derivation on K. Or, for any σ the map $\delta_\alpha = \alpha(\sigma - 1_K)$ with $\alpha \in K$ is a pseudo-derivation as well and has a sense of a difference operator on K.

Remark 1: In studing the nonlinear control systems we have to assume that $\alpha \in \mathbf{R}$. For it is necessary to satisfy the commutativity of a pseudo-derivation with the differential operator d, see [11] for details.

Definition 2: A σ-differential field is a triple (K, σ, δ) where K is a field, σ is an automorphism of K and δ is a pseudo-derivation.

Any automorphism σ and pseudo-derivation δ induce a non-commutative skew polynomial ring.

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Definition 3: The left skew polynomial ring given by σ and δ is the ring $K[p; \sigma, \delta]$ of polynomials in p over K with the usual addition, and the (non-commutative) multiplication given by the commutation rule

$$pa = \sigma(a)p + \delta(a) \qquad (2)$$

for any $a \in K$.

Elements of such a ring are called skew polynomials or non-commutative polynomials or Ore polynomials [17], [18]. The commutation rule (2) actually represents the action of the corresponding operator on polynomials.

Example 1: If K with a derivation $\frac{\mathrm{d}}{\mathrm{d}t}$ is a differential field, then $K[D; 1_K, \frac{\mathrm{d}}{\mathrm{d}t}]$ is the ring of linear ordinary differential operators and we interpret (2) as a rule for differentiation: $Da = aD + \dot{a}$ for any $a \in K$.

If K is a difference field and σ over K is the automorphism which takes t to $t + 1$, then $K[E; \sigma, 0]$ is the ring of linear ordinary shift operators, while $K[E; \sigma, \Delta]$, where $\Delta = \sigma - 1_K$, is the ring of linear ordinary difference operators [1], [5].

Definition 4: Let V be a vector space over a field K. A map $\theta : V \to V$ is called pseudo-linear if

$$\theta(u + v) = \theta(u) + \theta(v)$$
$$\theta(au) = \sigma(a)\theta(u) + \delta(a)u \qquad (3)$$

for any $a \in K$, $u, v \in V$.

Pseudo-linear maps, in comparison to pseudo-derivations, allow to handle differential, difference and also shift structures from a unique standpoint. Note that any field K is a vector space itself. Hence, we can consider pseudo-linear maps over K assuming that (3) holds for any $a, u, v \in K$.

Obviously, any pseudo-derivation δ over K is a pseudo-linear map, simply by letting $\theta = \delta$.

If $\delta = 0$ then $\theta = \sigma$ and (3) is clearly satisfied or we can equivalently associate a difference operator $\Delta = \sigma - 1_K$. Then $\theta = \Delta$ and (3) is again satisfied, $\Delta(au) = \sigma(a)\Delta(u) + \Delta(a)u = \sigma(a)(\sigma(u) - u) + (\sigma(a) - a)u = \sigma(au) - au$. This represents two alternative ways of describing "shift" structures.

Any pseudo-linear map $\theta : V \to V$ induces the action denoted by $*$

$$* : K[p; \sigma, \delta] \times V \to V; \left(\sum_{i=0}^{n} a_i p^i \right) * u = \sum_{i=0}^{n} a_i \theta^i(u)$$

for any $u \in V$. For the sake of simplicity, the symbol $*$ is often dropped. So the elements of $K[p; \sigma, \delta]$ can be viewed as operators acting on a vector space V.

Note also that multiplication in $K[p; \sigma, \delta]$ corresponds to the composition of operators and $(rs)u = r(su)$ for any $r, s \in K[p; \sigma, \delta]$ and $u \in V$.

B. Control systems

The nonlinear control systems considered in this paper are objects of the form

$$\begin{aligned} x^{\langle 1 \rangle} &= f(x, u) \\ y &= g(x, u) \end{aligned} \qquad (4)$$

where the entries of f and g are meromorphic functions, which we think of as elements of the quotient field of the ring of analytic functions. The symbol $x^{\langle 1 \rangle}$ stands for a pseudo-linear operator: $x^{\langle 1 \rangle} = \theta(x)$. It can be a derivation, $\theta(x) = \dot{x}$, that corresponds to the continuous-time case, a shift, $\theta(x) = \sigma(x)$, or a difference, $\theta(x) = \alpha(\sigma(x) - x)$ with $\alpha \in \mathbf{R}$, that correspond to two alternative discrete-time cases.

In (4), $x \in \mathbf{R}^n$, $u \in \mathbf{R}$ and $y \in \mathbf{R}$ denote respectively the state, the input and the output of the system.

Let $\mathcal{K}$ denote now the field of meromorphic functions of variables $\{x, u^{\langle k \rangle}; k \geq 0\}$.

We assume that system (4) is generically submersive, i.e.

$$\mathrm{rank}_{\mathcal{K}} \frac{\partial \sigma(x)}{\partial(x, u)} = n \qquad (5)$$

Under (5), σ is an automorphism of $\mathcal{K}$ and there exists, up to an isomorphism, a unique difference field $\mathcal{K}^*$ called the inversive closure of $\mathcal{K}$ [6]. Here we assume that the inversive closure is given and by abuse of notation we use the same symbol $\mathcal{K}$ for both. An explicit construction of the inversive closure follows the same line as in [4].

Next, define the vector space $\mathcal{E}$ of one-forms spanned over $\mathcal{K}$ by differentials of elements of $\mathcal{K}$, that is

$$\mathcal{E} = \mathrm{span}_{\mathcal{K}}\{\mathrm{d}\xi; \xi \in \mathcal{K}\}$$

The pseudo-linear operator θ acts on $\mathcal{K}$ and $\mathcal{E}$ as follows

$$\theta(\xi) = \begin{cases} \delta(\xi) & \text{if } \delta \neq 0 \\ \sigma(\xi) & \text{if } \delta = 0 \end{cases}$$

$$\theta(c\mathrm{d}\xi) = \sigma(c)\mathrm{d}(\theta(\xi)) + \delta(c)\mathrm{d}\xi$$

where $\xi \in \mathcal{K}$ and $c\mathrm{d}\xi \in \mathcal{E}$. For more details, see [11].

C. Polynomial system description

The behaviour of nonlinear control systems of the form (4) can be now described by two skew polynomials over the σ-differential field $\mathcal{K}$ that act on differentials of system inputs and outputs, see [11].

Let

$$y^{\langle n \rangle} = F\left(y, \ldots, y^{\langle n-1 \rangle}, u, \ldots, u^{\langle s \rangle}\right) \qquad (6)$$

where $F \in \mathcal{K}$, be the corresponding input-output equation of the control system (4). After differentiating (6) we get

$$\mathrm{d}y^{\langle n \rangle} - \sum_{i=1}^{n-1} \frac{\partial F}{\partial y^{\langle i \rangle}} \mathrm{d}y^{\langle i \rangle} = \sum_{i=0}^{s} \frac{\partial F}{\partial u^{\langle i \rangle}} \mathrm{d}u^{\langle i \rangle}$$

or, alternatively

$$a(p)\mathrm{d}y = b(p)\mathrm{d}u \qquad (7)$$

where

$$a(p) = p^n - \sum_{i=1}^{n-1} \frac{\partial F}{\partial y^{\langle i \rangle}} p^i$$

$$b(p) = \sum_{i=0}^{s} \frac{\partial F}{\partial u^{\langle i \rangle}} p^i$$

and $a(p), b(p) \in \mathcal{K}[p; \sigma, \delta]$.

Example 2: Consider the system $\ddot{y} = \dot{y}u$. After differentiating $\mathrm{d}\ddot{y} = u\mathrm{d}\dot{y} + \dot{y}\mathrm{d}u$ we get

$$(s^2 - us)\mathrm{d}y = \dot{y}\mathrm{d}u$$

where $a(s) = s^2 - us$ and $b(s) = \dot{y}$ are polynomials in $\mathcal{K}[s; 1_{\mathcal{K}}, \frac{\mathrm{d}}{\mathrm{d}t}]$.

Example 3: Consider the discrete-time system $y^{++} = u + y^+ u^+$ where y stands for $y(t)$, y^+ for $y(t+1)$ etc. Now, after differentiating $\mathrm{d}y^{++} = \mathrm{d}u + u^+\mathrm{d}y^+ + y^+\mathrm{d}u^+$ we get

$$(\delta^2 - u^+\delta)\mathrm{d}y = (y^+\delta + 1)\mathrm{d}u$$

where $a(\delta) = \delta^2 - u^+\delta$ and $b(\delta) = y^+\delta + 1$ are polynomials in $\mathcal{K}[\delta; \sigma, 0]$ with σ taking t to $t+1$ (forward shift operator).

III. ADJOINT POLYNOMIALS

Adjoint polynomials [1] represent in some sense dual objects to skew polynomials. We can get them by moving the indeterminate on the *left* of each summand. Formally, they are defined as follows.

Definition 5: The adjoint of a skew polynomial ring $K[p; \sigma, \delta]$ is defined as the skew polynomial ring $K[p^*; \sigma^*, \delta^*]$ where

$$\sigma^* = \sigma^{-1}, \qquad \delta^* = -\delta\sigma^{-1}.$$

Let $a(p) = a_n p^n + \ldots + a_1 p + a_0$ be a polynomial in $K[p; \sigma, \delta]$. The adjoint polynomial a^* is then defined by the formula

$$a^*(p^*) = p^{*n}a_n + \ldots + p^*a_1 + a_0 \in K[p^*; \sigma^*, \delta^*] \quad (8)$$

Note that products $p^{*i}a_i$ must be computed in the skew polynomial ring $K[p^*; \sigma^*, \delta^*]$, following the commutation rule

$$p^*a = \sigma^*(a)p^* + \delta^*(a)$$

Example 4: Consider the polynomials from Example 3

$$\begin{aligned} a(\delta) &= \delta^2 - u^+\delta \\ b(\delta) &= y^+\delta + 1 \end{aligned}$$

The adjoint of $\mathcal{K}[\delta; \sigma, 0]$ is the ring $\mathcal{K}[\delta^*; \sigma^*, 0]$ where $\sigma^* = \sigma^{-1}$; that is, the backward shift operator. So, the comutation rule in $\mathcal{K}[\delta^*; \sigma^{-1}, 0]$ reads

$$\delta^*a = a^-\delta^*$$

Thus, in according to the formula (8), the adjoint polynomials can be found by computing

$$\begin{aligned} a^*(\delta^*) &= \delta^{*2} - \delta^*u^+ = \\ &= \delta^{*2} - u\delta^* \\ b^*(\delta^*) &= \delta^*y^+ + 1 = \\ &= y\delta^* + 1 \end{aligned}$$

This is, in fact, the formalization of the idea of moving the indeterminate on the *left* of each summand in original polynomials

$$\begin{aligned} a(\delta) &= \delta^2 - \delta u \\ b(\delta) &= \delta y + 1 \end{aligned}$$

Notice that we get the coefficients of adjoint polynomials.

Example 5: Similarly, we can find adjoint polynomials in continuous-time case. Consider, for instance, the polynomial $a(s)$ from Example 2

$$a(s) = s^2 - us$$

The adjoint of $\mathcal{K}[s; 1_{\mathcal{K}}, \frac{\mathrm{d}}{\mathrm{d}t}]$ is the ring $\mathcal{K}[s^*; 1_{\mathcal{K}}, -\frac{\mathrm{d}}{\mathrm{d}t}]$ with the comutation rule $s^*a = as^* - \dot{a}$. Thus, the adjoint polynomial can be found as

$$\begin{aligned} a^*(s^*) &= s^{*2} - s^*u \\ &= s^{*2} - us^* + \dot{u} \end{aligned}$$

Or, again, by moving s on the left of each summand in $a(s)$

$$a(s) = s^2 - su + \dot{u}$$

Remark that in commutative case; that is, the case of linear systems when all coefficients are in $\mathbf{R}$, a polynomial and its adjoint are identical objects.

Finally, remark also that the adjoint is a bijective mapping and

$$(\sigma^*)^* = \sigma, \qquad (\delta^*)^* = \delta$$

Moreover

$$(a^*)^* = a, \qquad (ab)^* = b^*a^*$$

for any $a, b \in K[p; \sigma, \delta]$. For more details see [1].

IV. SYNTHESIS OF OBSERVER

The use of an observer is necessary whenever the state employed in a feedback loop is not directly measurable. Here, we recall some well known basic results on the observer design for system (4). The solution is based on the system linearization by the state coordinate transformation up to the input-output injection and on the standard Luenberger observer design for the linearized system.

A. Linearization and input-output injection

Consider the system (4) and suppose it to be observable. The aim is to find, if possible, a state transformation $\xi = \phi(x)$ such that in the new coordinates the system (4) is in the observer form

$$\begin{aligned} \xi_1^{\langle 1 \rangle} &= \xi_2 + \varphi_1(y, u) \\ &\vdots \\ \xi_{n-1}^{\langle 1 \rangle} &= \xi_n + \varphi_{n-1}(y, u) \\ \xi_n^{\langle 1 \rangle} &= \varphi_n(y, u) \\ y &= \xi_1 \end{aligned} \quad (9)$$

where $\varphi_i \in \mathcal{K}$ are the so-called input-output injections. From such a form the synthesis of the observer is quite straightforward. Note that the system (9) is, in fact, of the form

$$\begin{aligned} \xi^{\langle 1 \rangle} &= A\xi + \varphi(y, u) \\ y &= C\xi \end{aligned}$$

with constant matrices (C, A) in canonical observer form. An estimate $\hat{\xi}$ of the state ξ can be now obtained as

$$\hat{\xi}^{\langle 1 \rangle} = A\hat{\xi} + \varphi(y, u) + K(C\hat{\xi} - y)$$

provided that K is chosen such that the estimation error $e = \hat{\xi} - \xi$ goes asymptotically to zero. Note that $e^{\langle 1 \rangle} = (A + KC)e$ is linear, which is, in fact, the purpose of the transformation of the system equations (4) to the form (9). See for instance [7].

The solution to the problem thus reduces to finding, if possible, input-output injections φ_i's such that the input-output equation of the system (4)

$$y^{\langle n \rangle} = F\left(y, \ldots, y^{\langle n-1 \rangle}, u, \ldots, u^{\langle s \rangle}\right) \qquad (10)$$

can be rewritten as

$$y^{\langle n \rangle} = \varphi_1^{\langle n-1 \rangle}(y, u) + \ldots + \varphi_{n-1}^{\langle 1 \rangle}(y, u) + \varphi_n(y, u) \quad (11)$$

which is clearly the input-output equation of the transformed system (9).

B. Transformation to the observer form

Classical approach to the transformation of the system into the observer form consists in examining the exactness of certain one-forms derived stepwise from the input-output mapping F, see for instance [7] for the continuous-time case and [12], [13] for the discrete-time case.

Alternatively, as we will demonstrate below, such one-forms can be directly found from polynomial system description (7) by employing the notion of adjoint polynomials, forming the main ingredient of this paper.

After differentiating (10) we get $a(p)\mathrm{d}y = b(p)\mathrm{d}u$ where

$$a(p) = p^n + a_{n-1}p^{n-1} + \ldots + a_1 p + a_0$$
$$b(p) = b_{n-1}p^{n-1} + \ldots + b_s p^s + \ldots + b_1 p + b_0$$

with $b_{s+1}, \ldots, b_{n-1} = 0$, are polynomials in $K[p; \sigma, \delta]$. So

$$p^n \mathrm{d}y = -a_{n-1}p^{n-1}\mathrm{d}y - \ldots - a_1 p\mathrm{d}y - a_0 \mathrm{d}y +$$
$$+ b_{n-1}p^{n-1}\mathrm{d}u + \ldots + b_1 p\mathrm{d}u + b_0\mathrm{d}u$$

Now let

$$a^*(p^*) = p^{*n} + a_{n-1}^* p^{*n-1} + \ldots + a_1^* p^* + a_0^*$$
$$b^*(p^*) = b_{n-1}^* p^{*n-1} + \ldots + b_1^* p^* + b_0^*$$

in $K[p^*; \sigma^*, \delta^*]$ be the adjoints of $a(p)$, $b(p)$ respectively. Then clearly

$$p^n \mathrm{d}y = -p^{n-1}a_{n-1}^* \mathrm{d}y - \ldots - pa_1^* \mathrm{d}y - a_0^* \mathrm{d}y +$$
$$+ p^{n-1}b_{n-1}^* \mathrm{d}u + \ldots + pb_1^* \mathrm{d}u + b_0^* \mathrm{d}u$$

which can be rewritten as

$$p^n \mathrm{d}y = p^{n-1}\omega_1 + \ldots + p\omega_{n-1} + \omega_n$$

with

$$\omega_i = b_{n-i}^* \mathrm{d}u - a_{n-i}^* \mathrm{d}y \qquad (12)$$

for $i = 1, \ldots, n$.

Finally, the polynomial description of (11) reads as

$$p^n \mathrm{d}y = p^{n-1}\mathrm{d}\varphi_1 + \ldots + p\mathrm{d}\varphi_{n-1} + \mathrm{d}\varphi_n$$

We have thus proved

Proposition 1: The nonlinear system (4) is locally equivalent to the system (9) under a state transformation $\xi = \phi(x)$ if and only if

$$\mathrm{d}\omega_i = 0$$

for $i = 1, \ldots, n$, where the one-forms ω_i are defined by (12).

Remark 2: One can easily prove that for continuous- and, respectively, discrete-time cases the list $\{\omega_i; i = 1, \ldots, n\}$ coincides with the one derived within the classical approach, see [7] and [13]. Note, however, that the list of one-forms $\tilde{\omega}_i$ in [12] differs with the relationship $\omega_{i-1} = \tilde{\omega}_i - \tilde{\omega}_{i-1}$, as proved in [13].

Example 6: Consider the system from Example 2 where

$$s^2 \mathrm{d}y = us\mathrm{d}y + \dot{y}\mathrm{d}u$$

Coefficients of adjoint polynomials can be found by moving the indeterminate on the left of each summand

$$s^2 \mathrm{d}y = (su - \dot{u})\mathrm{d}y + \dot{y}\mathrm{d}u$$
$$s^2 \mathrm{d}y = s\omega_1 + \omega_2$$

and $\omega_1 = u\mathrm{d}y$, $\omega_2 = \dot{y}\mathrm{d}u - \dot{u}\mathrm{d}y$. Or, alternatively, following the formal definition

$$a(s) = s^2 - us$$
$$b(s) = \dot{y}$$

and the adjoint polynomials are

$$a^*(s^*) = s^{*2} - us^* + \dot{u}$$
$$b^*(s^*) = \dot{y}$$

The one-forms can be stated as $\omega_1 = b_1^* \mathrm{d}u - a_1^* \mathrm{d}y = u\mathrm{d}y$, $\omega_2 = b_0^* \mathrm{d}u - a_0^* \mathrm{d}y = \dot{y}\mathrm{d}u - \dot{u}\mathrm{d}y$. Note that neither ω_1 nor ω_2 are exact.

Example 7: Consider the system from Example 3 where

$$(\delta^2 - u^+\delta)\mathrm{d}y = (y^+\delta + 1)\mathrm{d}u$$

We have now

$$\delta^2 \mathrm{d}y = u^+\delta\mathrm{d}y + y^+\delta\mathrm{d}u + \mathrm{d}u$$
$$\delta^2 \mathrm{d}y = \delta(u\mathrm{d}y + y\mathrm{d}u) + \mathrm{d}u$$

Both $\omega_1 = u\mathrm{d}y + y\mathrm{d}u$ and $\omega_2 = \mathrm{d}u$ are exact. Thus $\varphi_1 = yu$, $\varphi_2 = u$ and the system can be transformed into the input-output injection form

$$\xi_1^+ = \xi_2 + yu$$
$$\xi_2^+ = u$$
$$y = \xi_1$$

Note that the approach given here covers directly also difference, q-shift or q-difference operators. Hence, it is possible to design observers also for those types of control systems. For instance, some discrete-time systems with the varying sampling period can be in some cases modeled using q-shift operators, as depicted in the following example.

Example 8: Consider the system

$$
\begin{aligned}
x_1(2t) &= x_2(t) \\
x_2(2t) &= x_1(t)u(t) + x_1^2(t) \\
y(t) &= x_1(t)
\end{aligned}
$$

with the input-output equation $y(4t) = y(t)u(t) + y^2(2t)$ which can be modeled over the σ-differential field $(\mathcal{K}, \sigma, 0)$ where σ takes t to $2t$ as

$$
\begin{aligned}
x_1^{\langle 1 \rangle} &= x_2 \\
x_2^{\langle 1 \rangle} &= x_1 u + x_1^2 \\
y &= x_1
\end{aligned}
$$

with the input-output equation

$$
y^{\langle 2 \rangle} = yu + (y^{\langle 1 \rangle})^2
$$

The related polynomial description can be found as

$$
\begin{aligned}
\mathrm{d}y^{\langle 2 \rangle} &= u\mathrm{d}y + y\mathrm{d}u + 2y^{\langle 1 \rangle}\mathrm{d}y^{\langle 1 \rangle} \\
(p^2 - 2y^{\langle 1 \rangle}p - u)\mathrm{d}y &= y\mathrm{d}u
\end{aligned}
$$

where $a(p) = p^2 - 2y^{\langle 1 \rangle}p - u$ and $b(p) = y$ are polynomials in $\mathcal{K}[p; \sigma, 0]$. The adjoint polynomials can be computed as

$$
\begin{aligned}
a^*(p^*) &= p^{*2} - 2yp^* - u \\
b^*(p^*) &= y
\end{aligned}
$$

from which $\omega_1 = b_1^*\mathrm{d}u - a_1^*\mathrm{d}y = 2y\mathrm{d}y$ and $\omega_2 = y\mathrm{d}u + u\mathrm{d}y$, both exact $\varphi_1 = y^2$ and yu. Finally, the transformed system is

$$
\begin{aligned}
\xi_1(2t) &= \xi_2(t) + y^2(t) \\
\xi_2(2t) &= y(t)u(t) \\
y(t) &= \xi_1(t)
\end{aligned}
$$

C. Implementation in Maple

Described procedure for synthesis of observer can be easily implemented in computer algebra system Maple, since it supports directly many procedures for handling Ore rings and skew polynomials, including adjoints.

Example 9: The following system representing a DC motor was considered in [7]

$$
\begin{aligned}
\dot{x}_1 &= -K_m x_1 x_2 - \frac{R_a + R_f}{K}x_1 + u \\
\dot{x}_2 &= -\frac{B}{J}x_2 - x_3 + \frac{K_m}{J}Kx_1^2 \\
\dot{x}_3 &= 0 \\
y &= \ln x_1
\end{aligned}
$$

Input-output equation can be found as

$$
\begin{aligned}
y^{(3)} &= -2\frac{K_m^2 K}{J}e^{2y}\dot{y} - \frac{B}{J}e^{-y}u\dot{y} - 2e^{-y}\dot{y}\dot{u} + \\
&+ e^{-y}\dot{y}^2 u - \frac{B}{J}\ddot{y} + e^{-y}\ddot{u} - e^{-y}\ddot{y}u + \frac{B}{J}e^{-y}\dot{u}
\end{aligned}
$$

and the related polynomial description as $a(s)\mathrm{d}y = b(s)\mathrm{d}u$ with

$$
\begin{aligned}
a(s) &= s^3 + (e^{-y}u + \frac{B}{J})s^2 + \\
&+ (2\frac{K_m^2 K}{J}e^{2y} + 2e^{-y}\dot{u} - 2e^{-y}u\dot{y} + \frac{B}{J}e^{-y}u)s + \\
&+ e^{-y}\ddot{u} + 4\frac{K_m^2 K}{J}e^{2y}\dot{y} - 2e^{-y}\dot{y}\dot{u} + e^{-y}\dot{y}^2 u + \\
&+ \frac{B}{J}e^{-y}\dot{u} - e^{-y}\ddot{y}u - \frac{B}{J}e^{-y}u\dot{y}
\end{aligned}
$$

and

$$
\begin{aligned}
b(s) &= e^{-y}s^2 + (\frac{B}{J}e^{-y} - 2e^{-y}\dot{y})s + \\
&+ e^{-y}\dot{y}^2 - e^{-y}\ddot{y} - \frac{B}{J}e^{-y}\dot{y}
\end{aligned}
$$

The adjoint polynomials can be found by Maple procedure **AdjointOrePoly** as

$$
\begin{aligned}
a^*(s^*) &= s^{*3} + (e^{-y}u + \frac{B}{J})s^{*2} + (2\frac{K_m^2 K}{J}e^{2y} + \\
&+ \frac{B}{J}e^{-y}u)s^* \\
b^*(s^*) &= e^{-y}s^{*2} + \frac{B}{J}e^{-y}s^*
\end{aligned}
$$

from which $\omega_i = b_{n-i}^*\mathrm{d}u - a_{n-i}^*\mathrm{d}y$ for $i = 1, 2, 3$ are as follows

$$
\begin{aligned}
\omega_1 &= e^{-y}\mathrm{d}u - (e^{-y}u + \frac{B}{J})\mathrm{d}y \\
\omega_2 &= \frac{B}{J}e^{-y}\mathrm{d}u - (2\frac{K_m^2 K}{J}e^{2y} + \frac{B}{J}e^{-y}u)\mathrm{d}y \\
\omega_3 &= 0
\end{aligned}
$$

all of them exact. That is

$$
\begin{aligned}
\varphi_1 &= e^{-y}u - \frac{B}{J}y \\
\varphi_2 &= \frac{B}{J}e^{-y}u - \frac{K_m^2 K}{J}e^{2y} \\
\varphi_3 &= 0
\end{aligned}
$$

Thus, the system can be transformed into the following input-output injection form

$$
\begin{aligned}
\dot{\xi}_1 &= \xi_2 + e^{-y}u - \frac{B}{J}y \\
\dot{\xi}_2 &= \xi_3 + \frac{B}{J}e^{-y}u - \frac{K_m^2 K}{J}e^{2y} \\
\dot{\xi}_3 &= 0 \\
y &= \xi_1
\end{aligned}
$$

like in [7].

V. CONCLUSIONS

In this paper the problem of transforming the single-input single-output nonlinear control system into the nonlinear observer (input-output injection) form using the notion of adjoint polynomials was discussed. Such an approach resulted in the transparent and mathematically elegant way for

the computation of the one-forms one needs to solve the problem.

Our future task is to extend the results of transformability the state equations into the observer form using both the state and output coordinate transformations. Note that in the discrete-time case necessary and sufficient transformabilty conditions exist [13], but in the continuous-time case the corresponding conditions are missing. Though the necessary and sufficient solvability conditions exist also in the continuous-time case [16], these conditions are not directly computable from the system description like in the discrete-time case; they depend on the existence of a unknown function. The same holds for the necessary solvability conditions in [8].

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