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Asymptotic Solution of Linear-Quadratic Control Problem with Intermediate Points and Small Parameter in Performance Index

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Abstract— **The asymptotic expansion of the solution of a linear-quadratic optimal control problem for a descriptor system with intermediate points and a small parameter in a performance index has been constructed as series of non-negative integer powers of a small parameter. The estimates have been obtained for the proximity of the asymptotic approximate solutions to the exact one. The nice property is proved, namely, the values of the minimized functional do not increase when higher-order approximations to the optimal control are used. The numerical example is given in order to illustrate the proposed method.**

I. Introduction

In this paper, we study a linear–quadratic optimal control problem (1)-(3) from the second section. The presence of a small parameter in the cost (1) indicates the different significances of the addends. We will assume that admissible controls $u(\cdot)$ in the perturbed problem are piecewise continuous functions ensuring the solvability of a state equation with a given condition for the state variable, trajectories $x(\cdot)$ of a state equation are piecewise continuous functions satisfying the state equation almost everywhere such that $Ex(\cdot)$ are continuous.

We will construct the asymptotic expansion of the solution of the considered perturbed problem in the form of series of non-negative integer powers of a small parameter by substituting the postulated asymptotic expansions into the problem condition and then defining a series of optimal control problems in order to find the expansions terms. Such method for the construction of asymptotic solutions for optimal control problems was essentially developed in [1] for singularly perturbed continuous optimal control problems without restrictions for the values of the control. This method has been called the "direct scheme." Further applications of the direct scheme and the survey of the publications, devoted to optimal control problems with a small parameter, are presented in [2].

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The numerical example will be given in order to illustrate the proposed method.

We indicate some works concerning the optimal control problems with intermediate points in the performance index. The paper [3] deals with the necessary control optimality condition for nonlinear optimal control problems consisting in the approaching of the object to fixed points in the given order, when resources are limited and a state equation is resolved with respect to the derivative. The integral part in the performance index is absent in this case. Some problems with intermediate points and with a state equation resolved with respect to the derivative, when the control is one-dimensional, are considered in [4]. In the last paper, the problem for determining of the volume of water, containing in a lake, is described. This problem is reduced to a problem with intermediate points. The necessary and sufficient control optimality conditions in the Pontryagin's maximum principle form are given in [5] for some linear-quadratic optimal control problems for descriptor systems with intermediate points. The methods for finding an optimal control, which are used in [5], are different from the methods in [4].

II. Problem Formulation

Let us consider the problem P_ε of the following form

$$J_\varepsilon(u,x)=\frac{1}{2}\left\langle x(T)-\xi_{N+1}, E^*G_{N+1}E(x(T)-\xi_{N+1})\right\rangle+ \\ +\frac{\varepsilon}{2}\sum_{j=1}^{N}\left\langle x(t_j)-\xi_j, E^*G_jE(x(t_j)-\xi_j)\right\rangle+ \\ +\frac{1}{2}\int_0^T\left\langle\begin{pmatrix}x(t)\\u(t)\end{pmatrix},\begin{pmatrix}W(t) & S(t)\\ S(t)^* & R(t)\end{pmatrix}\begin{pmatrix}x(t)\\u(t)\end{pmatrix}\right\rangle dt\to\min, \tag{1}$$

$$\frac{d(Ex(t))}{dt}=A(t)x(t)+B(t)u(t)+f(t), \tag{2}$$

$$Ex(0)=x^0. \tag{3}$$

Here $\langle\cdot,\cdot\rangle$ denotes the inner product in appropriate spaces, $t\in[0,T]$, $0=t_0<t_1<\ldots<t_N<T$, $t_{N+1}=T$, t_j, $j=1,\ldots,N+1$, are fixed, $x(t)\in X$, $u(t)\in U$; X, U are real finite-dimensional Euclidean spaces; E, $A(t)$, G_j, $W(t)\in L(X)$; $B(t)$, $S(t)\in L(U,X)$; $R(t)\in L(U)$; the operators G_j, $j=1,\ldots,N+1$, $W(t)$, and $R(t)$ are

symmetric, $R(t)>0$, $\begin{pmatrix} W(t) & S(t) \\ S(t)^* & R(t) \end{pmatrix} \geq 0$, $G_j \geq 0$; the elements $x^0 \in ImE$ and $\xi_j \in X$, $j=1,\dots,N+1$, are given, G_j and E are independent of t, but the other operators and the function $f(\cdot)$ with values in X depend continuously on t; the superscript $*$ denotes an adjoint operator. For the definiteness, we will suppose that a small parameter ε is positive and all functions are continuous to the right in the points of the discontinuity. When $t=0$ and $t=T$ we assume the continuity to the right and to the left, respectively. The co-ordinate representation is used nowhere in this paper.

The argument t is further dropped almost everywhere, and the given relations are meant pointwise for all $t \in [0,T]$.

The asymptotic expansion of the solution of the problem (1)-(3) will be constructed using the direct scheme. We will seek a solution of the perturbed problem (1)-(3) in the series form

$$u(t)=\sum_{j\geq 0} \varepsilon^j u_j(t), \quad x(t)=\sum_{j\geq 0} \varepsilon^j x_j(t). \tag{4}$$

We substitute the relations (4) into (1)-(3), expand the right-hand sides of (1) and (2) in series in powers of ε, and then equate the coefficients of like powers of ε in (2) and (3). Then the functional to be minimized may be written in the form

$$J_\varepsilon(u,x)=\sum_{j\geq 0} \varepsilon^j J_j, \tag{5}$$

and relations (2), (3) yield the equations for the terms of the decomposition (4).

We will determine a series of optimal control problems in order to find the coefficients in (4).

Further, P and Q denote the orthogonal projectors of the space X onto $KerE$ and $KerE^*$ respectively.

Assumption 1^0. The operator

$$QA(t)P : KerE \to KerE^* \tag{6}$$

has the inverse operator and $\begin{pmatrix} PW(t)P & PS(t) \\ S(t)^*P & R(t) \end{pmatrix} > 0$ for all $t \in [0,T]$.

III. Formalism of Asymptotic Expansions Construction

When $\varepsilon=0$, we obtain from (1)-(3) the degenerate problem

$$P_0 : J_0 = \frac{1}{2}\langle x_0(T)-\xi_{N+1}, F_{N+1}(x_0(T)-\xi_{N+1})\rangle + \int_0^T \left(\frac{1}{2}\langle x_0, Wx_0\rangle + \langle x_0, Su_0\rangle + \frac{1}{2}\langle Ru_0, u_0\rangle\right)dt \to \min, \tag{7}$$

$$\frac{d(Ex_0)}{dt}=Ax_0+Bu_0+f, \tag{8}$$

$$Ex_0(0)=x^0. \tag{9}$$

Here and further, we denote the operator E^*G_jE by F_j, $j=1,\dots,N+1$.

We can obtain the degenerate problem if we substitute the relations (4) into (1)-(3) and equate the coefficients of ε^0.

The solution of the linear-quadratic optimal control problem P_0 can be found from (8), (9) and from the following relations

$$E^*\frac{d\psi_0}{dt}=Wx_0-A^*\psi_0+Su_0, \tag{10}$$

$$E^*\psi_0(T)=-F_{N+1}(x_0(T)-\xi_{N+1}), \tag{11}$$

$$0=-S^*x_0+B^*\psi_0-Ru_0 \tag{12}$$

(see e.g. [6]).

Substituting the relations (4) into (2) and (3) then equating coefficients with ε^j, we obtain the initial values problems for x_j

$$\frac{d(Ex_j)}{dt}=Ax_j+Bu_j, \quad Ex_j(0)=0, \quad j\geq 1. \tag{13}$$

We write down the coefficient J_1 from (5).

$$J_1=\langle x_0(T)-\xi_{N+1}, F_{N+1}x_1(T)\rangle + \frac{1}{2}\sum_{j=1}^{N}\langle x_0(t_j)-\xi_j, F_j(x_0(t_j)-\xi_j)\rangle + \int_0^T\left(\langle x_1, Wx_0+Su_0\rangle + \langle Ru_0+S^*x_0, u_1\rangle\right)dt.$$

We transform the last expression, substituting the relation for Wx_0+Su_0, obtained from (10), and the relation for $Ru_0+S^*x_0$, obtained from (12). Using (13), (11), we get

$$J_1=\langle F_{N+1}(x_0(T)-\xi_{N+1}), x_1(T)\rangle + \frac{1}{2}\sum_{j=1}^{N}\langle x_0(t_j)-\xi_j, F_j(x_0(t_j)-\xi_j)\rangle + \int_0^T\left(\left\langle x_1, E^*\frac{d\psi_0}{dt}+A^*\psi_0\right\rangle + \langle B^*\psi_0, u_1\rangle\right)dt =$$

$$= -\langle E^*\psi_0(T), x_1(T)\rangle + \frac{1}{2}\sum_{j=1}^{N}\langle x_0(t_j)-\xi_j, F_j(x_0(t_j)-\xi_j)\rangle + \int_0^T\left(\left\langle Ex_1, \frac{d\psi_0}{dt}\right\rangle + \left\langle \frac{d(Ex_1)}{dt}, \psi_0\right\rangle\right)dt =$$

$$= -\langle E^*\psi_0(T), x_1(T)\rangle + \frac{1}{2}\sum_{j=1}^{N}\langle x_0(t_j)-\xi_j, F_j(x_0(t_j)-\xi_j)\rangle + \langle Ex_1, \psi_0\rangle\Big|_0^T = \frac{1}{2}\sum_{j=1}^{N}\langle x_0(t_j)-\xi_j, F_j(x_0(t_j)-\xi_j)\rangle.$$

So, the coefficient J_1 is known after the problem P_0 has been solved.

The coefficient J_2 from (5) has the following form:

$$J_2 = \tfrac{1}{2}\langle x_1(T), F_{N+1}x_1(T)\rangle + \langle x_2(T), F_{N+1}(x_0(T)-\xi_{N+1})\rangle + \\ + \sum_{j=1}^{N}\langle x_1(t_j), F_j(x_0(t_j)-\xi_j)\rangle + \\ + \int_0^T(\tfrac{1}{2}\langle x_1, Wx_1\rangle + \langle x_1, Su_1\rangle + \tfrac{1}{2}\langle Ru_1, u_1\rangle + \\ + \langle x_2, Wx_0+Su_0\rangle + \langle Ru_0 + S^* x_0, u_2\rangle)dt. \tag{14}$$

We realize the transformations in the relation (14) for J_2, as for J_1. Taking into consideration (10)-(12) and (13) for $j=2$, we obtain the different form for J_2, which will be denoted further by $\widetilde{J}_1(u_1, x_1)$.

To determine the pair of functions (u_1, x_1), we consider the linear-quadratic control problem with intermediate points of the form

$$P_1 : \widetilde{J}_1(u_1, x_1) = \tfrac{1}{2}\langle x_1(T), F_{N+1}x_1(T)\rangle + \\ + \sum_{j=1}^{N}\langle x_1(t_j), F_j(x_0(t_j)-\xi_j)\rangle + \\ + \int_0^T(\tfrac{1}{2}\langle x_1, Wx_1\rangle + \langle x_1, Su_1\rangle + \tfrac{1}{2}\langle Ru_1, u_1\rangle)dt \to \min,$$

$$\frac{d(Ex_1)}{dt} = Ax_1 + Bu_1, \quad Ex_1(0) = 0.$$

Further, in order to determine the pair of the functions (u_k, x_k) for $k \ge 2$, we define the following problems

$$P_k : \widetilde{J}_k(u_k, x_k) = \tfrac{1}{2}\langle x_k(T), F_{N+1}x_k(T)\rangle + \\ + \sum_{j=1}^{N}\langle x_k(t_j), F_j x_{k-1}(t_j)\rangle + \\ + \int_0^T(\tfrac{1}{2}\langle x_k, Wx_k\rangle + \langle x_k, Su_k\rangle + \tfrac{1}{2}\langle Ru_k, u_k\rangle)dt \to \min,$$

$$\frac{d(Ex_k)}{dt} = Ax_k + Bu_k, \quad Ex_k(0) = 0. \tag{15}$$

The solution of the problem P_k can be found from (15) and from the following relations

$$E^* \frac{d\psi_k}{dt} = Wx_k - A^*\psi_k + Su_k, \quad t \ne t_j, \tag{16}$$

$$E^*(\psi_k(t_j-0) - \psi_k(t_j+0)) = -F_j(x_{k-1}(t_j) - \zeta_{j,k-1}), \\ j = 1, \dots, N, \tag{17}$$

$$E^*\psi_k(T) = -F_{N+1}x_k(T), \tag{18}$$

$$0 = -S^* x_k + B^*\psi_k - Ru_k, \tag{19}$$

where $\zeta_{j,k-1} = \begin{cases} \xi_j, & k=1, \\ 0, & k>1. \end{cases}$

Theorem 1. The coefficient J_{2k-1} from (5) is known after problem P_{k-1} has been solved. The performance index $\widetilde{J}_k(u_k, x_k)$ in the problem P_k $(k \ge 1)$ is the transformed expression for the coefficient J_{2k}.

Proof. The theorem statement has already been proved for $k=1$. Let us suppose that it is true for $1 \le k < n$.

The coefficient J_{2n} from (5) has the form

$$J_{2n} = \sum_{j=1}^{N}(\sum_{m=1}^{n-1}\langle x_m(t_j), F_j x_{2n-1-m}(t_j)\rangle + \\ + \langle x_{2n-1}(t_j), F_j(x_0(t_j)-\xi_j)\rangle) + \\ + \langle x_{2n}(T), F_{N+1}(x_0(T)-\xi_{N+1})\rangle + \\ + \sum_{m=1}^{n-1}\langle x_m(T), F_{N+1}x_{2n-m}(T)\rangle + \tfrac{1}{2}\langle x_n(T), F_{N+1}x_n(T)\rangle + \\ + \int_0^T\left(\frac{1}{2}\langle x_n, Wx_n\rangle + \langle x_n, Su_n\rangle + \frac{1}{2}\langle Ru_n, u_n\rangle + \right. \\ \left. + \sum_{m=0}^{n-1}\left(\langle x_{2n-m}, Wx_m + Su_m\rangle + \langle u_{2n-m}, S^* x_m + Ru_m\rangle\right)\right)dt.$$

Using (10), (16), (12), (19), (13), we get the following equalities

$$\int_0^T \sum_{m=0}^{n-1}\left(\langle x_{2n-m}, Wx_m + Su_m\rangle + \langle u_{2n-m}, S^* x_m + Ru_m\rangle\right)dt = \\ = \sum_{j=1}^{N+1}\sum_{m=0}^{n-1}\int_{t_{j-1}}^{t_j}(\left\langle x_{2n-m}, E^*\frac{d\psi_m}{dt} + A^*\psi_m\right\rangle + \\ + \langle u_{2n-m}, B^*\psi_m\rangle)dt = \sum_{j=1}^{N+1}\sum_{m=0}^{n-1}\left(\langle x_{2n-m}, E^*\psi_m\rangle\Big|_{t_{j-1}}^{t_j} + \right. \\ \left. + \int_{t_{j-1}}^{t_j}\left\langle -\frac{d(Ex_{2n-m})}{dt} + Ax_{2n-m} + Bu_{2n-m}, \psi_m\right\rangle dt\right) = \\ = \sum_{j=1}^{N+1}\sum_{m=0}^{n-1}\langle x_{2n-m}(t), E^*\psi_m(t)\rangle\Big|_{t_{j-1}}^{t_j} = \\ = \sum_{j=1}^{N}\sum_{m=0}^{n-1}\langle x_{2n-m}(t_j), E^*(\psi_m(t_j-0) - \psi_m(t_j+0))\rangle + \\ + \sum_{m=0}^{n-1}\langle x_{2n-m}(T), E^*\psi_m(T)\rangle.$$

From here, taking into account (17), (11), (18), we reduce the expression for J_{2n} to the following form

$$\tfrac{1}{2}\left\langle x_n(T),\, F_{N+1}x_n(T)\right\rangle + \sum_{j=1}^{N}\left\langle x_n(t_j),\, F_j x_{n-1}(t_j)\right\rangle +$$

$$+\int_0^T (\tfrac{1}{2}\left\langle x_n, Wx_n\right\rangle + \left\langle x_n, Su_n\right\rangle + \tfrac{1}{2}\left\langle Ru_n, u_n\right\rangle)dt,$$

which is coincident with $\widetilde{J}_n(u_n, x_n)$.

We write down the coefficient J_{2n-1} from (5).

$$J_{2n-1} = \langle x_{2n-1}(T), F_{N+1}(x_0(T)-\xi_{N+1})\rangle +$$

$$+\sum_{m=1}^{n-1}\langle x_m(T), F_{N+1}x_{2n-1-m}(T)\rangle +$$

$$+\sum_{j=1}^{N}\sum_{m=1}^{n-2}\langle x_m(t_j), F_j x_{2n-2-m}(t_j)\rangle + \sum_{j=1}^{N}\langle x_{2n-2}(t_j), F_j(x_0(t_j)-\xi_j)\rangle +$$

$$+\tfrac{1}{2}\sum_{j=1}^{N}\langle x_{n-1}(t_j), F_j x_{n-1}(t_j)\rangle +$$

$$+\int_0^T\sum_{m=0}^{n-1}\Big(\langle x_{2n-1-m}, Wx_m + Su_m\rangle + \langle u_{2n-1-m}, S^* x_m + Ru_m\rangle\Big)dt.$$

Realizing the transformations which are similar to the transformations for J_{2n}, we get

$$J_{2n-1} = \tfrac{1}{2}\sum_{j=1}^{N}\left\langle x_{n-1}(t_j),\, F_j x_{n-1}(t_j)\right\rangle.$$

So, the coefficient J_{2n-1} is known after the problem P_{n-1} has been solved.

The theorem 1 is proved.

It should be noted that Theorem 1 is valid without the assumption on the invertibility of the operator (6).

IV. Estimates of Approximate Solution

Let us assume that the solutions (u_j, x_j) have been found for the problems P_j, $j = 0,\ldots,n$. We shall estimate the approximate solution of the perturbed problem P_ε:

$$\widetilde{u}_n(t) = \sum_{j=0}^{n}\varepsilon^j u_j(t), \quad \widetilde{x}_n(t) = \sum_{j=0}^{n}\varepsilon^j x_j(t).$$

It is not difficult to see that the function $\widetilde{x}_n(t)$ is a solution of the problem (2), (3) when $u(t) = \widetilde{u}_n(t)$.

We will denote the solution of the problem P_ε by (u_*, x_*). It satisfies the following system

$$\frac{d(Ex_*(t))}{dt} = A(t)x_*(t) + B(t)u_*(t) + f(t), \tag{20}$$

$$Ex_*(0) = x^0, \tag{21}$$

$$E^*\frac{d\psi(t)}{dt} = W(t)x_*(t) - A(t)^*\psi(t) + S(t)u_*(t), \quad t \neq t_j, \tag{22}$$

$$E^*(\psi(t_j - 0) - \psi(t_j + 0)) = -\varepsilon F_j(x_*(t_j) - \xi_j), \tag{23}$$

$j = 1,\ldots,N$,

$$E^*\psi(T) = -F_{N+1}(x_*(T) - \xi_{N+1}), \tag{24}$$

$$0 = -S(t)^* x_*(t) + B(t)^*\psi(t) - R(t)u_*(t). \tag{25}$$

It should be noted that it is simpler to find the solutions of the problems P_j than the problem P_ε solution. In particular, if $W = 0$, $S = 0$, and $F_{N+1} = 0$, we have the problem for finding $\psi_j(t)$ and $u_j(t)$ which does not depend on $x_j(t)$.

Further, we introduce the notations

$$\begin{aligned}&\Delta x(t) = x_*(t) - \widetilde{x}_n(t), \quad \Delta u(t) = u_*(t) - \widetilde{u}_n(t),\\ &\Delta\psi(t) = \psi(t) - \widetilde{\psi}_n(t),\end{aligned} \tag{26}$$

where $\widetilde{\psi}_n(t) = \sum_{j=0}^{n}\varepsilon^j\psi_j(t)$, $\psi_j(t)$ is the conjugate variable for the problem P_j.

In view of (20) – (25), (8), (12), (15) – (19), we have for Δx, Δu, $\Delta\psi$ the system

$$\frac{d(E\Delta x)}{dt} = A\Delta x + B\Delta u, \tag{27}$$

$$E\Delta x(0) = 0, \tag{28}$$

$$E^*\frac{d\Delta\psi}{dt} = W\Delta x - A^*\Delta\psi + S\Delta u, \quad t \neq t_j, \tag{29}$$

$$E^*(\Delta\psi(t_j - 0) - \Delta\psi(t_j + 0)) = -\varepsilon F_j\Delta x(t_j) - \varepsilon^{n+1}F_j x_n(t_j), \tag{30}$$

$j = 1,\ldots,N$,

$$E^*\Delta\psi(T) = -F_{N+1}\Delta x(T), \tag{31}$$

$$0 = -S^*\Delta x + B^*\Delta\psi - R\Delta u. \tag{32}$$

If Assumption 1^0 holds then the operator, having the matrix representation $\begin{pmatrix} QAP & 0 & QB \\ PWP & -PA^*Q & PS \\ -S^*P & B^*Q & -R \end{pmatrix}$, is invertible (see, for example, [7]). (The invertibility of the last operator means that the system following from the control optimality condition, has index one. The general conditions ensuring index one for this system, are given in [6].) In view of [7] (Theorem 5.1), the system (27), (29), (32) provides an explicit non-negative differential Hamiltonian system for the pair $(Ex, (I-Q)\psi)$ which is of the form

$$\frac{d}{dt}\begin{pmatrix} E\Delta x \\ (I-Q)\Delta\psi \end{pmatrix} = \begin{pmatrix} E_1 & E_2 \\ E_3 & -E_1^* \end{pmatrix}\begin{pmatrix} E\Delta x \\ (I-Q)\Delta\psi \end{pmatrix}, \quad t \neq t_j, \tag{33}$$

where $E_2 = E_2^* \geq 0$, $E_3 = E_3^* \geq 0$.

The relations (30), (31) are equivalent to the following equalities:

$$\begin{aligned}&(I-Q)\Delta\psi(t_j - 0) - (I-Q)\Delta\psi(t_j + 0) =\\ &= -\varepsilon(I-Q)G_j E\Delta x(t_j) - \varepsilon^{n+1}(I-Q)G_j Ex_n(t_j),\end{aligned} \tag{34}$$

$$(I-Q)\Delta\psi(T) = -(I-Q)G_{N+1}E\Delta x(T). \tag{35}$$

Analogously to the theorem 2.3 proof from [5], we can establish the unique solvability of the boundary value problem (33), (28), (34), (35) for all $\varepsilon \ge 0$. From here and from (32), it follows that the estimates

$$\| \Delta x(t) \|, \| \Delta \psi(t) \|, \| \Delta \mathrm{u}(t) \| \le c\varepsilon^{n+1}, \quad (36)$$

where a constant c does not depend on t and ε, take place for all $t \in [0, T]$ and sufficiently small $\varepsilon > 0$.

We will transform the difference
$\Delta J = J_\varepsilon(\widetilde{u}_n, \widetilde{x}_n) - J_\varepsilon(u_*, x_*)$.

Using (1), (26), (36), we have

$$\Delta J = O(\varepsilon^{2(n+1)}) + \left\langle \widetilde{x}_n(T) - x_*(T), F_{N+1}(x_*(T) - \xi_{N+1}) \right\rangle +$$
$$+ \varepsilon \sum_{j=1}^{N} \left\langle \widetilde{x}_n(t_j) - x_*(t_j), F_j(x_*(t_j) - \xi_j) \right\rangle +$$
$$+ \int_0^T (< \widetilde{x}_n - x_*, W x_* + S u_* > + < \widetilde{u}_n - u_*, S^* x_* + R u_* >) dt.$$

Further, in view of (22), (25), (2), (24), (23), (3), we obtain

$$\Delta J = O(\varepsilon^{2(n+1)}) + \left\langle \widetilde{x}_n(T) - x_*(T), F_{N+1}(x_*(T) - \xi_{N+1}) \right\rangle +$$
$$+ \varepsilon \sum_{j=1}^{N} \left\langle \widetilde{x}_n(t_j) - x_*(t_j), F_j(x_*(t_j) - \xi_j) \right\rangle +$$
$$+ \sum_{j=1}^{N+1} \int_{t_{j-1}}^{t_j} \left(\left\langle \widetilde{x}_n - x_*, E^* \frac{d\psi}{dt} + A^* \psi \right\rangle + \left\langle \widetilde{u}_n - u_*, B^* \psi \right\rangle \right) dt =$$
$$= O(\varepsilon^{2(n+1)}) + \left\langle \widetilde{x}_n(T) - x_*(T), F_{N+1}(x_*(T) - \xi_{N+1}) \right\rangle +$$
$$+ \varepsilon \sum_{j=1}^{N} \left\langle \widetilde{x}_n(t_j) - x_*(t_j), F_j(x_*(t_j) - \xi_j) \right\rangle +$$
$$+ \sum_{j=1}^{N+1} (< E(\widetilde{x}_n(t) - x_*(t)), \psi(t) > \Big|_{t_{j-1}}^{t_j} +$$
$$+ \int_{t_{j-1}}^{t_j} < -\frac{d}{dt}(E(\widetilde{x}_n - x_*)) + A(\widetilde{x}_n - x_*) + B(\widetilde{u}_n - u_*), \psi > dt) =$$
$$= O(\varepsilon^{2(n+1)}) + \left\langle \widetilde{x}_n(T) - x_*(T), F_{N+1}(x_*(T) - \xi_{N+1}) \right\rangle +$$
$$+ \varepsilon \sum_{j=1}^{N} \left\langle \widetilde{x}_n(t_j) - x_*(t_j), F_j(x_*(t_j) - \xi_j) \right\rangle +$$
$$+ \sum_{j=1}^{N+1} < E(\widetilde{x}_n(t_j) - x_*(t_j)), \psi(t_j - 0) - \psi(t_j + 0) > =$$
$$= O(\varepsilon^{2(n+1)}) + \left\langle \widetilde{x}_n(T) - x_*(T), F_{N+1}(x_*(T) - \xi_{N+1}) \right\rangle +$$
$$+ \varepsilon \sum_{j=1}^{N} \left\langle \widetilde{x}_n(t_j) - x_*(t_j), F_j(x_*(t_j) - \xi_j) \right\rangle +$$
$$+ \sum_{j=1}^{N} < (\widetilde{x}_n(t_j) - x_*(t_j)), E^*(\psi(t_j - 0) - \psi(t_j + 0)) > +$$
$$+ < \widetilde{x}_n(T) - x_*(T), E^* \psi(T) > = O(\varepsilon^{2(n+1)}).$$

So, we have proved

Theorem 2. The following estimates

$$\| x_*(t) - \widetilde{x}_n(t) \|, \| u_*(t) - \widetilde{u}_n(t) \| \le c\varepsilon^{n+1},$$
$$J_\varepsilon(\widetilde{u}_n, \widetilde{x}_n) - J_\varepsilon(u_*, x_*) \le c\varepsilon^{2(n+1)},$$

where a constant c does not depend on t and ε, are true for all $t \in [0, T]$ and sufficiently small $\varepsilon > 0$.

It follows from this theorem that $\{\widetilde{u}_n(t)\}$ is a minimizing sequence for the considered functional (1).

Further, we will prove that the sequence $\{J_\varepsilon(\widetilde{u}_i, \widetilde{x}_i)\}$ is decreasing for fixed ε.

Theorem 3. For sufficiently small $\varepsilon > 0$, we have

$$J_\varepsilon(\widetilde{u}_i, \widetilde{x}_i) \le J_\varepsilon(\widetilde{u}_{i-1}, \widetilde{x}_{i-1}), \quad i = 1, \ldots, n. \quad (37)$$

If $u_i \ne 0$ then (37) is a strict inequality.

Proof. If $u_i = 0$, (37) is obvious. Let us consider the case when $u_i \ne 0$. Expanding $J_\varepsilon(\widetilde{u}_s, \widetilde{x}_s)\,(s = i-1, i)$ in series (5) and using Theorem 1, we obtain that the first 2i terms in expansions $J_\varepsilon(\widetilde{u}_{i-1}, \widetilde{x}_{i-1})$ and $J_\varepsilon(\widetilde{u}_i, \widetilde{x}_i)$ are identical. The pair (u_i, x_i) is a solution of the linear-quadratic problem P_i, the performance index of which is a transformed expression for J_{2i}. Hence, the value of this coefficient J_{2i}, corresponding to the decomposition of $J_\varepsilon(\widetilde{u}_i, \widetilde{x}_i)$, is strictly less than the respective value for $J_\varepsilon(\widetilde{u}_{i-1}, \widetilde{x}_{i-1})$. Therefore, (37) is true for sufficiently small $\varepsilon > 0$.

V. Illustrative Example

As the obtained results are new for problems with a state equation resolved with respect to the derivative, we consider the problem P_ε of minimizing the functional

$$J_\varepsilon(u, x) = \frac{1}{2}(x_1(3) - 30)^2 + \frac{\varepsilon}{2}((x_1(1) + 1000)^2 +$$
$$+ (x_1(2) - 1000)^2 + (x_2(1) - 200)^2 + (x_2(2) + 400)^2) +$$
$$+ \frac{1}{2}\int_0^3 u^2 dt$$

on trajectories of the system

$$\begin{aligned} \dot{x}_1 &= x_2, \quad x_1(0) = 60, \\ \dot{x}_2 &= u, \quad x_2(0) = 10, \end{aligned} \quad (38)$$

when $\varepsilon = 0.1$.

Taking into account the method developed in this paper we find the solutions of the problems P_0 and P_1.

Then we obtain the zero and first order approximations $(\mathrm{u}_0, \mathrm{x}_0)$ and $(\widetilde{u}_1, \widetilde{x}_1)$ for the solution of the problem P_ε. The exact and approximate solutions are given in Fig.1 for x_1, in Fig.2 for x_2 and in Fig. 3 for u.

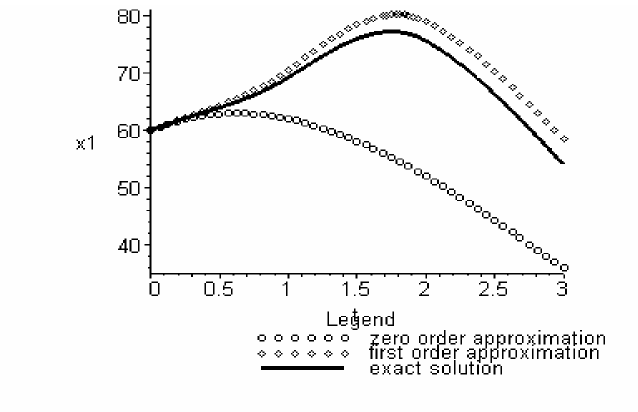

Fig.1. The graphs of exact and approximate solutions for $x_1 = x_1(t)$.

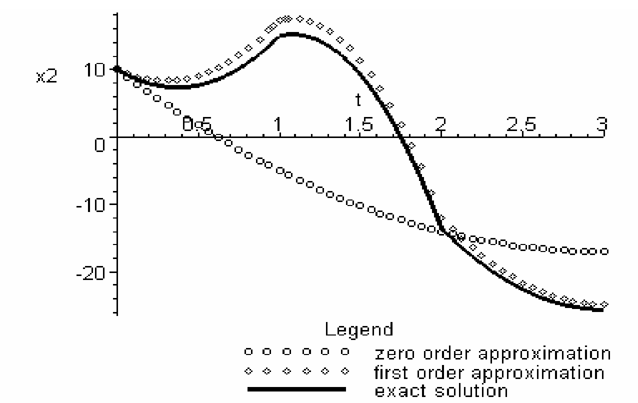

Fig.2. The graphs of exact and approximate solutions for $x_2 = x_2(t)$.

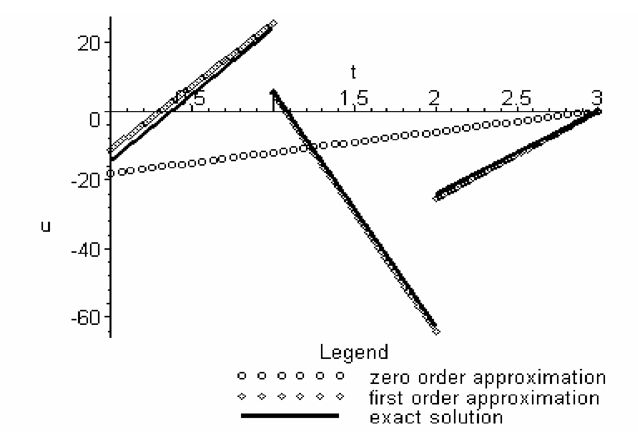

Fig.3. The graphs of exact and approximate solutions for $u = u(t)$.

Using the solutions of the system (38) when $u = u_0$ and $u = \tilde{u}_1$, we evaluate $J_\varepsilon(u_0, x_0)$ and $J_\varepsilon(\tilde{u}_1, \tilde{x}_1)$ accordingly. The obtained results and the optimal value of the performance index are given in Table I.

TABLE I
VALUES OF PERFORMANCE INDEX

PERFORMANCE INDEX	VALUE
$J_\varepsilon(u_0, x_0)$	111058.45
$J_\varepsilon(\tilde{u}_1, \tilde{x}_1)$	110141.9986
$J_\varepsilon(u_*, x_*)$	110127.4015

VI. CONCLUSION

So, we have constructed the asymptotic solution for the problem (1)-(3) using the direct scheme method.

The estimates have been obtained for the proximity of the asymptotic approximate solutions to the exact one in terms of the control, the trajectory and the functional.

Using the direct scheme method we have proved the nice property, that the values of the minimized functional do not increase when higher-order approximations for the optimal control are utilized. The direct scheme method advantage is also the possibility to apply standard programs for solving optimal control problems in order to find the terms of asymptotic expansions.

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