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Global Robust Synchronization of the Duffing System and Van der Pol Oscillator

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Abstract— **In this paper, we study the global robust synchronization problem of the controlled Duffing system and the Van der Pol oscillator. By employing the internal model approach, we first show that the problem can be converted into a global robust stabilization problem of a time-varying nonlinear system in lower triangular form. Then we show that the global stabilization problem of the lower triangular system is solvable, thus leading to the solution of the global robust synchronization problem.**

I. INTRODUCTION

In this paper, we consider the problem of global robust synchronization of the controlled Duffing system and Van der Pol oscillator with the controlled Duffing system as the slave system and Van der Pol system as the master system. A local version of the same problem was studied in [8]. It is known that the robust synchronization of the controlled Duffing system and Van der Pol oscillator can be formulated as a robust output regulation problem for a particular type of output feedback systems with nonlinear exosystem [8]. The robust output regulation problem is typically handled by the internal model approach. The internal model approach consists of two steps. In the first step, an appropriate dynamic compensator called internal model is designed. Attachment of the internal model to the given plant leads to an augmented system. The internal model has the property that the stabilization solution of the augmented system will lead to the output regulation solution of the given plant and the exosystem [2], [3]. The successful accomplishment of the first step relies on the satisfaction of two key conditions. The first one is the availability of the solution of a set of nonlinear partial differential equations called regulator equations and the second one is that the solution of the regulator equations has to satisfy what is called immersion condition. When the exosystem is nonlinear, there is no systematic way for verifying these two conditions. Nevertheless, it was shown in [8] that this step can be accomplished for the current problem and the resulting augmented system is a time-varying nonlinear system. In general, there is no clue to globally stabilize a time-varying nonlinear system. Thus, only a local stabilizing controller was given in [8] which can only guarantee the output synchronization of the controlled Duffing system and Van der Pol oscillator for sufficiently small initial conditions of the closed-loop system and sufficiently small uncertain parameter.

This paper aims to give a global solution of the robust output synchronization of the controlled Duffing system and Van der Pol oscillator. To accomplish this goal, we need to first convert the augmented system into a lower triangular form by a dynamic extension and time-varying coordinate transformation. The lower triangular system is also time-varying and we have managed to globally stabilize this system. As a result, we are able to construct an output feedback controller to solve the problem under consideration.

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II. PRELIMINARIES

Consider an uncertain controlled Duffing system described as follows

$$\ddot{y} + y^3 - y + (\sigma + w)\dot{y} = \delta cos\,\omega t + u \tag{1}$$

where $\sigma + w$ is the damping coefficient with σ the nominal value and w an unknown parameter, and $\delta cos\,\omega t$ a harmonic excitation with $\delta, \omega > 0$. Also consider the following Van der Pol oscillator

$$\begin{aligned} \dot{v}_1 &= v_2 \\ \dot{v}_2 &= -av_1 + b(1 - v_1^2)v_2 \end{aligned} \tag{2}$$

where $a > 0$ and $b > 0$.

By robust synchronization problem of systems (1) and (2), we mean the design of a feedback control law such that the output y of the Duffing system asymptotically approaches the output v_1 of the Van der Pol oscillator while maintaining the boundedness of the solution of the closed-loop system composed of Duffing system, Van der Pol oscillator and the controller for any sufficiently small initial condition in the presence of any sufficiently small parameter variation of the Duffing system. If both the initial condition and unknown parameter can be arbitrarily large, then the problem is called global robust synchronization.

It is shown in [8] that this problem can be formulated as a robust output regulation problem for a nonlinear system subject to a nonlinear exosystem. In fact, the Duffing system

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can be put in the following form

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -(\sigma+w)x_2 + x_1 - x_1^3 + v_3 + u \\ \dot{v}_3 &= -\omega v_4 \\ \dot{v}_4 &= \omega v_3 \\ y &= x_1 \end{aligned} \tag{3}$$

where $x_1 = y, x_2 = \dot{y}$ with $v_3(0) = \delta$ and $v_4(0) = 0$.

Let $x = [x_1, x_2]^T$, $v = [v_1, v_2, v_3, v_4]^T$, and $e = x_1 - v_1$. Then (2)-(3) can be put in the following form

$$\begin{aligned} \dot{x} &= f(x, u, v, w) \\ \dot{v} &= a(v) \\ e &= h(x, u, v, w) \end{aligned} \tag{4}$$

where

$$\begin{aligned} f(x,u,v,w) &= \begin{bmatrix} x_2 \\ -(\sigma+w)x_2 + x_1 - x_1^3 + v_3 + u \end{bmatrix}, \\ a(v) &= \begin{bmatrix} v_2 \\ -av_1 + b(1-v_1^2)v_2 \\ -\omega v_4 \\ \omega v_3 \end{bmatrix}, \end{aligned}$$

and $h(x, u, v, w) = x_1 - v_1$.

Thus, the robust synchronization problem described above can be viewed as a robust output regulation problem studied in [2] that aims to regulate the error output e of the composite system (4) to the origin asymptotically.

It is also shown in [8] that the regulator equations of the composite system (4) admit a globally defined solution as follows

$$\begin{aligned} \mathbf{x}_1(v,w) &= v_2 \\ \mathbf{x}_2(v,w) &= v_1 \\ \mathbf{u}(v,w) &= \mathbf{u_c}(v) + \mathbf{u}_1(v,w) \end{aligned} \tag{5}$$

where $\mathbf{u_c}(v) = -(a+1)v_1 + b(1-v_1^2)v_2 + v_1^3 - v_3$, and $\mathbf{u}_1(v,w) = (\sigma+w)v_2$. It can be seen that the function $\mathbf{u}(v,w)$ consists of two parts. While the second part $\mathbf{u_1}(v,w)$ depends on the unknown parameter w, the first part $\mathbf{u_c}(v)$ does not. Performing an input transformation $u = u_1 + \mathbf{u_c}(v)$ on the Duffing system gives the following system:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -(\sigma+w)x_2 + x_1 - x_1^3 + v_3 + \mathbf{u_c}(v) + u_1 \\ \dot{v} &= a(v) \\ e &= x_1 - v_1. \end{aligned} \tag{6}$$

Clearly, if we can find a controller u_1 to solve the output regulation problem of this system, then the control law $u = \mathbf{u_c}(v) + u_1$ can solve the output synchronization problem of the Duffing system and Van der Pol oscillator.

It is known that the first step for solving the robust output regulation problem for system (6) is to ascertain whether or not the system has an appropriate internal model. It is shown in [8] that system (6) admits an internal model which can be constructed as follows.

Decompose the function $a(v)$ as follows

$$a(v) = A_1 v + A_2 v a_2(v) \tag{7}$$

with

$$A_1 v = \begin{bmatrix} v_2 \\ -av_1 + bv_2 \\ -\omega v_4 \\ \omega v_3 \end{bmatrix}, A_2 v = \begin{bmatrix} 0 \\ -bv_2 \\ 0 \\ 0 \end{bmatrix}, a_2(v) = v_1^2.$$

Denote

$$\tau(v,w) = \mathrm{col}(\mathbf{u}_1(v,w), L_{A_1 v}\mathbf{u}_1(v,w)) \tag{8}$$

where the notation $L_{A_1 v}\mathbf{u}_1(v,w)$ means the Lie derivative of the function $\mathbf{u}_1(v,w)$ with w held as a constant along the vector field $A_1 v$.

Let

$$\Phi = \begin{bmatrix} 0 & 1 \\ -a & b \end{bmatrix}, \quad \Phi_1 = \begin{bmatrix} -b & 0 \\ -b^2 & 0 \end{bmatrix}, \quad \Psi = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

Let $\theta(v,w) = T\tau(v,w)$ with $T \in R^{2\times 2}$ any nonsingular matrix, $\phi(v) = \Phi + \varphi(v)$ with $\varphi(v) = \Phi_1 a_2(v)$, and

$$\begin{aligned} \alpha(\theta(v,w),v) &= T\phi(v)T^{-1}\theta(v,w), \\ \beta(\theta(v,w),v) &= \Psi T^{-1}\theta(v,w). \end{aligned} \tag{9}$$

Then, it can be verified that, for all v, and all w,

$$\begin{aligned} \frac{\partial\theta(v,w)}{\partial v}a(v) &= \alpha\left(\theta\left(v,w\right),v\right), \\ \mathbf{u}_1(v,w) &= \beta\left(\theta\left(v,w\right),v\right). \end{aligned} \tag{10}$$

Next, let $M \in R^{2\times 2}$ be any Hurwitz matrix and $N \in R^2$ be any column matrix such that (M,N) is controllable. Then there is a unique, nonsingular matrix T satisfying the Sylvester equation

$$T\Phi - MT = N\Psi \tag{11}$$

since the spectra of the matrices Φ and M are disjoint, and the pair (Ψ, Φ) is observable.

An internal model for the composite system (6) is now defined as the following dynamic compensator

$$\dot{\eta} = (M + T\varphi(v)T^{-1})\eta + Nu_1. \tag{12}$$

Applying the following coordinate and input transformation

$$\begin{aligned} \bar{x}_1 &= x_1 - v_1 \\ \bar{x}_2 &= x_2 - v_2 \\ \bar{\eta} &= \eta - \theta(v,w) \\ \bar{u}_1 &= u_1 - \Psi T^{-1}\eta \end{aligned} \tag{13}$$

to the system composed of (6) and the internal model (12) gives a new system as follows:

$$\begin{aligned} \dot{\bar{x}}_1 &= \bar{x}_2, \\ \dot{\bar{x}}_2 &= -(\sigma+w)\bar{x}_2 + \bar{x}_1 - (\bar{x}_1 + v_1)^3 + v_1^3 + \bar{u}_1 \\ &\quad + \Psi T^{-1}\bar{\eta}, \\ \dot{\bar{\eta}} &= (M + N\Psi T^{-1} + T\varphi(v)T^{-1})\bar{\eta} + N\bar{u}_1, \\ e &= \bar{x}_1. \end{aligned} \tag{14}$$

System (14) is called an augmented system for (6). It is known that the unforced augmented system has the property that its equilibrium point and the error output is identically zero for all v and all w [2]. Thus it suffices to solve the robust stabilization problem of the system (14) in order to solve the robust output regulation problem of the given plant (4).

III. MAIN RESULT

Since $\varphi(v(t))$ is time-varying, system (14) is a time-varying nonlinear system. The global robust stabilization problem for system (14) is not transparent. In this section, we will show that it is possible to globally solve the global robust stabilization problem for this system. For this purpose, let us perform another coordinate transformation as follows

$$z = \bar{x}_2, \ \tilde{\eta} = \bar{\eta} - Nz. \tag{15}$$

Then we have

$$\begin{aligned} \dot{z} &= [-(\sigma+w) + \Psi T^{-1}N]z + \Psi T^{-1}\tilde{\eta} + e \\ &\quad -(e+v_1)^3 + v_1^3 + \bar{u}_1 \end{aligned}$$

and

$$\begin{aligned} \dot{\tilde{\eta}} &= [M + T\varphi(v)T^{-1} + (\sigma+w)I]Nz \\ &\quad +(M + T\varphi(v)T^{-1})\tilde{\eta} - N[e - (e+v_1)^3 + v_1^3]. \end{aligned}$$

Letting $\zeta = \begin{bmatrix} z & \tilde{\eta} \end{bmatrix}^T$, we obtain, in the new coordinate ζ, e, the following system

$$\begin{aligned} \dot{\zeta} &= F_a(v,w)\zeta + \widetilde{G}_e(e,v,w) + g_a(w)\bar{u}_1 \\ \dot{e} &= H_a(w)\zeta, \end{aligned} \tag{16}$$

where
$F_a(v,w) = \begin{bmatrix} -(\sigma+w) + \Psi T^{-1}N & \Psi T^{-1} \\ [M + T\varphi(v)T^{-1} + (\sigma+w)I]N & M + T\varphi(v)T^{-1} \end{bmatrix}$,
$\widetilde{G}_e(e,v,w) = \begin{bmatrix} 1 \\ -N \end{bmatrix}(e - (e+v_1)^3 + v_1^3)$,
$g_a(w) = \begin{bmatrix} 1 & 0_{1\times 2} \end{bmatrix}^T$ and $H_a(w) = \begin{bmatrix} 1 & 0_{1\times 2} \end{bmatrix}$.

Remark 1: If $F_a(v,w)$ had not depended on the time function $v(t)$ and were a Hurwitz matrix for all w, then system (16) were in the standard output feedback form [7]. The robust stabilization problem of such systems has been well studied in [7]. A key technique in handling such systems is to convert them into lower triangular form through a dynamic extension and a time invariant coordinate transformation. For our current case, it is still possible to convert system (16) into the lower triangular form through a dynamic extension and a time varying coordinate transformation.

In fact, define the dynamical extension

$$\dot{\xi}_1 = -\lambda_1\xi_1 + \bar{u}_1 \tag{17}$$

with λ_1 being a positive number. It can be verified that the following time-varying coordinate transformation $\bar{\zeta} = \zeta - D\xi_1 - h(v,w)e$, where $\bar{\zeta} = \mathrm{col}(\bar{\zeta}_1, \bar{\zeta}_2)$, $\bar{\zeta}_1 \in R, \bar{\zeta}_2 \in R^2$, and

$$\begin{aligned} D &= \begin{bmatrix} 1 \\ 0_{2\times 1} \end{bmatrix}, \\ h(v,w) &= \begin{bmatrix} -(\sigma+w) + \Psi T^{-1}N + \lambda_1 \\ [M + T\varphi(v)T^{-1} + (\sigma+w)I]N \end{bmatrix} \end{aligned}$$

converts (16) and (17) into the following lower triangular form

$$\begin{aligned} \dot{\bar{\zeta}} &= F(v,w)\bar{\zeta} + G(e,v,w) \\ \dot{e} &= H(w)\bar{\zeta} + K(e,v,w) + \xi_1 \\ \dot{\xi}_1 &= -\lambda_1\xi_1 + \bar{u}_1 \end{aligned} \tag{18}$$

where $H(w) = \begin{bmatrix} 1 & 0_{1\times 2} \end{bmatrix}$, $K(e,v,w) = [-(\sigma+w) + \Psi T^{-1}N + \lambda_1]e$, $F(v,w) = \begin{bmatrix} -\lambda_1 & \Psi T^{-1} \\ 0_{2\times 1} & M + T\varphi(v)T^{-1} \end{bmatrix}$,
and $G(e,v,w) = \begin{bmatrix} G_1(e,v,w) \\ G_2(e,v,w) \end{bmatrix} =$

$$\begin{aligned} &\begin{bmatrix} -\lambda_1 & \Psi T^{-1} \\ 0_{2\times 1} & M + T\varphi(v)T^{-1} \end{bmatrix} \\ &\times \begin{bmatrix} -(\sigma+w) + \Psi T^{-1}N + \lambda_1 \\ [M + T\varphi(v)T^{-1} + (\sigma+w)I]N \end{bmatrix} e \\ &- \begin{bmatrix} 0 \\ \frac{\partial T\varphi(v)T^{-1}N}{\partial v}a(v) \end{bmatrix} e + \begin{bmatrix} 1 \\ -N \end{bmatrix}(e - (e+v_1)^3 + v_1^3). \end{aligned}$$

Remark 2: It now suffices to globally stabilize system (18) in order to solve the global robust output regulation problem of the composite system (6). As system (18) is in lower triangular form, we will consider to use the back-stepping method [6]. Since the matrix $F(v,w)$ contains a submatrix $M + T\varphi(v)T^{-1}$ which depends on a time-varying trajectory $v(t)$, we need to first establish a stability result as follows.

Proposition 1: Consider the following system

$$\dot{z} = (M + T\varphi(v(t))T^{-1})z \tag{19}$$

where $v(t)$ is any trajectory generated by the exosystem $\dot{v} = a(v)$ starting from any initial condition $v(0)$. Then there exist matrices M and N (hence T) independent of $v(0)$ such that system (19) is exponentially stable.

Proof: We will first show that, for any $v_1(0)$ and $v_2(0)$, the trajectory of the Van del Pol system is bounded. In fact, let $U(v_1, v_2) = \frac{a}{2}v_1^2 + \frac{1}{2}v_2^2$. Then, $\dot{U} = b(1 - v_1^2)v_2^2$. Since $b > 0$, $\dot{U} \leq 0$ whenever $v_1(t) \geq 1$. Thus, the trajectory of the Van del Pol system is bounded, and so is any trajectory of the exosystem since v_3 and v_4 are sinusoidal functions.

Now let M and N be in control canonical form, i.e.,

$$M = \begin{bmatrix} 0 & 1 \\ -q_1 & -q_2 \end{bmatrix}, \quad N = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{20}$$

where $q_1 > 0, q_2 > 0$.

Solving the Sylvester equation (11) gives a nonsingular matrix $T = \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}$ where

$$\begin{aligned} a' &= \frac{-q_1 - q_2}{q_1 - q_2 - q_1 * q_2 - 1 - q_1 * q_1 - q_2 * q_2}, \\ b' &= \frac{1 + q_2}{q_1 - q_2 - q_1 * q_2 - 1 - q_1 * q_1 - q_2 * q_2}, \\ c' &= \frac{-(1 + q_2)}{q_1 - q_2 - q_1 * q_2 - 1 - q_1 * q_1 - q_2 * q_2}, \\ d' &= \frac{1 - q_1}{q_1 - q_2 - q_1 * q_2 - 1 - q_1 * q_1 - q_2 * q_2}. \end{aligned}$$

Thus,

$$\begin{aligned} M + T\varphi(v)T^{-1} &= M + \begin{bmatrix} -\frac{d'(a'+b')}{a'd'-b'c'}v_1^2 & \frac{b'(a'+b')}{a'd'-b'c'}v_1^2 \\ -\frac{d'(c'+d')}{a'd'-b'c'}v_1^2 & \frac{b'(c'+d')}{a'd'-b'c'}v_1^2 \end{bmatrix} \\ &= \begin{bmatrix} -\frac{d'(a'+b')}{a'd'-b'c'}v_1^2 & 1 + \frac{b'(a'+b')}{a'd'-b'c'}v_1^2 \\ -q_1 - \frac{d'(c'+d')}{a'd'-b'c'}v_1^2 & -q_2 + \frac{b'(c'+d')}{a'd'-b'c'}v_1^2 \end{bmatrix}. \end{aligned}$$

Letting $q_1 = 1$ and $q_2 = 2$ yields

$$M + T\varphi(v)T^{-1} = \begin{bmatrix} 0 & 1 \\ -1 & -c(v) \end{bmatrix}$$

where $c(v) = (2 + v_1^2)$.

Since for any initial condition $v(0)$, $v(t)$ is bounded for all $t \geq 0$, so is $c(v(t))$. Also $c(v(t)) \geq 2$. Thus, A simple exercise (see, for example, Exercise 4.37 of [5]) shows that there exist real numbers $\beta_2 \geq \beta_1 > 0$, $\alpha_2 \geq \alpha_1 > 0$ and matrices $R(t)$ and $P(t)$ satisfying $0 < \beta_1 I \leq R(t) \leq \beta_2 I, \forall t \geq 0$ and $0 < \alpha_1 I \leq P(t) \leq \alpha_2 I, \forall t \geq 0$ such that

$$\begin{aligned} &\dot{P}(t) + P(t)(M + T\varphi(v)T^{-1}) + \\ &(M + T\varphi(v)T^{-1})^T P(t) = -R(t). \end{aligned} \tag{21}$$

We are now ready to stabilize system (18) using the backstepping method [6].

Notice that since $G_1(e, v, w)$, $G_2(e, v, w)$ and $K(e, v, w)$ are sufficiently smooth functions and $G_1(0, v, w) = 0, G_2(0, v, w) = 0, K(0, v, w) = 0$, there exist sufficiently smooth functions $q_i(v, w) \geq 1$ and $a_i(e) \geq 1$, $i = 1, 2, 3$, such that, for all $e \in R$, $v \in R^4$ and $w \in R$,

$$\begin{aligned} &|G_1(e, v, w)|^2 \leq q_1(v, w)a_1(e)e^2 \\ &|G_2(e, v, w)|^2 \leq q_2(v, w)a_2(e)e^2 \\ &|K(e, v, w)|^2 \leq q_3(v, w)a_3(e)e^2. \end{aligned} \tag{22}$$

It is ready to design a robust stabilization controller for system (18) with the following steps.

step 0. Denote $V_0 = \frac{1}{2}\bar{l}\bar{\zeta}_1^2 + \bar{h}\bar{\zeta}_2^T P(t)\bar{\zeta}_2$ where $\bar{l}$, $\bar{h}$ are positive constants to be determined later, and $P(t)$ is the positive solution to the Lyapunov equation (21). The time derivative of V_0 along the trajectory of $\bar{\zeta}$ subsystem of (18) is given by

$$\begin{aligned} \dot{V}_0 \;\leq\; & -[\bar{l}(\lambda_1 - \epsilon_1)]\bar{\zeta}_1^2 - [\bar{h}(\beta_1 - \epsilon_2\alpha_2^2) \\ & -\bar{l}\frac{1}{2\epsilon_1}\|\Psi T^{-1}\|^2]\|\bar{\zeta}_2\|^2 \\ & +\bar{l}\frac{1}{2\epsilon_1}q_1(v, w)a_1(\tilde{x}_1)\tilde{x}_1^2 + \bar{h}\epsilon_2^{-1}q_2(v, w)a_2(\tilde{x}_1)\tilde{x}_1^2 \\ \leq\; & -l\bar{\zeta}_1^2 - h\|\bar{\zeta}_2\|^2 + q_0(v, w)a_0(\tilde{x}_1)\tilde{x}_1^2. \end{aligned}$$

where $l = \bar{l}(\lambda_1 - \epsilon_1), h = \bar{h}(\beta_1 - \epsilon_2\alpha_2^2) - \bar{l}\frac{1}{2\epsilon_1}\|\Psi T^{-1}\|^2, q_0(v, w) = \bar{l}\frac{1}{2\epsilon_1}q_1(v, w) + \bar{h}\epsilon_2^{-1}q_2(v, w)$, and $a_0(\tilde{x}_1) \geq \max\ (a_1(\tilde{x}_1), a_2(\tilde{x}_1))$, $\forall \epsilon_1 > 0, \forall \epsilon_2 > 0$.

step 1. Define $\tilde{x}_1 = e$. Then

$$\dot{\tilde{x}}_1 = H(w)\bar{\zeta} + K(\tilde{x}_1, v, w) + \xi_1. \tag{23}$$

Define

$$\begin{aligned} \alpha_1(\tilde{x}_1, k) &= -k\rho(\tilde{x}_1)\tilde{x}_1 \\ \dot{k} &= \rho(\tilde{x}_1)\tilde{x}_1^2 \\ \tilde{x}_2 &= \xi_1 - \alpha_1 \end{aligned} \tag{24}$$

where $\rho(\tilde{x}_1)$ is some smooth nonnegative function to be specified later.

Define

$$V_1 = V_0 + \frac{1}{2}\tilde{x}_1^2 + \frac{1}{2}(k - \bar{k})^2 \tag{25}$$

where $\bar{k}$ is some positive constant to be determined later. The time derivative of V_1 along the trajectory of (23) and (24) is

$$\begin{aligned} \dot{V}_1 \;\leq\; & -l\bar{\zeta}_1^2 - h\|\bar{\zeta}_2\|^2 + q_0(v, w)a_0(\tilde{x}_1)\tilde{x}_1^2 + \frac{\epsilon_3}{2}\tilde{x}_1^2 \\ & +\frac{1}{2\epsilon_3}\bar{\zeta}_1^2 + \frac{\epsilon_3}{2}\tilde{x}_1^2 + \frac{1}{2\epsilon_3}\|K(\tilde{x}_1, v, w)\|^2 + \tilde{x}_1\tilde{x}_2 \\ & -\bar{k}\tilde{x}_1^2\rho(\tilde{x}_1) \\ \leq\; & -(l - \frac{1}{2\epsilon_3})\bar{\zeta}_1^2 - h\|\bar{\zeta}_2\|^2 + \tilde{x}_1\tilde{x}_2 \\ +\; & (q_0(v, w)a_0(\tilde{x}_1) + \epsilon_3 + \frac{1}{2\epsilon_3}q_3(v, w)a_3(\tilde{x}_1) \\ & -\bar{k}\rho(\tilde{x}_1))\tilde{x}_1^2 \end{aligned}$$

step 2. Define

$$\begin{aligned} \alpha_2(\tilde{x}_1, \tilde{x}_2, k) &= \lambda_1\xi_1 + \frac{\partial\alpha_1}{\partial k}\dot{k} - (\tilde{x}_1 - \frac{\partial\alpha_1}{\partial\tilde{x}_1}\xi_1) \\ & \quad -\tilde{x}_2 - \frac{1}{2}(\frac{\partial\alpha_1}{\partial\tilde{x}_1})^2\tilde{x}_2 \\ \tilde{x}_3 &= \bar{u}_1 - \alpha_2 \\ V_2 &= V_1 + \frac{1}{2}\tilde{x}_2^2. \end{aligned} \tag{26}$$

Using

$$\begin{aligned} \tilde{x}_2\dot{\tilde{x}}_2 \;=\; & \tilde{x}_2\tilde{x}_3 - \tilde{x}_1\tilde{x}_2 + \frac{\partial\alpha_1}{\partial\tilde{x}_1}\tilde{x}_2\xi_1 - \tilde{x}_2^2 - \frac{1}{2}(\frac{\partial\alpha_1}{\partial\tilde{x}_1})^2\tilde{x}_2^2 \\ & -\tilde{x}_2\frac{\partial\alpha_1}{\partial\tilde{x}_1}(H(w)\bar{\zeta} + K(\tilde{x}_1, v, w) + \xi_1) \\ \leq\; & \tilde{x}_2\tilde{x}_3 - \tilde{x}_1\tilde{x}_2 - \tilde{x}_2^2 + \bar{\zeta}_1^2 + q_3(v, w)a_3(\tilde{x}_1)\tilde{x}_1^2 \end{aligned}$$

gives

$$\begin{aligned}\dot{V}_2 &\leq -(l-\frac{1}{2\epsilon_3})\bar{\zeta}_1^2 - h\|\bar{\zeta}_2\|^2 + (q_0(v,w)a_0(\tilde{x}_1)+\epsilon_3 \\ &\quad +\frac{1}{2\epsilon_3}q_3(v,w)a_3(\tilde{x}_1) - \bar{k}\rho(\tilde{x}_1))\tilde{x}_1^2 + \tilde{x}_1\tilde{x}_2 + \tilde{x}_2\tilde{x}_3 \\ &\quad -\tilde{x}_1\tilde{x}_2 - \tilde{x}_2^2 + \bar{\zeta}_1^2 + q_3(v,w)a_3(\tilde{x}_1)\tilde{x}_1^2 \\ &= -[l-\frac{1}{2\epsilon_3}-1]\bar{\zeta}_1^2 - h\|\bar{\zeta}_2\|^2 \\ &\quad -[\bar{k}\rho(\tilde{x}_1) - q_0(v,w)a_0(\tilde{x}_1) - \epsilon_3 \\ &\quad -(\frac{1}{2\epsilon_3}+1)q_3(v,w)a_3(\tilde{x}_1)]\tilde{x}_1^2 + \tilde{x}_2\tilde{x}_3 - \tilde{x}_2^2.\end{aligned}$$

By taking $\bar{u}_1 = \alpha_2$, i.e., $\tilde{x}_3 = 0$, we obtain

$$\begin{aligned}\dot{V}_2 &\leq -[l-\frac{1}{2\epsilon_3}-1]\bar{\zeta}_1^2 - h\|\bar{\zeta}_2\|^2 \\ &\quad -[\bar{k}\rho(\tilde{x}_1) - q_0(v,w)a_0(\tilde{x}_1) - \epsilon_3 \\ &\quad -(\frac{1}{2\epsilon_3}+1)q_3(v,w)a_3(\tilde{x}_1)]\tilde{x}_1^2 - \tilde{x}_2^2. \qquad (27)\end{aligned}$$

Let positive constants λ_1 and ϵ_1 be such that $\lambda_1 - \epsilon_1 > 0$. Choose a positive constant $\bar{l}$ satisfying $\bar{l}(\lambda_1-\epsilon_1) - \frac{1}{2\epsilon_3} \geq 2$ for any fixed positive constant ϵ_3, which implies that $l - \frac{1}{2\epsilon_3} - 1 \geq 1$. Let ϵ_2 be such that $\beta_1 - \epsilon_2\alpha_2^2 > 0$. Choosing $\bar{h} \geq \frac{1+\bar{l}\frac{1}{2\epsilon_1}\|\Psi T^{-1}\|^2}{(\beta_1-\epsilon_2\alpha_2^2)}$ yields $h \geq 1$. Let $\overline{\beta}(v,w,\tilde{x}_1) = q_0(v,w)a_0(\tilde{x}_1) + \epsilon_3 + (\frac{1}{2\epsilon_3}+1)q_3(v,w)a_3(\tilde{x}_1) + 1$. Then there exist smooth functions $\bar{q}(v,w) \geq 1$, and $\rho(\tilde{x}_1) \geq 1$ such that $\bar{q}(v,w)\rho(\tilde{x}_1) \geq \overline{\beta}(v,w,\tilde{x}_1)$. Since, for any initial state $v(0)$, $v(t)$ is bounded for all $t \geq 0$, we can choose a finite constant $\bar{k}$ such that $\bar{k} \geq \sup_{t\geq 0}\bar{q}(v(t),w)$ to make $\bar{k}\rho(\tilde{x}_1) - [q_0(v,w)a_0(\tilde{x}_1) + \epsilon_3 + (\frac{1}{2\epsilon_3}+1)q_3(v,w)a_3(\tilde{x}_1)] \geq 1$.

Thus we have

$$\dot{V}_2 \leq -\|\bar{\zeta}\|^2 - \tilde{x}_1^2 - \tilde{x}_2^2. \qquad (28)$$

Remark 3: The functions $a_i(e)$, $i = 1,2,3$, can be given as $a_1(e) = a_2(e) = 1 + e^2 + e^4 \geq 1, a_3(e) = 1$. As a result, ρ can be derived as follows.

$$\begin{aligned}\overline{\beta}(v,w,\tilde{x}_1) &= q_0(v,w)a_0(\tilde{x}_1) + \epsilon_3 \\ &\quad +(\frac{1}{2\epsilon_3}+1)q_3(v,w)a_3(\tilde{x}_1) + 1 \\ &\leq \bar{q}(v,w)(1+\tilde{x}_1^2+\tilde{x}_1^4) \\ &\leq \bar{q}(v,w)\rho(\tilde{x}_1) \qquad (29)\end{aligned}$$

where $\bar{q}(v,w) = q_0(v,w) + \epsilon_3 + (\frac{1}{2\epsilon_3}+1)q_3(v,w) + 1 \geq 1$ and $\rho(\tilde{x}_1) = (1+\tilde{x}_1^2)^2$.

We are now ready to formally summarize our main result as follows.

Theorem 1: There exists a feedback control law of the following form:

$$\begin{aligned}u_1 &= \alpha_2(\tilde{x}_1,\tilde{x}_2,k) + \Psi T^{-1}\eta \\ \dot{\eta} &= (M + T\varphi(v)T^{-1})\eta + N(\alpha_2(\tilde{x}_1,\tilde{x}_2,k) + \Psi T^{-1}\eta) \\ \dot{k} &= (1+\tilde{x}_1^2)^2\tilde{x}_1^2 \\ \dot{\xi}_1 &= -\lambda_1\xi_1 + \alpha_2(\tilde{x}_1,\tilde{x}_2,k), \qquad (30)\end{aligned}$$

where $\tilde{x}_1 = e$, and $\alpha_2(\tilde{x}_1,\tilde{x}_2,k) = \lambda_1\xi_1 + \frac{\partial\alpha_1}{\partial k}\dot{k} - (\tilde{x}_1 - \frac{\partial\alpha_1}{\partial\tilde{x}_1}\xi_1) - \tilde{x}_2 - \frac{1}{2}(\frac{\partial\alpha_1}{\partial\tilde{x}_1})^2\tilde{x}_2$ with α_1 and $\tilde{x}_2$ being defined in (24) such that, for any initial condition of the closed-loop system, and any value of uncertain parameter w of the Duffing system, the solution of the closed-loop system is bounded for all $t \geq 0$ and the synchronization error $e(t)$ approaches 0 as $t \to \infty$.

Proof: Noting that V_2 is globally positive definite and $\dot{V}_2$ is globally negative semi-definite as well as the fact that $v(t)$ is bounded for any $v(0)$ shows that the solution and the derivative of the solution of the closed-loop system are bounded for all $t \geq 0$. Furthermore, using (28) shows that $\bar{\zeta}$, $\tilde{x}_1$, and $\tilde{x}_2$ are square integrable on $[0,\infty)$. By Barbalat's lemma, $(\bar{\zeta},\tilde{x}_1,\tilde{x}_2)$ approaches 0 as $t \to \infty$.

The performance of the control law is illustrated by computer simulation with the initial condition being $x_1(0) = 2, x_2(0) = 1, v_1(0) = -1, v_2(0) = 2, v_3(0) = 0.8, v_4(0) = 0, \eta(0) = \mathrm{col}(1,0), k(0) = 1, \xi_1(0) = 2$. Other parameters are $a = 1, b = 1$, $\lambda_1 = 1$, $\sigma = 0.3, w = 3, \delta = 0.8, \omega = 1.2$. Figures 1 and 2 show the phase portraits of the uncontrolled Duffing system and Van der Pol oscillator, respectively. It can be seen that the uncontrolled Duffing system displays a chaotic motion while the solution of the Van der Pol oscillator approaches a limit cycle asymptotically. Figures 3 to 6 show the response of the closed-loop system. It can be seen that the controller has a satisfactory performance.

IV. CONCLUSION

In this paper, we have investigated the synchronization problem of the controlled Duffing system and Van der Pol oscillator via the internal model approach. The problem is solved by globally stabilizing a time-varying nonlinear system. The design philosophy can also be applied to the synchronization problem of other pairs of nonlinear systems.

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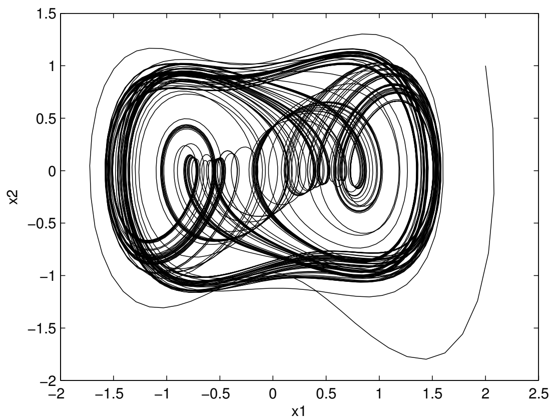

Fig. 1. Chaotic behavior of the Duffing system.

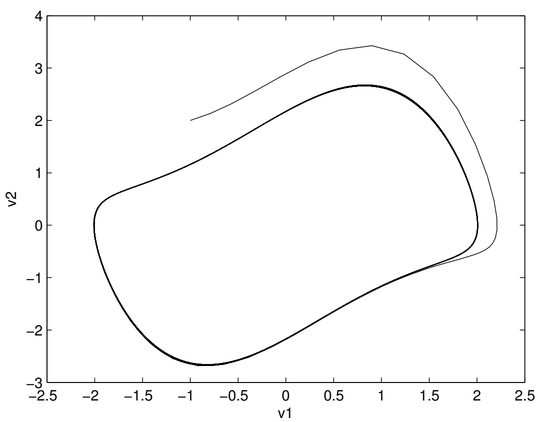

Fig. 2. Limit cycle of the Van der Pol oscillator

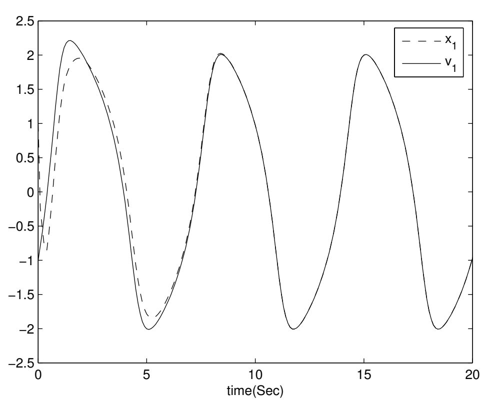

Fig. 3. x_1 vs. v_1.

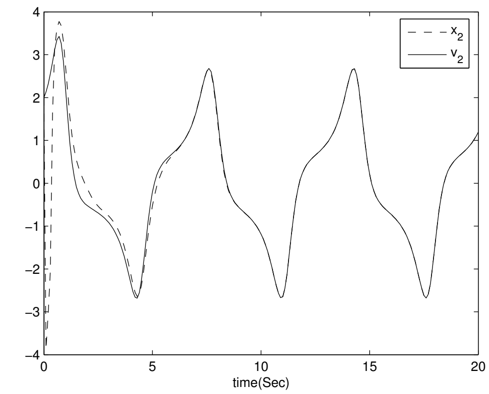

Fig. 4. x_2 vs. v_2.

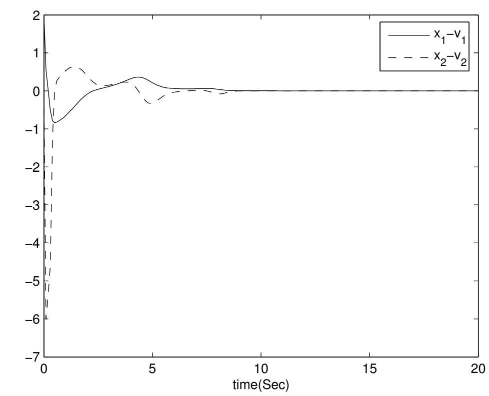

Fig. 5. The synchronization error.

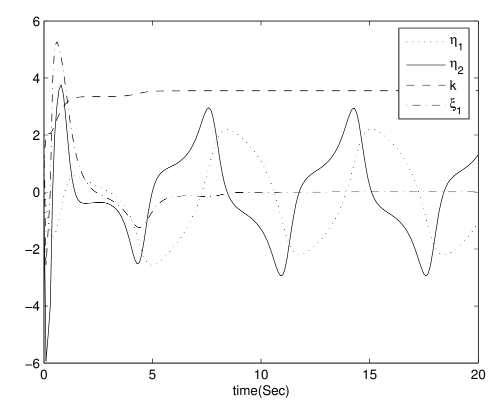

Fig. 6. Profile of the states of the controller.