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Stability Analysis of Quantized Feedback Systems Including Optimal Dynamic Quantizers

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Abstract— **This paper characterizes the stability of quantized feedback systems which contains optimal dynamic quantizers recently proposed by the authors. First, it is shown that the separation property of the quantizer-controller design, which is similar to the well-known separation property of the observer-controller design, holds in the quantized feedback systems. Next, based on this property, a necessary and sufficient condition for the stability is derived, where the stability is characterized by the poles/zeros of a linear feedback system to be quantized. Finally, we present suboptimal dynamic quantizers for which the resulting quantized feedback systems are always stable.**

I. INTRODUCTION

The quantization, i.e., the map from a continuous-valued signal set to a discrete-valued signal set, is one of the most important topics also in the systems and control field. As is seen, input/output signals of controlled plants have to be quantized in various practical situations. In fact, digital controllers are commonly used, and discrete-valued actuators (e.g., on/off actuators) are often employed for industrial plants and embedded systems. Furthermore, digital networks are utilized for connecting controllers to plants.

This topic has been actively studied so far, e.g., [1]–[10]. The authors also have considered it and have obtained one of the key results in [11], [12]. There, the following problem is considered: when a plant and a controller are given for the quantized feedback system in Fig. 1 (a), find a dynamic quantizer such that the system in (a) *optimally* approximates the usual (ideal) feedback system in Fig. 1 (b), in terms of the input-output relation. To this problem, we have derived a *closed form* solution, which clarifies an optimal structure and the performance limitation of (a general class of) dynamic quantizers. However, since the optimization problem is considered without taking stability into account, the resulting quantized feedback system (a) are often unstable. Thus, it is important to characterize the plant and controller for which the optimally quantized system is stable.

This paper thus addresses the stability of the quantized feedback systems including the optimal dynamic quantizers. The main contributions of this paper are as follows.

First, it is proven that the quantized feedback system (a) is stable if and only if the ideal feedback system (b) is stable and the dynamic quantizer itself is stable. Similarly to the well-known separation property of the observer-controller design [13], this separation property plays an important role in analysis and design of quantized feedback systems. Second, based on this property, the stability of the optimally

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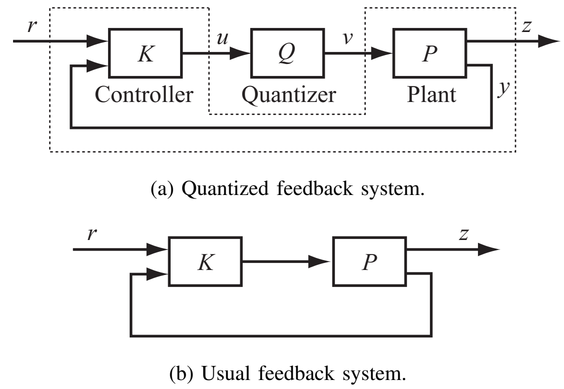


Fig. 1. Quantized and unquantized feedback systems.

quantized feedback system (a) is characterized by the poles and zeros of the ideal feedback system (b). This will be useful to capture the easiness/hardness for dynamic quantization in terms of typical system characteristics. Finally, as an alternative to the *un*-stabilizing quantizers, suboptimal dynamic quantizers, which guarantee the stability of the resulting feedback systems, are presented.

Notation: Let $\mathbf{R}$, $\mathbf{R}_+$, and $\mathbf{N}$ be the real number field, the set of positive real numbers, and the set of nonnegative integers, respectively. We denote by 0 and I the zero matrix and the identity matrix of appropriate dimensions, respectively. For the matrix M, let $\mathrm{abs}(M)$ denote the matrix composed of the absolute value of each element. For the vector sequences $X := (x_1, x_2, \dots)$ and $Y := (y_1, y_2, \dots)$, we denote by $X - Y$ the vector sequence $(x_1 - y_1, x_2 - y_2, \dots)$. For the vector x, the matrix M, and the vector sequence X, the symbols $\|x\|$, $\|M\|$, and $\|X\|$ express their ∞-norms. For the linear time-invariant system S with the parameters (A, B, C, D), a minimal realization is expressed by

$$S_{mr} : \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right]$$

and the transfer function is denoted by $S(\mathsf{z})$ with the shift operator z. Finally, the linear system S is said to be *stable* if the *system poles* of S are stable. So note that the following statements are equivalent: (a) S_{mr} is stable, (b) the system poles of S_{mr} are stable, and (c) the transmission poles of S are stable.

II. OPTIMAL DYNAMIC QUANTIZERS FOR DISCRETE-VALUED INPUT CONTROL

We briefly review the results in [11], [12], which motivates us to analyze the stability of quantized feedback systems.

A. Dynamic Quantizer Design Problem

Consider the feedback system Σ_Q shown in Fig. 2 (a), which is composed of the discrete-time linear system G and the dynamic quantizer Q.

The system G is given by

$$G : \begin{cases} x(k+1) &=& Ax(k) + B_1 r(k) + B_2 v(k), \\ z(k) &=& C_1 x(k) + D_1 r(k), \\ u(k) &=& C_2 x(k) + D_2 r(k) \end{cases} \qquad (1)$$

where $x \in \mathbf{R}^n$ is the state, $r \in \mathbf{R}^p$ and $v \in \mathbf{R}^m$ are the inputs, $z \in \mathbf{R}^l$ and $u \in \mathbf{R}^m$ are the outputs, $k \in \mathbf{N}$ is the time, and A, B_1, B_2, C_1, C_2, D_1, and D_2 are constant matrices of appropriate dimensions. The initial state is given as $x(0) = x_0$ for $x_0 \in \mathbf{R}^n$.

The dynamic quantizer Q is given in the following form:

$$Q : \begin{cases} \xi(k+1) &=& \mathcal{A}\xi(k) + \mathcal{B}_1 u(k) + \mathcal{B}_2 v(k), \\ v(k) &=& q(\mathcal{C}\xi(k) + u(k)) \end{cases} \qquad (2)$$

where $\xi \in \mathbf{R}^{\mathcal{N}}$ is the state of dimension $\mathcal{N}$, $u \in \mathbf{R}^m$ and $v \in \mathbf{V}^m := \{0, \pm d, \pm 2d, \dots\}^m$ are the input and the output, respectively. Furthermore, $\mathcal{A}$, $\mathcal{B}_1$, $\mathcal{B}_2$, and $\mathcal{C}$ are constant matrices of appropriate dimensions, and the function $q : \mathbf{R}^m \to \mathbf{V}^m$ is the static quantizer which is the nearest type toward $-\infty$[1]. The set $\mathbf{V}$ is the discrete set on which each output takes values, and $d \in \mathbf{R}_+$ is the quantization interval which specifies $\mathbf{V}$. The initial state is given as $\xi(0) = 0$ for guaranteeing that Q is drift-free, i.e., $v(k) = 0$ for $u(k) = 0$ $(k = 0, 1, \dots)$. Note that the input-output relation of Q is invariant under the (linear) state transformation and the minimal realization.

It is remarked that Σ_Q is a generalized version of the quantized feedback system in Fig. 1 (a). In fact, Σ_Q is equivalent to the system in Fig. 1 (a) by regarding the part indicated by the dotted line frame in Fig. 1 (a) as G. Thus the discussion in this paper holds not only for the feedback system in Fig. 1 (a) but also for various types of systems including Q.

Then the design problem of Q is formulated as follows.

For the system Σ_Q, let $Z_Q(x_0, R)$ denote the controlled output sequence $(z(1), z(2), \dots, z(\infty))$ for the initial state x_0 and the reference input $R := (r_0, r_1, \dots) \in \ell_\infty^p$, and let $z_Q(k, x_0, R)$ be the output at time k. We consider the feedback system Σ in Fig. 2 (b) as an ideal system, for which the symbols $Z(x_0, R)$ and $z(k, x_0, R)$ are similarly defined.

Problem 1: For the system Σ_Q, find a dynamic quantizer Q (i.e., find $\mathcal{N}$, $\mathcal{A}$, $\mathcal{B}_1$, $\mathcal{B}_2$, and $\mathcal{C}$) minimizing

$$E(Q) := \sup_{(x_0,R)\in\mathbf{R}^n\times\ell_\infty^p} \|Z_Q(x_0,R) - Z(x_0,R)\|. \qquad (3)$$

■

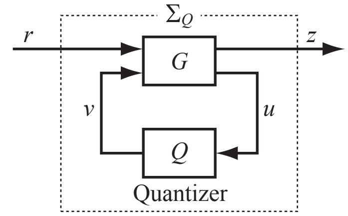
In Problem 1, the performance index $E(Q)$ corresponds to the difference between the system Σ_Q in Fig. 2 (a) and the ideal system Σ in Fig. 2 (b), in terms of the input-output relation. In other words, if $E(Q)$ is small, the input-output relation of the ideal system Σ is almost preserved in Σ_Q.

This provides us a practical method of control systems design with discrete-valued signal constraints. For example, consider the feedback system in Fig. 1 (a), and suppose that P is a linear plant which has to be actuated by discrete-valued signal. Then Σ_Q would have good performance for

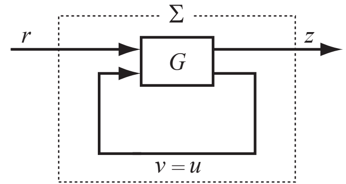
- a controller K achieving desirable performance in the ideal system in Fig. 1 (b),
- a dynamic quantizer Q such that $E(Q)$ is small.

So, with a solution to Problem 1, controllers designed by the conventional theory can be applied to the quantized systems.

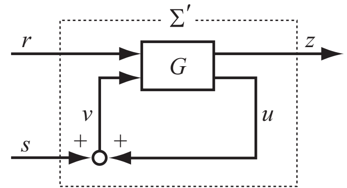
[1] For $\mu \in \mathbf{R}^m$, $q(\mu)$ equals to the smallest vector (in the sense of the sum of the all elements) of the optimal solutions to $\min_{v\in\mathbf{V}^m}(v-\mu)^\top(v-\mu)$.



(a) Quantized feedback system Σ_Q.



(b) Usual feedback system Σ.



(c) Usual feedback system Σ' with new input.

Fig. 2. General expressions of quantized and unquantized feedback systems.

B. A Closed Form Solution

We next show a solution to Problem 1, derived in [12].

Let us introduce the system Σ' in Fig. 2 (c). This is defined by adding a new external input to Σ, and its impulse response matrices from s to z are given by $C_1\tilde{A}^k B_2$ $(k = 0, 1, \dots)$ for $\tilde{A} := A + B_2 C_2$. Then the following result is obtained [12].

Lemma 1: For the system Σ_Q, assume that

(A1) rank $D_2 = m$ (D_2 is full row rank),

(A2) $l = m$ and there exists an integer $k \in \mathbf{N}$ such that $C_1\tilde{A}^0 B_2 = C_1\tilde{A}^1 B_2 = \cdots = C_1\tilde{A}^{k-1}B_2 = 0$ holds and $C_1\tilde{A}^k B_2$ is nonsingular.

Then a solution to Problem 1 is given by

$$Q^* : (\mathcal{N}, \mathcal{A}, \mathcal{B}_1, \mathcal{B}_2, \mathcal{C}) := (n, \tilde{A}, -B_2, B_2, -(C_1\tilde{A}^\tau B_2)^{-1}C_1\tilde{A}^{\tau+1}), \qquad (4)$$

and the minimum value of $E(Q)$ is given by

$$\min_Q E(Q) = E(Q^*) = \|C_1\tilde{A}^\tau B_2\|\frac{d}{2} \qquad (5)$$

where τ is the value of k satisfying the condition in (A2). ■

Lemma 1 provides an analytical solution to Problem 1. The performance of Q^* is remarkable as follows.

Example 1: Consider the system Σ_{Q^*} for the feedback system in Fig. 1 (a). Here, P and K are the discrete-time plant and controller obtained from the continuous-time ones

$$P_c : \begin{cases} \dot{x}_P(t) = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} x_P(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} v(t), \\ z(t) \;\; = C_{11} x_P(t), \\ y(t) \;\; = [1 \;\; 0] x_P(t), \end{cases}$$

$$K_c : \begin{cases} \dot{x}_K(t) = \begin{bmatrix} -9 & 1 \\ -37.2 & -6 \end{bmatrix} x_K(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} r(t) + \begin{bmatrix} 9 \\ 27.9 \end{bmatrix} y(t), \\ u(t) \;\; = -[12 \;\; 7] x_K(t) + r(t) \end{cases}$$

and the zero-order hold with the sampling period $h := 0.1$. For Q^*, the quantization interval is given by $d := 2$.

Fig. 3 shows the simulation result on the time responses of the system Σ_{Q^*} for the output matrix

$$C_{11} := [1 \;\; 2], \tag{6}$$

where $x_0 := [0.1 \;\; -0.5 \;\; 0 \;\; 0]^\top$ ($x := [x_P^\top \; x_K^\top]^\top$) and $r(k) \equiv 0$. In addition, the output response of Σ in Fig. 2 (b) (Fig. 1 (b)) is also shown by the thin line in the third figure, where x_0 and r are set to the same values. Though the discrete-valued signal is applied to P in Σ_{Q^*}, we see that the output behavior of Σ_{Q^*} is almost same as that of Σ. ■

Meanwhile, such a good result is not always obtained.

Example 2: Consider again Σ_{Q^*} in Example 1, and suppose that C_{11} is given as follows (instead of (6)):

$$C_{11} := [1 \;\; -1]. \tag{7}$$

The simulation result is illustrated in Fig. 4 in the same fashion. Although a similar result is obtained for the output z, it is observed that u and v diverge. This suggests that Σ_{Q^*} becomes unstable and Q^* cannot be always applied to Σ_Q. ■

So the following question arises.

Problem 2: *For what G is the optimally quantized system Σ_{Q^*} stable?*

This question is fairly basic but still open. The main purpose of this paper is to address this stability problem.

Remark 1: Problem 1 generally has multiple solutions, while if $m = l = 1$, Q^* is the unique solution except for the similar Q with respect to the input-output relation. In this sense, Q^* is the most fundamental solution to Problem 1. ■

III. Stability Analysis

This section presents a solution to Problem 2.

For simplicity of notation, it is assumed that Q is minimally realized. So $(\mathcal{A}, [\mathcal{B}_1 \;\; \mathcal{B}_2], \mathcal{C})$ is controllable and observable. The same assumption is made for Q^*.

A. Stability Notion and Separation Property

The following stability notion is considered.

Definition 1: (i) The quantized feedback system Σ_Q is said to be *stable* if the state (x, ξ) is bounded for every initial state $x_0 \in \mathbf{R}^n$ and input sequence $R \in \ell_\infty^p$ (note $\xi(0) = 0$).

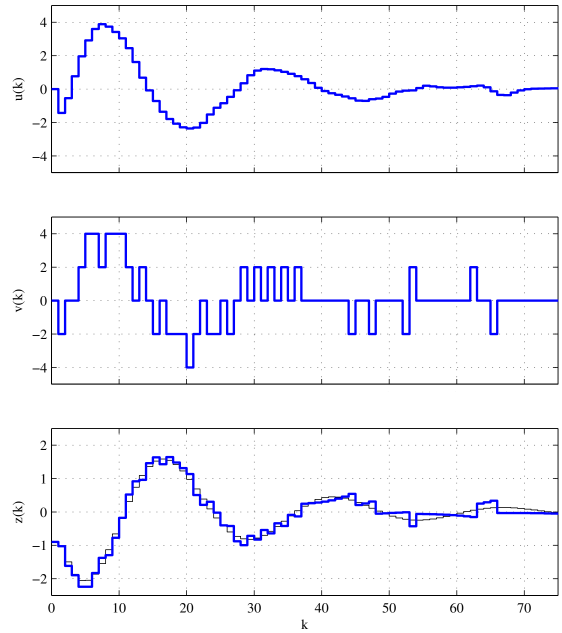


Fig. 3. Responses of Σ_{Q^*} for $C_{11} := [1 \;\; 2]$.

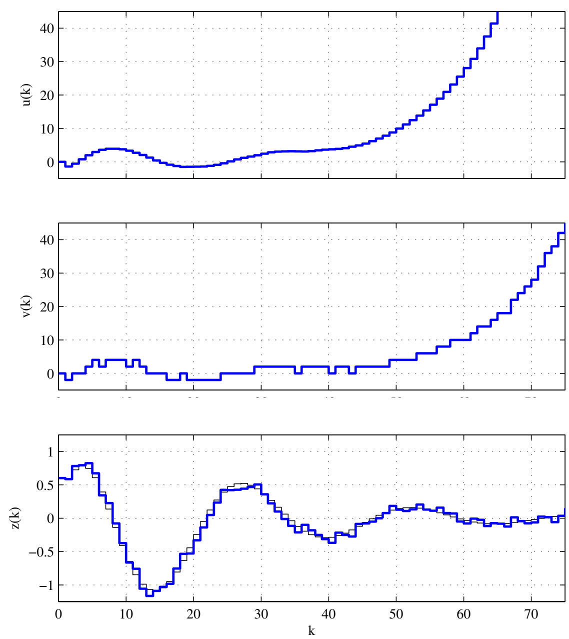


Fig. 4. Responses of Σ_{Q^*} for $C_{11} := [1 \;\; -1]$.

(ii) The dynamic quantizer Q is said to be *stable* if the state ξ is bounded for every input sequence $U \in \ell_\infty^m$. ■

For the system Σ_Q, the boundedness of the state is considered as a stability criterion. Such a notion is quite natural for quantized systems. In fact, since the signal v takes values on a uniform lattice $\mathbf{V}^m$, it is not necessarily possible for Σ_Q to be asymptotically stable with any Q. As for the quantizer Q, a similar stability notion is introduced.

Now, let us characterize the stability of Σ_Q.

To express the quantization error of the static quantizer q in (2), we introduce the new variable $w \in [-d/2, d/2]^m$:

$$w(k) := q(\mathcal{C}\xi(k) + u(k)) - (\mathcal{C}\xi(k) + u(k)). \tag{8}$$

With this variable, Q is represent as

$$Q : \begin{cases} \xi(k+1) = (\mathcal{A}+\mathcal{B}_2\mathcal{C})\xi(k) + (\mathcal{B}_1+\mathcal{B}_2)u(k)+\mathcal{B}_2 w(k), \\ \quad v(k) \quad = \mathcal{C}\xi(k)+u(k) + w(k), \end{cases} \tag{9}$$

and Σ_Q is illustrated as Fig. 5 (a), where H is the linear system given by

$$H : \left[\begin{array}{c|cc} \mathcal{A}+\mathcal{B}_2\mathcal{C} & I & \mathcal{B}_2 \\ \hline \mathcal{C} & 0 & I \end{array} \right]. \tag{10}$$

Then if the condition

$$\mathcal{B}_1 = -\mathcal{B}_2 \tag{11}$$

holds ((11) holds for Q^*), Σ_Q is regarded as the cascade connection of the linear systems Σ' and

$$H_{ws} : \left[\begin{array}{c|c} \mathcal{A}+\mathcal{B}_2\mathcal{C} & \mathcal{B}_2 \\ \hline \mathcal{C} & I \end{array} \right] \tag{12}$$

as shown in Fig. 5 (b), where H_{ws} corresponds to the system H with $u \equiv 0$ (the input and output of H_{ws} are w and s). Although w depends on ξ and u as seen in (8), the following result is obtained (the proof is given in Appendix I).

Theorem 1: For the system Σ_Q, assume (A1) and (11). Then the following statements hold.
(i) Σ_Q is stable if and only if Σ' and Q are stable.
(ii) Q is stable if and only if H_{ws} is stable. ■

In (i), the stability of Σ_Q is characterized by the stability of its two components Σ' and Q (see Fig. 5 (b)). By considering that a controller is contained in Σ' (i.e., G) as a design object, this implies that a kind of separation property of the quantizer-controller design holds. As well as the well-known separation property of the observer-controller design [13], this property is fundamental and useful in analysis and design of the quantized feedback systems. On the other hand, in (ii), the stability of Q is characterized by the stability of the linear system H_{ws}. Thus, using (i) and (ii), we can check the stability of Σ_Q by calculating the system poles of Σ' and of H_{ws}. Note that the stability notion for the linear systems Σ' and H_{ws} is defined in the end of Section I.

Although a solution to Problem 2 is provided by Theorem 1 with $Q := Q^*$, it cannot clarify the relation between the stability of Σ_{Q^*} and some system property of G. So, in the next subsection, we derive a simple stability condition.

B. Stability Condition for Optimal Dynamic Quantizers

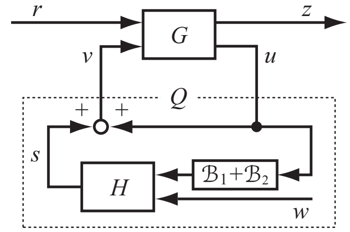
In this section, under (A1) and (A2), the stability of Q^* is characterized by a system property of G. The outline is shown in Fig. 6, where the relation between Q and Q^* and Theorem 1 (ii) are indicated by (a) and (b).

By applying (4) to (12), the system H_{ws} for $Q := Q^*$ is given by

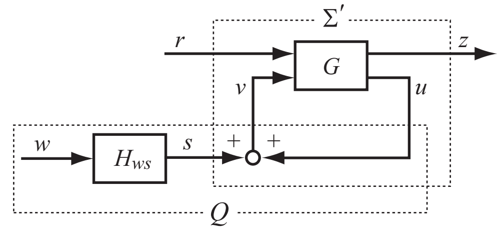
$$H_{ws}^* : \left[\begin{array}{c|c} \tilde{A} - B_2(C_1\tilde{A}^{\tau}B_2)^{-1}C_1\tilde{A}^{\tau+1} & B_2 \\ \hline -(C_1\tilde{A}^{\tau}B_2)^{-1}C_1\tilde{A}^{\tau+1} & I \end{array} \right], \tag{13}$$

which gives the relations (c) and (d) in Fig. 6. Then the inverse of the system H_{ws}^* is expressed as

$$H_{inv}^* : \left[\begin{array}{c|c} \tilde{A} & B_2 \\ \hline (C_1\tilde{A}^{\tau}B_2)^{-1}C_1\tilde{A}^{\tau+1} & I \end{array} \right], \tag{14}$$



(a) System Σ_Q.



(b) System Σ_Q with (11).

Fig. 5. Equivalent expression.

for which we have the following relation:

$$\begin{aligned} H_{inv}^*(\mathtt{z}) &= \left(\sum_{k=0}^{\infty} \frac{(C_1\tilde{A}^{\tau}B_2)^{-1}C_1\tilde{A}^{\tau+1}\tilde{A}^kB_2}{\mathtt{z}^{k+1}} \right) + I \\ &= (C_1\tilde{A}^{\tau}B_2)^{-1}\mathtt{z}^{\tau+1} \left(\sum_{k=0}^{\infty} \frac{C_1\tilde{A}^kB_2}{\mathtt{z}^{k+1}} \right) \\ &= (C_1\tilde{A}^{\tau}B_2)^{-1}\mathtt{z}^{\tau+1}\Sigma'_{sz}(\mathtt{z}) \end{aligned}$$

where Σ'_{sz} is the system Σ' with $r \equiv 0$, and $H_{inv}^*(\mathtt{z})$ and $\Sigma'_{sz}(\mathtt{z})$ express their transfer functions. So, as shown by (e) and (f) in Fig. 6, it follows that H_{ws}^* is equal to the inverse of the system $(C_1\tilde{A}^{\tau}B_2)^{-1}\mathtt{z}^{\tau+1}\Sigma'_{sz}(\mathtt{z})$. More specifically, the transmission poles of H_{ws}^* are composed of both "0" and the transmission zeros of Σ'_{sz}, and the transmission zeros of H_{ws}^* are equal to the transmission poles of Σ'_{sz}. Furthermore, the transmission poles/zeros of H_{ws} are equal to its system poles/zeros, since Q (i.e., H_{ws}) is a minimal realization. Therefore, a stability condition for Q^* is obtained as follows.

Theorem 2: For the system Σ_Q, assume (A1) and (A2). Then Q^* is stable if and only if the all transmission zeros of Σ'_{sz} (i.e., the all transmission zeros of G) are stable. ■

From Theorem 2, it turns out that the stability of Q^* is characterized by the transmission zeros of Σ'_{sz}, i.e., of G (note that the zeros are feedback invariant). This can be verified in Examples 1 and 2, where Σ'_{sz} in Example 1 has the stable zeros $\{0.39\pm0.26i,\ 0.95\}$, while Σ'_{sz} in Example 2 has the stable and unstable zeros $\{0.39 \pm 0.26i,\ 1.10\}$. Theorems 1 and 2 provide a stability condition for Σ_{Q^*} based on system properties of G.

In addition to Theorem 2, two interesting observations are obtained from the above discussion.

First, in the quantized feedback system Σ_{Q^*}, Q^* plays a role to cancel the transmission poles and zeros of Σ' and to reduce the signal transfer level from the static quantization

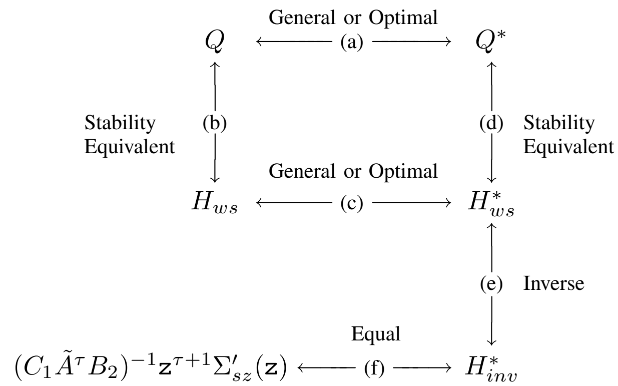


Fig. 6. Relation between Q^* and Σ' (a unity feedback system with G).

error w to the output z. In fact, if Σ_{Q^*} is regarded as the cascade connection of Σ' and H^*_{ws} as shown in Fig. 5 (b), the relations (e) and (f) in Fig. 6 imply that the pole-zero cancellation occurs between Σ'_{sz} and H^*_{ws}, i.e., between subsystems of Σ' and Q^*. Then the transfer function from w to z is equal to $C_1\tilde{A}^\tau B_2/\mathbf{z}^{\tau+1}$.

Second, Theorem 2, together with Lemma 1, captures the easiness/hardness for dynamic quantization in terms of the zeros of G. This is explained as follows. Assume $m=l=1$, which guarantees that Q^* is a unique solution to Problem 1 (see Remark 1). Then Theorem 2 and Lemma 1 imply that if G has no unstable transmission zero

$$\min_{Q:\text{stable}} E(Q) = \min_Q E(Q) = \|\text{abs}(C_1\tilde{A}^\tau B_2)\|\frac{d}{2} \qquad (15)$$

holds; otherwise

$$\min_{Q:\text{stable}} E(Q) > \min_Q E(Q) = \|\text{abs}(C_1\tilde{A}^\tau B_2)\|\frac{d}{2} \qquad (16)$$

holds, where $\min_{Q:\text{stable}} E(Q)$ expresses the minimum value of $E(Q)$ with respect to stable Q, and the inequality is obtained from the uniqueness of Q^*. Thus, similarly to the well-known performance limitation of feedback control (see, e.g. [13]), the achievable performance by the dynamic quantizers is limited by the presence of unstable zeros.

Finally, this section is concluded with our answer to Problem 2.

Theorem 3: For the system Σ_Q, assume (A1) and (A2). Then Σ_{Q^*} is stable if and only if the all system poles and transmission zeros of Σ'_{sz} are stable. ■

IV. Suboptimal Dynamic Quantizers with Stability Consideration

Throughout the discussion in Section III, it turns out that the following problem has to be considered especially for G with an unstable transmission zero.

Problem 3: For the system Σ_Q, find a *stable* dynamic quantizer Q minimizing $E(Q)$. ■

Unlike Problem 1, Problem 3 must be difficult to analytically solve, because a good special structure of Problem 1 [12], which gives Lemma 1, is lost by the introduction of the stability specification. So we present here an alternative solution under (A1) and (A2).

The performance index $E(Q)$ can be expressed as

$$E(Q) = \begin{cases} \left\| \displaystyle\sum_{k=0}^{\infty} \text{abs}\left([C_1\ \ 0] \begin{bmatrix} \tilde{A} & B_2\mathcal{C} \\ 0 & \mathcal{A}+\mathcal{B}_2\mathcal{C} \end{bmatrix}^k \begin{bmatrix} B_2 \\ \mathcal{B}_2 \end{bmatrix} \right) \right\| \dfrac{d}{2} \\ \qquad \text{if } [C_1\ \ 0] \begin{bmatrix} \tilde{A} & B_2\mathcal{C} \\ 0 & \mathcal{A}+\mathcal{B}_2\mathcal{C} \end{bmatrix}^k \begin{bmatrix} 0 \\ \mathcal{B}_1+\mathcal{B}_2 \end{bmatrix} = 0 \ \ (\forall k \in \mathbf{N}), \\ \infty \qquad \text{otherwise} \end{cases} \qquad (17)$$

under (A1) (cf. [12], [14]). If Q satisfies

(C1) $\mathcal{N}=n$, $\mathcal{A}=\tilde{A}$, $\mathcal{B}_1=-B_2$, and $\mathcal{B}_2=B_2$,

the first case in (17) holds and the main component of $E(Q)$ is represented as

$$[C_1\ \ 0] \begin{bmatrix} \tilde{A} & B_2\mathcal{C} \\ 0 & \mathcal{A}+\mathcal{B}_2\mathcal{C} \end{bmatrix}^k \begin{bmatrix} B_2 \\ \mathcal{B}_2 \end{bmatrix} = C_1(\tilde{A}+B_2\mathcal{C})^k B_2$$
$$= \begin{cases} 0 & \text{if } 0 \le k \le \tau-1, \\ C_1\tilde{A}^\tau B_2 & \text{if } k=\tau, \\ C_1(\tilde{A}+B_2\mathcal{C})^k B_2 & \text{if } \tau+1 \le k. \end{cases} \qquad (18)$$

In addition to (C1), if Q (in particular, $\mathcal{C}$) satisfies

(C2) the all controllable poles of the pair $(\tilde{A}+B_2\mathcal{C}, B_2)$ are stable and the all (transmission) poles of

$$W : \left[\begin{array}{c|c} \tilde{A}+B_2\mathcal{C} & B_2 \\ \hline C_1 & 0 \end{array} \right]$$

are zero,

then Q is stable (Theorem 1) and the following relation holds:

$$C_1(\tilde{A}+B_2\mathcal{C})^k B_2 = 0 \quad \text{if } \nu \le k \qquad (19)$$

where $\nu \in \{\tau+1, \tau+2, \ldots, n\}$ is the number of the poles of W which are set to zero. Note that there exists a $\mathcal{C}$ satisfying (C2). From (17)–(19), the following result is obtained.

Lemma 2: Let Q° denote a minimal realization of Q satisfying (C1) and (C2). Then Q° is stable, and its performance is given by

$$E(Q^\circ) = \left\| \text{abs}(C_1\tilde{A}^\tau B_2) + \sum_{k=\tau+1}^{\nu-1} \text{abs}(C_1(\tilde{A}+B_2\mathcal{C})^k B_2) \right\| \frac{d}{2} \qquad (20)$$

where ν is a finite number determined by the selection of $\mathcal{C}$. ■

Lemma 2 provides a stable Q for which $E(Q) < \infty$. So it follows that Problem 3 is feasible for every G.

The performance $E(Q^\circ)$ consists of two components. In the norm of (20), the first term, i.e., $\text{abs}(C_1\tilde{A}^\tau B_2)$, corresponds to the minimum value of $E(Q)$ shown in (5), and the other term corresponds to the degradation.

If G has no unstable zero, $E(Q^\circ) = E(Q^*)$ holds by setting $\mathcal{C}$ as (4). Then Q° is an optimal solution to Problem 3. Meanwhile, if G has an unstable zero, Q° is not always an optimal solution. However, it can be shown that Q° is a suboptimal solution, as follows.

For $R(Q^\circ) := \|\sum_{k=\tau+1}^{\nu-1} \text{abs}(C_1(\tilde{A}+B_2\mathcal{C})^k B_2)\|(d/2)$, the following inequality holds:

$$\min_Q E(Q) \le \min_{Q:\text{stable}} E(Q) \le E(Q^\circ) \le \min_Q E(Q) + R(Q^\circ). \qquad (21)$$

Then if G is a discrete-time model given from the continuous-time system $(A^c, [B_1^c\ \ B_2^c], [(C_1^c)^\top\ \ (C_2^c)^\top]^\top, [(D_1^c)^\top\ \ (D_2^c)^\top]^\top)$ and the sampling period h (i.e., $A := e^{A_c h}$, $B_2 := \int_0^h e^{A_c t} dt B_2^c$, $C_1 := C_1^c$), we have $R(Q^\circ) \leq \mu \| \int_0^h e^{A_c t} dt \|$ for a constant $\mu \in \mathbf{R}_+$ which does not depend on h (the proof is omitted). So $R(Q^\circ)$ is bounded by a monotone decreasing function with respect to h, and $\lim_{h \to 0} R(Q^\circ) = 0$. Therefore, for a sufficiently small h, (21) implies $\min_{Q:\text{stable}} E(Q) \simeq E(Q^\circ)$. In this sense, Q° is a sub-optimal solution to Problem 3.

Example 3: Recall Example 2. In stead of Q^*, let us apply Q° to Σ_Q, where $\mathcal{C}$ is chosen such that the (transmission) poles of W are $\{0, 0\}$. Note that the system poles and the transmission zeros of Σ'_{sw} are $\{0.239, 0.923 \pm 0.237i, 0.827\}$ and $\{1.10, 0.39 \pm 0.26i\}$, and the controllable poles of the pair $(\tilde{A} + B_2\mathcal{C}, B_2)$ are $\{0, 0, 0.39 \pm 0.26i\}$.

Fig. 7 shows the simulation result of the system Σ_{Q° under the same condition as Example 2. In addition, the output response of the ideal system Σ is also shown in the same way. Although u and v diverge in Example 2, they range practical values by virtue of Q°. In addition, it can be observed that the output behavior of Σ_{Q° approximates that of Σ. Thus the proposed suboptimal dynamic quantizer Q° is useful. ■

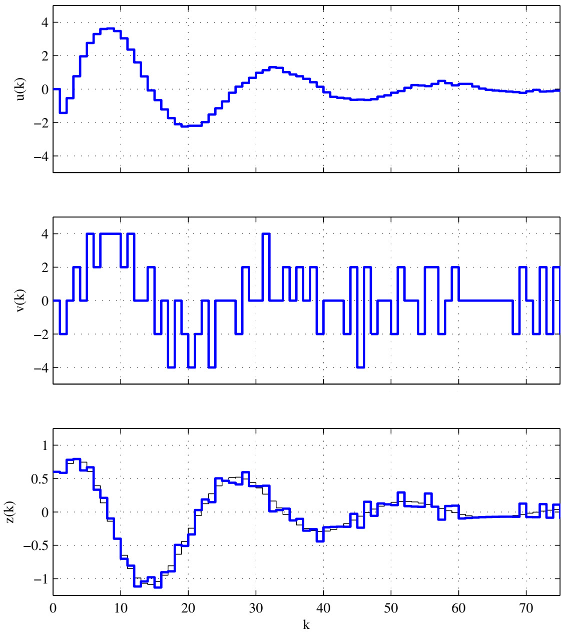


Fig. 7. Responses of Σ_{Q° for $C_{11} := [1\ \ -1]$.

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APPENDIX I

Due to the limited space, we only prove (ii) of Theorem 1. In a similar way, (i) can be proven.

$(\rightarrow)$ The contrapositive is shown as follows. If H_{ws} is not stable, there exists a vector $\zeta \in (-d/2, d/2)^m$ such that

$$\lim_{k \to \infty} \|(\mathcal{A} + \mathcal{B}_2\mathcal{C})^{k-1}\mathcal{B}_2\zeta\| = \infty. \tag{22}$$

For such a vector ζ, if the ℓ_∞-input sequence

$$u(k) = \begin{cases} -\zeta & \text{if } k = 0, \\ q(\mathcal{C}\xi(k)) - \mathcal{C}\xi(k) & \text{if } k \geq 1 \end{cases} \tag{23}$$

is applied to Q, the output is given by

$$v(k) = \begin{cases} q(-\zeta) & \text{if } k = 0, \\ q(q(\mathcal{C}\xi(k))) & \text{if } k \geq 1 \end{cases} \tag{24}$$

from (1), (2), and (11). Since $\xi(0) = 0$, $q(-\zeta) = 0$, and $q(q(\mathcal{C}\xi(k))) = q(\mathcal{C}\xi(k))$, it follows that $\xi(k) = (\mathcal{A} + \mathcal{B}_2\mathcal{C})^{k-1}\mathcal{B}_2\zeta$. This and (22) imply $\lim_{k \to \infty} \|\xi(k)\| = \infty$, which proves that Q is not stable.

$(\leftarrow)$ Consider Q in (9), and suppose that w is an independent external input to which the input sequence $W := \{w_1, w_2, \dots\} \in \ell_\infty^m$ is applied (namely, we ignore the relation (8)). Then the following statement holds:

(a) H_{ws} is stable if and only if ξ is bounded for every $(U, W) \in \mathbf{U} \times \mathbf{W}_1 := \ell_\infty^m \times \ell_\infty^m$.

On the other hand, Definition 1 (ii) implies that

(b) Q is stable if and only if ξ is bounded for every $(U, W) \in \mathbf{U} \times \mathbf{W}_2$

where $\mathbf{W}_2$ is the set of the signals $(w(0), w(1), \dots) \in \ell_\infty^m$ that are generated in Q for every $U \in \mathbf{U}$. Thus, since $\mathbf{U} \times \mathbf{W}_2 \subseteq \mathbf{U} \times \mathbf{W}_1$, (a) and (b) imply $(\leftarrow)$.

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