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Suboptimal Control of Hybrid Systems Using Approximate Multi-parametric MILP

Jalal Habibi and Ali Khaki Sedigh

***Abstract*— As a way to reduce the on-line computational burden, explicit solution to the problem of optimal control for some classes of hybrid systems can be found by reformulating the problem as multi-parametric MILP problems. The main contribution of this paper is the introduction of an approximation algorithm for solving a general class of mp-MILP problems. The algorithm wisely selects those binary sequences which make important improvement in the objective function if considered. It is shown that considerable reduction in computational complexity could be achieved by introduction of adjustable level of suboptimality. So a family of suboptimal controllers would be obtained for which the level of error and complexity can be adjusted by a tuning parameter. Several important theoretical results about approximate solutions to the mp-MILP problem are presented. It is shown that no part of the parameter space is lost during the approximation. Also it is proved that the error in the achieved approximate solutions is monotonically increasing function of the tuning parameter. The reduced complexity achieved by the proposed approach is clarified through an illustrative example.**

I. Introduction

In explicit approach to predictive control of hybrid systems a family of MILP problems can be reformulated as a multi-parametric MILP problem. As a known fact the optimizer function for the mp-MILP problem is a piecewise affine function of the parameter vector over many polyhedral regions in the parameter space [1]. So the explicit solution contains the critical regions in the parameter space and corresponding feedback gains and these can be stored in a look-up table. So the on-line computation is reduced to distinguishing the optimizer corresponding to the current value of parameters and picking it up from the look-up table.

The main drawback of the current state-of-the-art multi-parametric mixed-integer programming approaches is their computational complexity when facing real world problems. Both complexity of the exact solution and complexity of the procedures to find the exact solutions get prohibitive when the number of variables in the problem grows. This urges the research for development of approximation methods to solve these problems.

The main goal of this paper is to introduce an approximation algorithm for special class of multi-parametric mixed-integer linear programming problems and also to show how this algorithm can be used to design suboptimal controllers for hybrid systems. Parameters in this class are assumed to appear linearly in the constraint-defining inequalities. Also without loss of generality we assume the integer variables to be binary variables.

The paper is organized as follows. In the next section *exact* mp-MILP approaches are reviewed. The approximation algorithm for mp-MILP is described in section 3 as well as several theorems about optimality and complexity in the approximation approach. Section 4 contains a detailed illustrative example which further clarifies the proposed algorithm. Section 5 concludes the paper.

II. Multi-parametric MILP Algorithms

In general, the parameters in an mp-MILP problem might appear in different parts of the problem. Since solution to parametric problems in their full generality is intractable, special classes of parametric problems are usually considered.

Here we assume that the parameter vector θ appears linearly in the right-hand side of the constraints as follows:

$$z(\theta) = \min_{U_r, U_b} C_r U_r + C_b U_b \quad \text{(1a)}$$

$$\text{s.t. } G_r U_r + G_b U_b \leq W + E\theta \quad \text{(1b)}$$

$$\theta_{\min}(s) \leq \theta(s) \leq \theta_{\max}(s),\ s = 1, \ldots, n_\theta \quad \text{(1c)}$$

$U_r \in D_{U_r}$, $U_b \in \{0,1\}^{n_b}$ where U_r, U_b and θ are vectors of real and binary variables and parameters respectively. D_{U_r} is a polyhedron in R^n which denotes the region of interest for U_r and θ_{min} and θ_{max} are the vectors of lower and upper bounds on θ. The number of variables are denoted by n_r, n_b and n_θ. The vectors C_r, C_b and W and the matrices G_r, G_b and E have constant values with proper dimensions.

This family of problems is known as *the right-hand side problems* in the context of parametric programming [2]. In [3] the mp-MILP problem is solved by parametric branch and bound which explores the feasible region by solving relaxed mp-LP problems at each node of the tree.

In a more recent approach the problem is solved using a three-part procedure [4]. The feasible region is searched by successively solving a multi-parametric LP subproblem and an MILP subproblem. Since the proposed approximation algorithm is based on the mp-MILP approach from [4], the algorithm is briefly reviewed here.

Three parts in the exact mp-MILP procedure are initialization, mp-LP subproblem and MILP subproblem. The following MILP problem is solved in the initialization phase to find an initial feasible integer vector $\bar{U}_b$:

$$\min_{U_r, U_b, \theta} C_r U_r + C_b U_b \quad \text{(2a)}$$

J. Habibi is with Control and Intelligent Processing Center of Excellence, School of Electrical and Computer Engineering, University of Tehran, Tehran, Iran. j.habibi@ece.ut.ac.ir

A. Khaki Sedigh is with the Department of Electrical Engineering, K.N. Toosi University of Technology, Tehran, Iran. sedigh@kntu.ac.ir

$$\text{s.t. } G_r U_r + G_b U_b \leq W + E\theta \quad (2b)$$

$$\theta_{\min}(s) \leq \theta(s) \leq \theta_{\max}(s), \; s = 1, \ldots, n_\theta \quad (2c)$$

$U_r \in D_{U_r}$, $U_b \in \{0,1\}^{n_b}$. Here θ is considered as a vector of optimization variables. The obtained binary vector is denoted by $\bar{U}_b$. If this problem is infeasible, the procedure stops since the original parametric problem (1) is also infeasible.

In the mp-LP part of the procedure, the binary vector U_b is fixed to $\bar{U}_b$ and the space of real variables is partitioned by solving the following mp-LP problem:

$$\hat{z}(\theta) = \min_{U_r} C_r U_r + C_b \bar{U}_b \quad (3a)$$

$$\text{s.t. } G_r U_r + G_b \bar{U}_b \leq W + E\theta \quad (3b)$$

$$\theta_{\min}(s) \leq \theta(s) \leq \theta_{\max}(s), \; s = 1, \ldots, n_\theta \quad (3c)$$

$U_r \in D_{U_r}$. The optimizer and the optimal objective function for the mp-LP problem (3) are piecewise affine functions of parameter vector over polyhedral regions (which are called critical regions) as is reported in [5]. So the result of (3) is a set of linear parametric profiles $\hat{z}(\theta)^i$, a set of parametric optimizers $U_{r*}(\theta)^i$ which are affine function of the vector of parameters and corresponding critical regions, CR^i. Note that all of the optimizers $U_{r*}(\theta)^i$ correspond to the binary vector $U_b = \bar{U}_b$. Also note that $\hat{z}(\theta)^i$ s shape an upper bound for the optimal objective function.

In the third part of the algorithm an MILP problem is solved in each critical region CR^i to determine if there exists a feasible binary vector U_b yielding a lower objective function:

$$\min_{U_r, U_b, \theta} C_r U_r + C_b U_b \quad (4a)$$

$$\text{s.t. } G_r U_r + G_b U_b \leq W + E\theta \quad (4b)$$

$$C_r U_r + C_b U_b \leq \hat{z}(\theta)^i \quad (4c)$$

$$\sum_{j \in J^{ik}} (U_b)_j^{ik} - \sum_{j \in L^{ik}} (U_b)_j^{ik} \leq \left|J^{ik}\right| - 1, \quad (4d)$$

$k = 1, \ldots, K^i$, $\theta \in CR^i$, $U_r \in D_{U_r}$, $U_b \in \{0,1\}^{n_b}$. Note that the set of constraints (4d) on the elements of vector U_b is used to exclude already analyzed binary solutions from all binary candidates for the critical region CR^i [6]. Here, K^i is the number of binary solutions that have already been analyzed in CR^i and for each analyzed binary solution $(U_b)^{ik}$, the sets J^{ik} and L^{ik} are defined as follows:

$$J^{ik} = \{j \left| (U_b)_j^{ik} = 1 \right.\} \quad (5)$$

$$L^{ik} = \{j \left| (U_b)_j^{ik} = 0 \right.\} \quad (6)$$

We use $\left|J^{ik}\right|$ to denote the cardinality of the set J^{ik}.

If the MILP subproblem (4) is feasible the new candidate binary vector $\bar{U}_b$ is passed to the mp-LP part of the algorithm. This time the mp-LP is solved inside a region CR^i and the result is further partitioning of CR^i into M smaller critical regions and their corresponding profiles $\hat{z}(\theta)_m^i$ for $m = 1, \ldots, M$. In the next step the obtained parametric profiles for the region CR^i should be compared with the original profile $\hat{z}(\theta)^i$. Note that both sets of profiles are *parametric* profiles and the comparison should also be performed in a parametric manner. This in turn might provide smaller partitions. The upper bound in each region should be updated whenever a better bound is found.

The algorithm converges whenever infeasibility occurs for MILP subproblems in all critical regions. This means that no better values of the objective function can be found for the regions.

The region of parameters for which the problem is feasible is in general non-convex or even disconnected. Furthermore the optimizer and objective function, though affine in each CR^i, are in general discontinuous over the whole set of parameters.

III. Approximation Algorithm for mp-MILP Problem

Different approximation approaches might be proposed for the exact algorithm. One possibility is to use an approximation algorithm for the mp-LP subproblem as is reported in [7]. Alternatively the approximation methods could be utilized in the MILP subproblem as is selected in the current paper.

A. Basic algorithm

The proposed approach here is to neglect all feasible solutions in the MILP subproblem that do not provide *important* improvement. The idea is intuitively simple: in each critical region we continue solving the mp-LP problems only if there exists a binary feasible solution which improves the current upper bound of the objective function more than a pre-defined value μ.

For any value of μ we denote the critical regions obtained by approximation algorithm by $CR(\mu)^i$ and the number of these critical regions by $n_R(\mu)$. Also we denote the approximate parametric solution by $z(\theta, \mu)$ and parametric upper bound in region $CR(\mu)^i$ (which are obtained during the approximation algorithm) by $\hat{z}(\theta, \mu)^i$.

The approximation algorithm for the mp-MILP can be summarized as follows:

Algorithm 1: Approximation Algorithm for the mp-MILP

Step 0: Initialize the approximate solution $z(\theta, \mu)$ with ∞, the critical region $CR(\mu)^i$ with initial description of parameter space like $\theta_{\min}(s) \leq \theta(s) \leq \theta_{\max}(s)$, $s = 1, \ldots, n_\theta$.

Solve the MILP optimization problem (2). If the problem is not feasible, stop. The original mp-MILP problem is also infeasible. If (2) is feasible, choose the binary part of the solution as the initial $\bar{U}_b$.

Step 1: (mp-LP subproblem) For each critical region with an associated binary vector $\bar{U}_b$:

1-1 - Solve the mp-LP problem (3) to find a set of critical regions $CR(\mu)^i$ and their corresponding parametric solution $\hat{z}(\theta, \mu)^i$.

1-2 - For each critical region $CR(\mu)^i$, compare the parametric solution $\hat{z}(\theta, \mu)^i$ and $z(\theta, \mu)$. If $\hat{z}(\theta, \mu)^i \leq z(\theta, \mu)$, update the approximate solution for the region $CR(\mu)^i$.

Step 2: (MILP subproblem) for each critical region solve the following problem.

$$\min_{U_r, U_b, \theta} C_r U_r + C_b U_b \tag{7a}$$

$$\text{s.t. } G_r U_r + G_b U_b \leq W + E\theta \tag{7b}$$

$$C_r U_r + C_b U_b \leq z(\theta, \mu) - \mu \tag{7c}$$

$$\sum_{j \in J^{ik}} (U_b)_j^{ik} - \sum_{j \in L^{ik}} (U_b)_j^{ik} \leq \left|J^{ik}\right| - 1, \tag{7d}$$

$k = 1, \ldots, K^i$, $\theta \in CR(\mu)^i$, $U_r \in D_{U_r}$, $U_b \in \{0,1\}^{n_b}$. If the problem is infeasible the algorithm terminates in the current $CR(\mu)^i$. if not, return to step 1 with new binary solution $\bar{U}_b$ and critical region $CR(\mu)^i$. □

This algorithm speeds up the infeasibility in critical regions which in turn stops further partitioning those regions by mp-LP part of the procedure. Compared with the exact solution, the result would be a solution with fewer critical regions and thus simpler optimizers.

In the sequel some important properties of the proposed approach are introduced. The first property is about feasible region for approximate solutions which is stated below.

Theorem 1. *Assume that the exact multi-parametric MILP problem is feasible for a subset of parameter space like FR_θ. Then FR_θ is also the feasible region for the approximate parametric solutions $z(\theta, \mu)$ for any $\mu \geq 0$.*

Proof. Assume that θ_0 belongs to the feasible region of the exact mp-MILP solution, FR_θ. This means that there exists a pair of real and binary vector $(U_{r*}(\theta_0), (U_{b*}(\theta_0))$ which minimizes the mp-MILP problem for $\theta = \theta_0$.

Now consider the approximation algorithm. The $z(\theta, \mu)$ has been initialized with ∞. If a feasible solution (for a critical region containing θ_0) obtained in the initial MILP problem and first run of mp-LP problem, the theorem has been proved.

Otherwise the MILP problem in the approximation algorithm should run with $z(\theta, \mu)^i = \infty$. So for any value of μ, $(U_{r*}(\theta_0), (U_{b*}(\theta_0))$ is a feasible solution for the problem (7). This implies that the next run of mp-LP will result a feasible solution for the region containing θ_0. □

The theorem simply means that for any $\mu \geq 0$:

$$\bigcup_{i=1}^{n_R(\mu)} CR(\mu)^i = FR_\theta \tag{8}$$

This implies an important fact: no part of the feasible region of the parameter space is lost in the proposed approximation algorithm. Obviously a good approximation procedure should not exclude some parts of the feasible region, since this limits the applicability of the approximation algorithm.

We have already discussed the exact solution of the mp-MILP problem $z_*(\theta)$ and the approximate solution $z(\theta, \mu)$. To investigate the optimality aspects of the algorithm we need to define two other parametric profiles. The first one is the parametric profile $z_{relaxed}(\theta)$ which is defined to be the solution of mp-LP problem obtained by relaxation of the integrality condition on the binary variables in the mp-MILP problem. Considering the mp-MILP problem (1), we have:

$$z_{relaxed}(\theta) = \min_{U_r, U_b} C_r U_r + C_b U_b \tag{9a}$$

$$\text{s.t. } G_r U_r + G_b U_b \leq W + E\theta \tag{9b}$$

$$\theta_{\min}(s) \leq \theta(s) \leq \theta_{\max}(s), \ s = 1, \ldots, n_\theta \tag{9c}$$

$U_r \in D_{U_r}$, $U_b \in [0,1]^{n_b}$.

The second parametric function which should be defined is $z_{cover}(\theta)$ which is a parametric solution of mp-MILP which covers the FR_θ and could be obtained using the following algorithm. This algorithm aims at finding a feasible solution for mp-MILP with reduced level of complexity with respect to the exact algorithm. It will be shown later that this profile is an upper bound for all approximate profiles which has been introduced by the proposed approach.

Algorithm 2: Finding a feasible Solution for the mp-MILP problem

Step 0: Initialize the feasible solution $z_{cover}(\theta)$ with ∞, the critical region CR^i with initial parameter space like $\theta_{\min}(s) \leq \theta(s) \leq \theta_{\max}(s), \ s = 1, \ldots, n_\theta$.

Solve the MILP optimization problem (2). If the problem is not feasible, stop. The original mp-MILP problem is also infeasible. If (2) is feasible pick up the binary part of the solution as the initial $\bar{U}_b$.

Step 1: (mp-LP subproblem) for each critical region with an associated binary vector $\bar{U}_b$:

Solve the mp-LP problem (3) to find a set of critical regions CR^i and their corresponding upper bound $z_{cover}(\theta)$.

Step 2: (MILP subproblem) for each critical region if $z_{cover}(\theta)$ is still ∞, solve the problem (4). If the problem is infeasible the algorithm terminates in the current CR^i. if not, return to step 1 with new binary solution $\bar{U}_b$ and critical region CR^i. □

The algorithm simply tries to find a feasible bounded solution without improving the optimality. By choosing a *sufficiently large* value for the tuning parameter μ in algorithm 1, the algorithm 1 behaves exactly as the algorithm 2, i.e. it just considers the problem of finding a feasible bounded solution. Hence the $z_{cover}(\theta)$ is the $z(\theta, \mu)$ for sufficiently large values of μ. Quantitative description of *sufficiently large* here, will be mentioned later in Theorem 4.

Note that $z_{cover}(\theta)$ has been found with certain level of complexity (solving several mp-LP and MILP subproblems). An important open problem is to find a feasible solution for mp-MILP problem with minimum degree of computational complexity.

The following theorem states that introduced parametric profiles form an *ordered* set of profiles over FR_θ.

Theorem 2. *Consider the mp-MILP problem (1) and corresponding four parametric functions. Then:*

$\forall \mu \geq 0, \forall \theta \in FR_\theta$:

$$z_{relaxed}(\theta) \leq z_*(\theta) \leq z(\theta, \mu) \leq z_{cover}(\theta) \tag{10}$$

The proof is dropped here for the sake of brevity. □

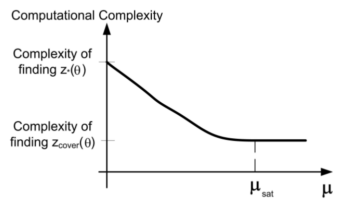

Fig. 1. Typical behavior of computational complexity in approximate solutions

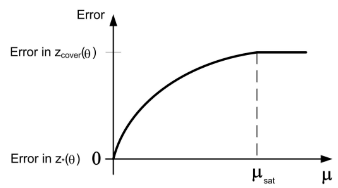

Fig. 2. Typical behavior of error in approximate solutions

Note that the above theorem introduces an upper bound and a lower bound for two main parametric solution i.e. $z_*(\theta)$ and $z(\theta,\mu)$. Finding these bounds requires less computations and this motivates to use these bounds to over-estimate the error in the approximate profile $z(\theta,\mu)$ as is discussed later.

The next theorem states monotonicity property of the error in the approximate profiles when the tuning parameter μ is changing.

Theorem 3. *Consider that two tuning parameters μ_1 and μ_2 have been used to obtain approximate parametric profiles $z(\theta,\mu_1)$ and $z(\theta,\mu_2)$ respectively. Then*:

$$\mu_1 \leq \mu_2 \Rightarrow z(\theta,\mu_1) \leq z(\theta,\mu_2), \forall\theta \in FR_\theta \qquad (11)$$

Again the proof is not mentioned here and will be reported elsewhere. □

Theorem 3 states that the approximate parametric profiles obtained by different values of μ form an *ordered* set of profiles over feasible region of the parameter space. An approximate profile with a larger value of μ is located above the other profile which is found by a smaller μ and so is worse for all values of the parameter vector.

Generally speaking the error and computational complexity of approximate parametric solutions are monotonic functions of tuning parameter μ as have been visualized in Figs. 1 and 2. Also note that the algorithm will be continued until whole of the feasible region is covered by the critical regions of the approximation algorithm. This fact determines a lower bound on the number of critical regions and when occurs the number of CR^is won't decrease any more by increasing μ and the approximation saturates.

In the sequel further topics about optimality and complexity of the approximation algorithm are discussed.

B. Measures of Suboptimality

Generally speaking the level of approximation can be controlled by the tuning parameter μ. Small values of μ result in small level of approximation and partial decrease in the number of regions. Large values of μ, *up to some extent* result in fewer regions and higher level of approximation. The approximation procedure saturates for large values of μ as will be expressed later.

In fact the designer can make a trade-off between *optimality* of the solution obtained from the approximation algorithm and the on-line and off-line *computational complexity* which affects practically important factors like storage memory and required computation time. Illustrative example mentioned in the following makes the mentioned trade-off more sensible.

For investigation of suboptimality level which is introduced due to the proposed approximation algorithm, some *measures of suboptimality* have to be defined as criteria. Here two measures are defined. First we introduce an upper bound for introduced suboptimality. This bound (which is shown to be a function of parameter μ) could be seen as *a priori measure of suboptimality*. With a known tolerable level of suboptimality, one can use this measure to select a suitable value for μ before running the approximation algorithm.

Secondly another function is introduced which is actually *a posteriori measure of suboptimality*. This measure could be computed after running the algorithm to analyze the achieved approximate parametric profile.

Before stating the upper bound theorem, we need to prove the following auxiliary lemma.

Lemma 1. *Consider the approximate parametric profile $z(\theta,\mu_0)$ obtained from proposed algorithm with fixed tuning parameter μ_0 and the exact solution of mp-MILP problem $z_*(\theta)$, then*:

$$z(\theta,\mu_0) - z_*(\theta) \leq \mu_0, \forall\theta \in FR_\theta \qquad (12)$$

Proof. The fact that the approximate profile $z(\theta,\mu_0)$ is the result of algorithm 1, implies that with this profile the MILP subproblem (7) is infeasible in all critical regions. Now for a proof based on contradiction suppose that there exists a parameter value like θ with $z(\theta,\mu_0) - z_*(\theta) > \mu_0$. This means that the the following inequality holds for the real and binary parts of $z_*(\theta)$ (i.e. $U_{r*}(\theta)$ and $U_{b*}(\theta)$):

$$C_r U_{r*}(\theta) + C_b U_{b*}(\theta) < z(\theta,\mu_0) - \mu_0 \qquad (13)$$

This in turn implies that the MILP subproblem in (7) is feasible and this contradicts the first assumption which has been made. □

Based on the previous lemma, the upper bound on the error in the approximate profiles is introduced in the following theorem. Here the saturation of the upper bound is considered.

Theorem 4. *Consider the approximate parametric profile $z(\theta,\mu)$ and the exact solution of the mp-MILP problem $z_*(\theta)$, then:*

$$z(\theta,\mu) - z_*(\theta) \leq SF_1(\mu), \forall\theta \in FR_\theta \qquad (14)$$

where:

$$SF_1(\mu) = \min\{\mu, \mu_{sat}\} \qquad (15)$$

and

$$\mu_{sat} = \max_{\theta} [z_{cover}(\theta) - z_{relaxed}(\theta)] \qquad (16)$$

Proof. Lemma 1 proves that the error is always less than μ. Here it should be proved that for $\mu > \mu_{sat}$ a better upper bound for error in the approximate profile can be provided. This can be easily seen from aforementioned fact about four parametric profiles: we have $z(\theta,\mu) \le z_{cover}(\theta)$ and also $-z_*(\theta) \le -z_{relaxed}(\theta)$. Summing up these inequalities and taking $\max(.)$ of the result yields the μ_{sat} relation. □

As mentioned earlier, $SF_1(\mu)$ could be used to select proper value of the tuning parameter μ for a specified level of error in the approximate parametric profile.

The second measure of suboptimality for the approximation algorithm is defined as follows:

$$SF_2(\mu) = \frac{\int\limits_{FR_\theta} |z_*(\theta) - z(\theta,\mu)|\, d\theta}{\int\limits_{FR_\theta} |z_*(\theta)|\, d\theta} \tag{17}$$

The numerator of SF_2 is the volume between two parametric profiles $z_*(\theta)$ and $z(\theta,\mu)$. The denominator is simply the volume defined by the optimal profile $z_*(\theta)$.

The value of SF_2 for the exact solution ($\mu = 0$) is zero as is expected and it means no suboptimality in the exact solution of the mp-MILP problem. Note that SF_2 is *a posteriori* criterion for the approximation level and could be used to analyze the quality of an approximate solution *after* running the algorithm.

C. Achieved complexity reduction

Decrease in the computational complexity in the approximation algorithm is achieved by reduced number of mp-LP and MILP problems which should be solved in original exact algorithm. These values can be treated as measures for complexity of the *algorithm*. The number of critical regions in the obtained parametric profile can be treated as a measure for complexity of the *solution*.

The following theorem states that these measures of complexity are monotonic decreasing functions of the tuning parameter μ.

Theorem 5. *Assume two approximate parametric profiles $z(\theta,\mu_1)$ and $z(\theta,\mu_2)$ have been obtained during the proposed approach with $n_R(\mu_1)$ and $n_R(\mu_2)$ critical regions respectively . Also assume that the number of mp-LP and MILP subproblems which have to be solved for these profiles be $n_{mpLP}(\mu_1)$, $n_{MILP}(\mu_1)$, $n_{mpLP}(\mu_2)$ and $n_{MILP}(\mu_2)$ respectively. Then*:

$$\begin{aligned} \mu_1 \le \mu_2 \Rightarrow n_R(\mu_1) &\ge n_R(\mu_2), \\ n_{mpLP}(\mu_1) &\ge n_{mpLP}(\mu_2), \\ n_{MILP}(\mu_1) &\ge n_{MILP}(\mu_2) \end{aligned} \tag{18}$$

As expected this theorem states that among all approximate solutions the exact solution of the mp-MILP problem $z_*(\theta) = z(\theta,\mu = 0)$ has the highest number of regions and requires highest number of mp-LP and MILP subproblems to be solved. It can also be shown that the reduction in the computational complexity will be saturated at some point and can not be improved when the approximate profile reaches $z_{cover}(\theta)$. The proofs are omitted here for the sake of brevity.

IV. An Illustrative Example

A typical case study for hybrid systems is selected here to highlight the power of the proposed approximate mp-MILP algorithm in obtaining low-complexity controllers. The control problem of a piece-wise affine system is considered and it is assumed that the system has linear second order dynamics in different regions of the state space as described below.

Region 1: $x_1 \ge x_2$ and $x_1 \ge -x_2$.
Region 2: $x_1 < x_2$ and $x_1 \ge -x_2$.
Region 3: $x_1 < x_2$ and $x_1 < x_2$.
Region 4: $x_1 \ge x_2$ and $x_1 < x_2$.

Corresponding second order dynamics for different regions are assumed as follows:
$H_1(s) = \frac{s+1}{s^2+5s+10}$, $H_2(s) = \frac{s+2}{s^2+13s+5}$,
$H_3(s) = \frac{s+2.5}{s^2+15s+8}$, $H_4(s) = \frac{s+5}{s^2+3s+3}$

The continuous subsystems are discretized with the sampling time 100 msec. The control action and the state vector elements are assumed to be subject to hard constraints as $|u| \le 1.5$ and $|x_i| \le 5\,(i = 1,2)$. Here the MLD framework is selected to model this system [8]. The modeling procedure is straight forward and an MLD model with 5 binary variables and 2 state variables and 1 input is obtained as follows.

$$\begin{aligned} &x[k+1] = Ax[k] + B_1u[k] + B_2\delta[k] + B_3z[k] \\ &E_2\delta[k] + E_3z[k] \le E_1u[k] + E_4x[k] + E_5 \end{aligned} \tag{19}$$

where the state, input and output of the system are denoted by x, u and y and binary and auxiliary variables by δ and z respectively.

The control task is assumed to be the regulation of the state vector toward the origin while fulfilling the input and state constraints. To do this with predictive control strategy we assume that the following objective function has to be minimized at each time instant and the first element of control input sequence obtained in the optimization is applied to the system.

$$\begin{aligned} &J(\{U\}_0^{N-1}, x(0\,|t)) = \sum_{k=0}^{N-1} \|u(k\,|t) - u_e\|_{Q_1}^1 + \\ &\|\delta(k|t) - \delta_e\|_{Q_3}^1 + \|z(k|t) - z_e\|_{Q_3}^1 + \|x(k|t) - x_e\|_{Q_4}^1 \end{aligned} \tag{20}$$

where N is the control horizon, $\{U\}_0^{N-1}$ denotes the optimization variables over control horizon, $x(0\,|t)$ is the measured state vector at time t and x_e, u_e, z_e and δ_e are steady state values for state, input and auxiliary variables.

The problem can be reformulated as an mp-MILP problem using standard optimization techniques and with $N = 3$ as control horizon the result is an mp-MILP problem with 15 binary variables, 30 real variables and 2 parameters. For solving mp-LP subproblems the MPT toolbox [9] and for MILP subproblems the CPLEX [10] have been used.

Now the result of exact solution to the current mp-MILP problem is the partitioning of the state space as shown in Fig. 3. The solution contains 284 critical regions. The value of μ_{sat} for this problem is about 7.5 which could be easily found from simulation. Partitioning of the state

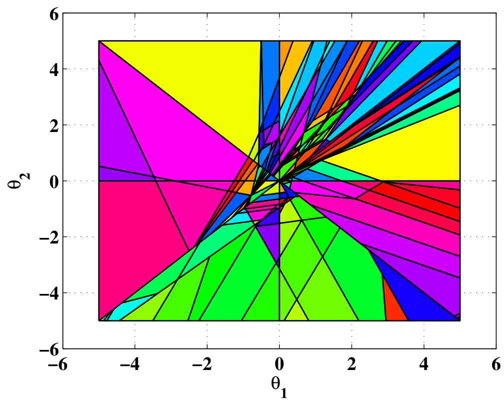

Fig. 3. Critical regions for the parametric solution of mp-MILP in the illustrative example

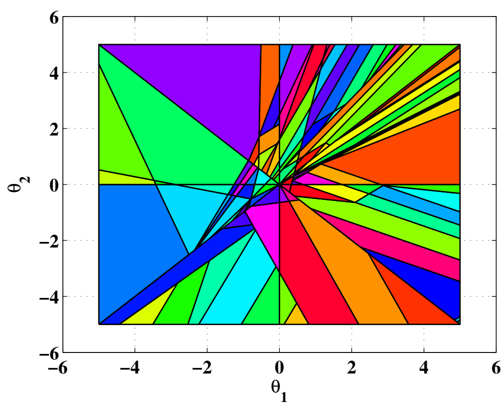

Fig. 4. Approximate critical regions obtained with $\mu = 7.5$, $n_R = 126$

space with this value has been depicted in Fig. 4. Here the approximate algorithm is run for several values of the tuning parameter μ. To show the computational benefits achieved by the proposed approach, the number of MILP subproblems (n_{MILP}), the number of mp-LP subproblems (n_{mpLP}), the time required to find the explicit solution ($T(\mu)$) and a posteriori measure of suboptimality ($SF_2(\mu)$) for several values of the tuning parameter have been compared in Table 1. The results are found using a machine with Intel Pentium M 1.6 GHz CPU with 500 MB of RAM. The values in the table clarify that as stated earlier the error in the approximate profiles is monotonically increasing function of the tuning parameter and the computational complexity measures are monotonically decreasing functions of μ.

TABLE I
PERFORMANCE MEASURES OF PROPOSED ALGORITHM FOR THE ILLUSTRATIVE EXAMPLE.

μ	$n_R(\mu)$	n_{MILP}	n_{mpLP}	$T(\mu)$ (sec)	$SF_2(\mu)$
0	284	2007	717	989.4	0
0.5	224	634	149	176.6	0.2 %
1	186	318	71	65.1	0.29 %
2	172	264	54	47.3	1.06 %
7.5	126	178	34	26.8	11.78 %

Also it can be seen that for this special example the proposed approach at the best case reduces the number of critical regions by a factor of 2.25 and also provides considerable reduction in the number of MILP and mp-LP subproblems. In the approximate solution obtained by $\mu = \mu_{sat}$ these parameters (n_{MILP} and n_{mpLP}) are respectively 11 and 21 times better than the exact solution. The time complexity in solving the mpMILP problem has also been reduced by a factor of 36. The important point is that whole these computational benefits have been achieved by loosing less that 12 % of optimality.

V. CONCLUSION

Although the basic idea in the proposed approximate multi-parametric MILP solver is intuitively simple, nice theoretical properties about approximate solutions have been presented in the present article. It is shown that the feasible region of the approximate solution coincides with the feasible region of the exact solution. Also it is proved that the error in the approximate solutions and also the computational complexity in finding the solution and in its final representation can be monotonically adjusted using the tuning parameter.

However as is pointed out in the paper, the problem of finding a feasible solution to the mp-MILP problem with least level of computational complexity is still an open important problem.

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