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On the use of semi-invariants for the stability analysis of planar systems

Laura Menini and Antonio Tornambè

***Abstract*— This paper is concerned with the concept of semi-invariant and its use for the stability analysis of the origin for nonlinear systems of order two. Well known tools from differential geometry such as (orbital) symmetries render the proposed results quite interesting from a computational point of view. The proper connections with center manifold theory are pointed out.**

I. INTRODUCTION

The goal of this paper is to propose a technique for the construction in closed-form of Lyapunov functions for stability analysis; time-invariant nonlinear systems of order two (planar systems) are considered. The proposed Lyapunov functions are based on some special functions, the semi-invariants, that generalize to nonlinear systems the concept of left-eigenvector for (the dynamic matrix of) linear systems. The semi-invariants are well known in nonlinear dynamics (see, e.g., [1], [2]) and are referred to with various names, e.g., second integrals or Darboux polynomials. In this paper, the very strong connection between semi-invariants and (orbital) symmetries is exploited to derive computationally attractive techniques for the stability analysis of nonlinear systems.

Symmetries are another well-known concept from differential geometry; the concept of (orbital) symmetry of a differential equation was introduced by S. Lie [3] in the second half of the 19-th century, as an attempt of generalizing the theory of Galois, and it was primarily used for the solution in closed-form of differential equations. In [4], S. Lie proved that a planar system, described by a pair of first-order time-invariant differential equations, (or, equivalently, one time-varying differential equation) admits an inverse integrating factor, whence by quadrature a (non-trivial) first integral, if and only if it admits a (non-trivial) orbital symmetry. Modern reference on the subject can be found, e.g., in [5]-[9].

II. NOTATIONS AND BACKGROUND

Consider two vector functions $f(x), g(x) \in \mathbb{R}^2$ and the corresponding planar systems described by the ordinary differential equations (from now on, the dependencies on times $t, \tau \in \mathbb{R}$ are omitted, if not necessary):

$$\frac{dx}{dt} = f(x), \quad x \in \mathcal{U}_f \subseteq \mathbb{R}^2, \tag{1}$$

$$\frac{dx}{d\tau} = g(x), \quad x \in \mathcal{U}_g \subseteq \mathbb{R}^2, \tag{2}$$

where $\mathcal{U}_f$ and $\mathcal{U}_g$ are open and connected subsets of $\mathbb{R}^2$; assume that $0 \in \mathcal{U}_f$, $f(0) = 0$ and that $\mathcal{U}_f \cap \mathcal{U}_g \neq \emptyset$. For the sake of simplicity, in the remainder of the paper, with $\mathcal{U}$ we will denote $\mathcal{U}_f$ or $\mathcal{U}_f \cap \mathcal{U}_g$ if both systems (1) and (2) are to be jointly considered; moreover, we will assume that all the functions are analytic in some open and connected subset of $\mathbb{R}^2$, not necessarily containing the origin of $\mathbb{R}^2$, unless otherwise specified. This implies that systems (1) and (2) have unique maximal solutions $x(t) = \Phi_f(t, x_0)$ and $x(\tau) = \Phi_g(\tau, x_0)$, respectively, which are defined in an open and connected subset $\mathcal{V} \subseteq \mathbb{R} \times \mathbb{R}^2$, including $\{0\} \times \mathcal{U}$. Let $A := \left.\frac{\partial f}{\partial x}\right|_{x=0}$; then, Ax is called the *linear part* of f; f is said to have *zero linear part* if $A = 0$.

The *Lie derivative* of a scalar function h by f is $L_f h := \frac{\partial h}{\partial x} f$; the *Lie bracket* $[f, g]$ of f by g is $[f, g] := \frac{\partial g}{\partial x} f - \frac{\partial f}{\partial x} g$. A *first integral* of system (1) is a scalar function $I(x) : \mathcal{U}^* \to \mathbb{R}$, analytic in $\mathcal{U}^*$, such that $L_f I = 0, \forall x \in \mathcal{U}^*$, with $\mathcal{U}^*$ being an open and connected subset of $\mathcal{U}$; if I is a constant, then the first integral is *trivial*.

The flow $y = \Phi_g(\tau, x)$ is a *symmetry* (respectively, an *orbital symmetry*) of system (1) and system (2) is its *infinitesimal generator* if $[f, g] = 0$ (respectively, $[f, g] = \mu f$), $\forall x \in \mathcal{U}^*$, with $\mathcal{U}^*$ being an open and connected subset of $\mathcal{U}$ and μ being a scalar function analytic in $\mathcal{U}^*$. With a little abuse of notation, g is also called a symmetry (respectively, an orbital symmetry) of f; g is said to be *non-trivial* if it is not colinear with f.

III. PRELIMINARY TECHNICAL RESULTS

We first recall some well known facts about the local integration of a one-form (see Chapter 1 of [10]).

A one-form in $\mathbb{R}^2$, $\beta = \begin{bmatrix} \beta_1 & \beta_2 \end{bmatrix}$, with $\beta_1(x), \beta_2(x) \in \mathbb{R}$, is *closed* if $\frac{\partial \beta_1}{\partial x_2} = \frac{\partial \beta_2}{\partial x_1}$; any closed one-form β is *locally exact*, i.e., there exists a (not necessarily unique) scalar function h such that $\frac{\partial h}{\partial x} = \beta$, in some open and connected subset of $\mathbb{R}^2$. A scalar function ω, not identically null, is an *inverse integrating factor* of the one-form β, if the one-form $\frac{1}{\omega}\begin{bmatrix} \beta_1 & \beta_2 \end{bmatrix}$ is closed; in the planar case, an inverse integrating factor always exists in a neighborhood of any regular point of the one-form (i.e., around any $x_0 \in \mathbb{R}^2$ such that $\beta(x_0) \neq 0$) (see Theorem 1.15 of [10]). A function ω, not identically null, is an *inverse integrating factor* of system (1) if the one-form $\frac{1}{\omega}\begin{bmatrix} f_2 & -f_1 \end{bmatrix}$ is closed: in such a case, there exists a first integral $I(x)$ of system (1) such that $\frac{\partial I}{\partial x_1} = \frac{f_2}{\omega}$, $\frac{\partial I}{\partial x_2} = -\frac{f_1}{\omega}$, in some open and connected subset of $\mathbb{R}^2$.

The following Lemmas 1-2 are classical (see, e.g. [1]).

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The authors are with Dip. Informatica, Sistemi e Produzione, Università di Roma Tor Vergata, Via del Politecnico, 1, 00133 Roma, Italy [menini,tornambe]@disp.uniroma2.it

Lemma 1: A function ω is an inverse integrating factor of system (1) if and only if one of the following two equivalent conditions holds:

$$\operatorname{div}\left(\tfrac{1}{\omega}f\right)=0, \quad \forall x\in\mathcal{U}:\omega(x)\neq 0, \tag{3}$$
$$\operatorname{div}(f)=\tfrac{1}{\omega}L_f\omega, \quad \forall x\in\mathcal{U}:\omega(x)\neq 0. \quad \square \tag{4}$$

Let $\omega=\det\left[\begin{array}{cc} f & g\end{array}\right]$; if ω is not identically equal to 0 (f and g are not colinear), then, by linear algebra, we have (see Statement (a) of Proposition 1.1 of [1])

$$[f,g]=\left(\operatorname{div}(g)-\tfrac{1}{\omega}L_g\omega\right)f+\left(-\operatorname{div}(f)+\tfrac{1}{\omega}L_f\omega\right)g;$$

if $[f,g]=\mu f$ holds, then (since f and g are not colinear) we have $\operatorname{div}(f)=\frac{1}{\omega}L_f\omega$ and ω is an inverse integrating factor of system (1). Such a reasoning allows one to prove Lemma 2, given by S. Lie in [4] (see also [3]).

Lemma 2: If g is an orbital symmetry of f and $\omega=\det\left[\begin{array}{cc} f & g\end{array}\right]$ is not identically equal to 0, then ω is an inverse integrating factor of system (1). If ω is an inverse integrating factor of system (1), then any g, such that $\det\left[\begin{array}{cc} f & g\end{array}\right]=\omega$ holds, is an orbital symmetry of f. $\square$

If ω is an inverse integrating factor and I is a first integral of system (1), then all the orbital symmetries of f are given by $g=\left[\begin{array}{cc} g_1 & g_2\end{array}\right]^T$, with g_1,g_2 such that $\omega G(I)=f_1g_2-f_2g_1$, where $G(I)$ is an arbitrary (not null) function of I; if f_1 is not the null function, then

$$g=\left[\begin{array}{cc} g_1 & \frac{\omega G(I)+f_2g_1}{f_1}\end{array}\right]^T \tag{5}$$

parameterizes all the (non-trivial) orbital symmetries of f, with g_1 being an arbitrary function of x (a similar expression can be obtained if f_2 is not the null function).

Since in the planar case an inverse integrating factor ω always exists, letting I be the first integral of system (1) such that $\frac{\partial I}{\partial x}=\frac{1}{\omega}\left[\begin{array}{cc} f_2 & -f_1\end{array}\right]$, then any planar system admits as orbital symmetries, for instance if f_1 is not the null function, all the g parameterized by (5). In particular, since a first integral of a planar system always exists in a neighborhood of any regular point x_0, $f(x_0)\neq 0$, then an orbital symmetry always exists around x_0. An orbital symmetry g may be not defined in a singular point x_0, $f(x_0)=0$, but this has not consequences for the subsequent developments: the only requirement is that g is analytic in some open and connected domain, not necessarily containing the equilibrium points of f. Nevertheless, the difficulty for the computation in closed-form of an orbital symmetry is the same as that for the computation in closed-form of an inverse integrating factor.

In the following, we describe some classes of planar systems for which an orbital symmetry can be easily found; note that once an orbital symmetry of f is known in closed-form, then we know in closed-form an inverse integrating factor ω, whence we know, by quadrature, a first integral I; then, for instance by (5), we can parameterize all the orbital symmetries of f. In addition, note that, for the mere computation of an orbital symmetry g, a common factor $\alpha(x)$ between the two entries f_1 and f_2 of f can be dropped out. Moreover, if, by a suitable change of coordinates $y=T(x)$, we bring $f(x)$ into a simpler form $\tilde{f}(y)$ and we are able to compute an orbital symmetry $\tilde{g}(y)$ of $\tilde{f}(y)$, then we have found an orbital symmetry $g(x)$ of $f(x)$ by expressing $\tilde{g}(y)$ into the original coordinates x.

S1) *Homogeneous systems*. Given a vector of integers $r=\left[\begin{array}{cc} r_1 & r_2\end{array}\right]^T$ (r_1 and r_2 are called weights), an *integer dilation* $\delta_\varepsilon^r x$ is defined as $\delta_\varepsilon^r x:=\left[\begin{array}{cc} \varepsilon^{r_1}x_1 & \varepsilon^{r_2}x_2\end{array}\right]^T$, for any scalar real $\varepsilon\neq 0$. A scalar function $h(x):\mathbb{R}^2\to\mathbb{R}$ is homogeneous of degree $m\in\mathbb{Z}$, with respect to $\delta_\varepsilon^r x$, if $h(\delta_\varepsilon^r x)=\varepsilon^m h(x)$, whenever defined. A vector function $f:=\left[\begin{array}{cc} f_1 & f_2\end{array}\right]^T:\mathbb{R}^2\to\mathbb{R}^2$ is homogeneous of degree $m\in\mathbb{Z}$, with respect to $\delta_\varepsilon^r x$, if f_i is homogeneous of degree r_i-m with respect to $\delta_\varepsilon^r x$, $i=1,2$ (see Sections 1.1 and 1.2 of [11]; see also [12], [13] and Section 5.3 of [14]), namely if $f_i(\varepsilon^{r_1}x_1,\varepsilon^{r_2}x_2)=\varepsilon^{r_i-m}f_i(x_1,x_2), i=1,2$. Let $g=\left[\begin{array}{cc} r_1x_1 & r_2x_2\end{array}\right]^T$; if f is homogeneous of degree m with respect to $\delta_\varepsilon^r x$ (with the integer m being possibly negative), then g is an orbital symmetry of f (a symmetry if $m=0$): in particular, $[f,g]=mf$ (see [15], [16]). If $[f,g]=mf$ for some g (not necessarily of the form $g=\left[\begin{array}{cc} r_1x_1 & r_2x_2\end{array}\right]^T$), with $m\in\mathbb{Z}$, then f is said to be homogeneous of degree m with respect to g [15]. Notice that a linear f is homogeneous of degree 0 with respect to the standard dilation $\delta_\varepsilon^r x$, with $r=\left[\begin{array}{cc} 1 & 1\end{array}\right]^T$. The inverse integrating factor corresponding to the pair f, $g=\left[\begin{array}{cc} r_1x_1 & r_2x_2\end{array}\right]^T$ such that $[f,g]=mf$, with $m\in\mathbb{Z}$, is $\omega=\det\left[\begin{array}{cc} f_1 & r_1x_1\\ f_2 & r_2x_2\end{array}\right]=r_2x_2f_1-r_1x_1f_2$: notice that, such an ω is homogeneous of degree r_1+r_2-m with respect to $\delta_\varepsilon^r x$, with $r=\left[\begin{array}{cc} r_1 & r_2\end{array}\right]^T$. Assume that all the weights r_i are positive. If the scalar function h is homogeneous of degree m with respect to $\delta_\varepsilon^r x$ and analytic at $x=0$, then $m\geq 0$ (actually, h in this case is polynomial). If the vector function f is homogeneous of degree m with respect to $\delta_\varepsilon^r x$ and analytic at $x=0$, then $r_i-m\geq 0$, $i=1,...,n$, which implies $m\leq\min\{r_1,...,r_n\}$. Any vector function f analytic at $x=0$ can be expanded in Taylor series about $x=0$ and all the resulting monomials can be grouped together according to their degree of homogeneity with respect to $\delta_\varepsilon^r x$ so that $f=\sum_{i\leq m^*}f^{[i]}$, where $m^*\leq\min\{r_1,...,r_n\}$ and $f^{[i]}$ is homogeneous of degree i with respect to $\delta_\varepsilon^r x$; the term of maximum degree $f^{[m^*]}$ is called the *first approximation* with respect to the dilation $\delta_\varepsilon^r x$. This first approximation is useful in the study of the stability of the origin, because (see the classical references [17], [18] when $m^*=0$ and the dilation is standard, Chapter III of [17] when the dilation is standard, [11], [13], [14] and references therein for the general case) if the origin is asymptotically stable for system $\dot{x}=f^{[m^*]}$, then it is also asymptotically stable for $\dot{x}=f$. The techniques proposed in this paper can be applied to the first approximation (which necessarily admits the orbital symmetry mentioned above) to find a Lyapunov function that can be used for the whole system.

S2) *Systems, with a semi-simple linear part, in the Poincaré-Dulac normal form.* A vector function $f(x)$ is in the Poincaré-Dulac normal form (briefly, PD-normal form) if $f(x) = Ax + h(x)$, with A being semi-simple and $h(x)$ having zero linear part and satisfying the condition $[Ax, h(x)] = 0$ (see [19]). Since condition $[Ax, h(x)] = 0$ implies $0 = [Ax, Ax + h(x)] = -[f(x), Ax]$, then $g = Ax$ is a symmetry of $f(x)$ (notice that $[Ax, h(x)] = 0 \iff h(e^{At}x) = e^{At}h(x)$, see equation (5.7) of [20]). Any f with a semi-simple linear part can be *formally* transformed into its normal form through a *formal* series $y = T(x)$; under some convergence conditions $T(x)$ is analytic (see [21]). If we do not require any convergence condition, then by the Borel Lemma [22], there exists a C^∞-transformation such that the transformed f differs from its PD-normal form for a *flat* vector function; whence, for any arbitrarily high integer $m > 0$, there exists a polynomial diffeomorphism such that the transformed f differs from its PD-normal form for terms of order higher than m. The inverse integrating factor corresponding to the pair $f(x) = Ax + h(x)$, $g(x) = Ax$ is $\omega = \det \begin{bmatrix} Ax + h(x) & Ax \end{bmatrix} = \det \begin{bmatrix} h(x) & Ax \end{bmatrix}$.

S3) *A first integral I associated with f is known.* Since $L_f I = 0$, then the gradient $\frac{\partial I}{\partial x}$ is colinear with $\begin{bmatrix} f_2 & -f_1 \end{bmatrix}$, i.e., there exists a non-null scalar function ω such that $\frac{\partial I}{\partial x} = \frac{1}{\omega} \begin{bmatrix} f_2 & -f_1 \end{bmatrix}$, namely ω is an inverse integrating factor. Then, if $\frac{\partial I}{\partial x_1}$ is not the null function, an orbital symmetry g of f is $g = \begin{bmatrix} \frac{1}{\frac{\partial I}{\partial x_1}} & 0 \end{bmatrix}^T = \begin{bmatrix} \frac{\omega}{f_2} & 0 \end{bmatrix}^T$; if $\frac{\partial I}{\partial x_2}$ is not the null function, then an orbital symmetry g of f is $g = \begin{bmatrix} 0 & \frac{1}{\frac{\partial I}{\partial x_2}} \end{bmatrix}^T = \begin{bmatrix} 0 & -\frac{\omega}{f_1} \end{bmatrix}^T$.

The classes of systems described above are just what is needed for the developments to follow. Other special classes of systems for which it is possible to derive interesting results include classes of systems in the Belitskii normal form [20], Hamiltonian systems, systems with the separations of variables property and classes of Lienard systems. As for the Belitskii normal form, it is a generalization of the PD-normal form, not requiring that A is semi-simple; notice that the three systems S^A, S^B and S^C studied in the subsequent Example 1 are in the Belitskii normal form.

IV. SEMI-INVARIANTS AND LYAPUNOV FUNCTIONS

The semi-invariants (using the name given in [1]) are widely studied in the literature under various names, such as: semi-invariants, second integrals, special integrals (polynomials), eigenpolynomials, Darboux polynomials (curves), algebraic invariant curves (manifolds), particular algebraic solutions; an introductory reference is Section 2.5 of [2] in case of polynomial semi-invariants, with polynomial characteristic function. In this section, the semi-invariants are used as elementary bricks for the construction, in closed-form, of Lyapunov functions, to be used for the stability analysis.

Definition 1: A *semi-invariant* of system (1) is a scalar function $\omega(x) : \mathcal{U} \to \mathbb{R}$, for some $\mathcal{U}$ open and connected, analytic in $\mathcal{U}$, such that

$$L_f \omega = \lambda \omega, \quad \forall x \in \mathcal{U},$$

with $\lambda(x) \in \mathbb{R}$ being a ratio of functions analytic in $\mathcal{U}$, $\lambda(x) = \frac{N(x)}{D(x)}$, where $D(x) \neq 0$ in $\mathcal{U}$; if $\omega(x)$ and $\lambda(x)$ are polynomial in $x \in \mathcal{U}$, then ω is said to be a *Darboux polynomial*; $\lambda(x)$ is called the *characteristic function* (respectively, *characteristic polynomial*) of the semi-invariant (respectively, of the Darboux polynomial); if $\omega(x)$ is a constant, then the semi-invariant is said to be *trivial*, *non-trivial* otherwise. Let $\mathcal{I}_\omega := \{x \in \mathcal{U} : \omega(x) = 0\}$. □

Since $\dot{\omega} = \lambda \omega$ along the dynamics of system (1), we have $\omega(x(t)) = \omega(x(0)) \exp\left(\int_0^t \lambda(x(\tau)) d\tau\right)$ for each $(t, x(0)) \in \mathcal{V}$. If $\omega(x(0)) = 0$, then $\omega(x(t)) = 0$ for each $t \geq 0$ such that $\exp\left(\int_0^t \lambda(x(\tau)) d\tau\right)$ is finite; since $x(\tau)$ is finite for each $(\tau, x(0)) \in \mathcal{V}$ and $D(x) \neq 0$, $\forall x \in \mathcal{I}_\omega$, if $\omega(x(0)) = 0$, then $\omega(x(t)) = 0$ for each $(t, x(0)) \in \mathcal{V}$, namely the set of points $\mathcal{I}_\omega$ is invariant for system (1).

The following theorem, restated from [2] (see, also, [23]), characterizes Darboux polynomials (notice that some of its statements, with proper amendments, also apply to non-polynomial semi-invariants).

Theorem 1: Assume that f is polynomial.

(1) If $I(x) = \frac{\omega_1(x)}{\omega_2(x)}$ is a (non-trivial) first integral of system (1), with ω_1 and ω_2 being coprime (non-constant) polynomials, then ω_1 and ω_2 are Darboux polynomials of system (1), with the same characteristic polynomials $\lambda_1(x) = \lambda_2(x)$.

(2) Let $\omega(x)$, $\omega_1(x)$ and $\omega_2(x)$ be Darboux polynomials of system (1) with respective characteristic polynomials $\lambda(x)$, $\lambda_1(x)$ and $\lambda_2(x)$; then, all the irreducible factors of $\omega(x)$ are Darboux polynomials of system (1), and the product $\omega_1^{n_1}(x)\omega_2^{n_2}(x)$ is a Darboux polynomial of system (1) for any pair of integers n_1, n_2, with characteristic polynomial $n_1\lambda_1(x) + n_2\lambda_2(x)$. □

Let

$$\omega(x) := \det \begin{bmatrix} f(x) & g(x) \end{bmatrix}; \tag{6}$$

if $\omega(x)$ is not the null function and g is an orbital symmetry of f, then $L_f \omega = \text{div}\,(f)\,\omega$, whence, since $\text{div}(f)$ is analytic in $\mathcal{U}$, ω is a semi-invariant, with characteristic function $\text{div}\,(f)$. If $\omega = \omega_1\omega_2$, then there exists λ_1 and λ_2 such that $L_f\omega_1 = \lambda_1\omega_1$, $L_f\omega_2 = \lambda_2\omega_2$ and $\lambda_1 + \lambda_2 = \text{div}(f)$: hence, if λ_i is analytic for each $x \in \mathcal{I}_{\omega_i}$, $i = 1, 2$, then ω_1 and ω_2 are semi-invariants with characteristic functions λ_1 and λ_2. If f and g are polynomial, then ω is a Darboux polynomial, as well as its irreducible factors.

There may be a strong connection between a semi-invariant (respectively, a Darboux polynomial) and the center manifold (see [24]-[19]). As already mentioned, if ω is a semi-invariant, then the manifold described by $\omega = 0$ is invariant. Assume that the matrix A of the linear part of f

has two real eigenvalues $\lambda_1 \neq 0$ and $\lambda_2 = 0$; call *center* the subspace of $\mathbb{R}^2$ spanned by the eigenvector with eigenvalue λ_2. If $\omega = 0$ is tangent with the center at $x = 0$, then $\omega = 0$ is the center manifold. Such planar systems can be studied easily either by the Shoshitaishvili Theorem [26], [27], or by using the PD-normal form. To this end, assume that the linear part of f is such that $A = diag\,(0, b)$, with $b \neq 0$. Assume that $f(x)$ can be transformed into its PD-normal form $\tilde{f}(y)$ by $y = T(x)$. The linear centralizer of A (i.e., the set of all matrices that commute with A) is spanned by $\{E, A\}$, being E the identity matrix; all the first integrals of $\dot{y} = Ay$ are of the form $I = G(y_1)$, with G being an arbitrary function of the argument. Then,

$$\tilde{f} = Ay + \psi y + \varphi Ay = \left[\begin{array}{cc} \psi y_1 & by_2 + y_2 (\psi + \varphi b) \end{array} \right]^T,$$

with ψ and φ being arbitrary functions of y_1. A symmetry $\tilde{g}$ of $\tilde{f}$ is given by $\tilde{g} = Ay = \left[\begin{array}{cc} 0 & by_2 \end{array} \right]^T$. The corresponding inverse integrating factor is $\omega = b\psi y_1 y_2$, giving two Darboux polynomials $\omega_1 = y_1$ and $\omega_2 = y_2$. The center manifold is described by $\omega_2 = 0$, and $\dot{y}_1 = y_1 \psi(y_1)$ is the corresponding reduced system. For $b < 0$, the origin is asymptotically stable for the given system if and only if it is such for the reduced system, i.e., if and only if $\psi(y_1) < 0$ for all $y_1 \neq 0$ belonging to a neighborhood of $y_1 = 0$; this can be verified, in the original coordinates, using as a Lyapunov function $V = \frac{1}{2}\omega_1^2 + \frac{1}{2}\omega_2^2$. As a consequence, if a given system (1) has a linear part with eigenvalues 0 and $b \neq 0$, then, if the transformation $T(x)$ is convergent [resp., formal], the system has at least two [resp., formal] semi-invariants that coincide with the entries of $T(x)$.

For linear systems, the Darboux polynomials are strictly correlated with the left eigenvectors of the dynamic matrix A: in particular, if $u^T A = \lambda u^T$, with $\lambda \in \mathbb{R}$ and $u \neq 0$, then $\omega = u^T x$ is a Darboux polynomial of system $\dot{x} = Ax$, with a constant characteristic function λ. Easy computations show that, letting $\omega(x) = \det[f(x)\ g(x)]$, with $f = Ax$ and $g = x$, one can use as a Lyapunov function for proving the (possibly asymptotic) stability of the origin either $V = \omega$ in the case of complex eigenvalues, or, in the case of real, negative and distinct, eigenvalues, the function $V = \frac{1}{2}(\omega_1^2 + \omega_2^2)$, being ω_1 and ω_2 the two irreducible factors of ω. The case of two real and coincident eigenvalues is to be dealt with by means of a different reasoning if A is not semi-simple. This way of using Darboux polynomials for the stability analysis can be extended to the nonlinear case, as shown in the following example, used to motivate the subsequent Theorem 2.

Example 1: Assume that f is polynomial and homogeneous of degree -2 with respect to the dilation $\delta_\varepsilon^r x$, with weights $r = \left[\begin{array}{cc} 1 & 3 \end{array} \right]^T$,

$$f = \left[\begin{array}{c} a_1 x_2 + a_2 x_1^3 \\ a_3 x_1^5 + a_4 x_1^2 x_2 \end{array} \right];$$

notice that the linear part of f is nilpotent (its linear approximation cannot be directly used for stability analysis). Letting $g = \left[\begin{array}{cc} x_1 & 3x_2 \end{array} \right]^T$, by construction $[f, g] = -2f$; then, a Darboux polynomial is given by the inverse integrating factor

$$\omega = \det \left[\begin{array}{cc} f & g \end{array} \right] = 3a_1 x_2^2 + (3a_2 - a_4)\, x_2 x_1^3 - a_3 x_1^6.$$

Since $\omega = 0$ is an invariant set, then the possible curves obtained by letting $\omega = 0$ divide the plane into open sectors such that if the initial state is in one of these sectors then the state remains there for all times. Other Darboux polynomials are given by the possible irreducible factors of ω, depending on the values of the parameters a_i's. Consider the following three cases:

$$\begin{array}{lcl} S^A & : & f^A(x) = \left[\begin{array}{cc} x_2 - x_1^3 & -x_2 x_1^2 \end{array} \right]^T, \\ S^B & : & f^B(x) = \left[\begin{array}{cc} x_2 - x_1^3 & -x_2 x_1^2 + x_1^5 \end{array} \right]^T, \\ S^C & : & f^C(x) = \left[\begin{array}{cc} x_2 - x_1^3 & -x_2 x_1^2 + \frac{8}{3} x_1^5 \end{array} \right]^T; \end{array}$$

the respective inverse integrating factors are:

$$\begin{array}{rcl} \omega^A & = & x_2 \left(3x_2 - 2x_1^3\right), \\ \omega^B & = & \left(3x_2 + x_1^3\right)\left(x_2 - x_1^3\right), \\ \omega^C & = & \frac{1}{3}\left(3x_2 + 2x_1^3\right)\left(3x_2 - 4x_1^3\right). \end{array}$$

By computing the irreducible factors of the inverse integrating factor, one has two Darboux polynomials in each case:

$$\begin{array}{rcl} \omega_1^A & = & x_2, \quad \omega_2^A = 3x_2 - 2x_1^3, \\ \omega_1^B & = & 3x_2 + x_1^3, \quad \omega_2^B = x_2 - x_1^3, \\ \omega_1^C & = & 3x_2 + 2x_1^3, \quad \omega_2^C = 3x_2 - 4x_1^3, \end{array}$$

with respective characteristic functions:

$$\begin{array}{rcl} \lambda_1^A & = & -x_1^2, \quad \lambda_2^A = -3x_1^2, \\ \lambda_1^B & = & 0, \quad \lambda_2^B = -4x_1^2, \\ \lambda_1^C & = & x_1^2, \quad \lambda_2^C = -5x_1^2. \end{array}$$

In Case A, choosing the Lyapunov function $V^A = \frac{1}{2}\left(\omega_1^A\right)^2 + \frac{1}{2}\left(\omega_2^A\right)^2 = \frac{1}{2}x_2^2 + \frac{1}{2}\left(3x_2 - 2x_1^3\right)^2$, one has $\dot{V}^A = -x_1^2\left(\omega_1^A\right)^2 - 3x_1^2\left(\omega_2^A\right)^2 = -x_1^2 x_2^2 - 3x_1^2\left(3x_2 - 2x_1^3\right)^2$, which is negative semi-definite and, therefore, shows that the origin is stable; the further remark that the origin is the largest invariant set contained in $\dot{V}^A = 0$ shows, by Krasowskii-LaSalle Theorem (see Theorems 3 and 4 of [28]; in the following, briefly, KLS theorem), that the origin is asymptotically stable (since V^A is radially unbounded, then the origin is globally asymptotically stable).

In case B, choosing the Lyapunov function $V^B = \frac{1}{2}\left(\omega_1^B\right)^2 + \frac{1}{2}\left(\omega_2^B\right)^2 = \frac{1}{2}\left(3x_2 + x_1^3\right)^2 + \frac{1}{2}\left(x_2 - x_1^3\right)^2$, one has $\dot{V}^B = -4x_1^2\left(\omega_2^B\right)^2 = -4x_1^2\left(x_2 - x_1^3\right)^2$, which is negative semi-definite and, therefore, shows that the origin is stable; since the curve described by $\omega_1^B = c$ (namely, $x_2 = -\frac{1}{3}x_1^3 + \frac{c}{3}$) is invariant for any real c (because $\dot{\omega}_1^B = 0$), it does not pass through the origin for $c \neq 0$, and for $c \neq 0$ arbitrarily small it passes through points arbitrarily close to $x = 0$, then the origin is not attractive.

In case C, instability of the origin can be proven by means of Chetaev's theorem (see [18]) using $V = \frac{1}{2}(\omega_1^C)^2 - \frac{1}{2}(\omega_2^C)^2$, since in the set $\mathcal{A} = \{x_1 > 0 \text{ and } x_2 > \frac{1}{3}x_1^3\}$ one has both $V > 0$ and $\dot{V} > 0$, and $V = 0$ for $x \in \partial\mathcal{A}$.

Notice that f^A contains monomials of degree less than or equal to 3 with respect to the standard dilation, whereas f^B and f^C are obtained from f^A by adding a term of higher degree with respect to the standard dilation; in particular, the origin of S^A, which is asymptotically stable, is rendered simply stable by adding to f^A the term $h^B = \begin{bmatrix} 0 & x_1^5 \end{bmatrix}^T$ ($f^B = f^A + h^B$) and unstable by adding to f^A the term $h^C = \begin{bmatrix} 0 & \frac{8}{3}x_1^5 \end{bmatrix}^T$ ($f^C = f^A + h^C$). Actually, this has been simply done because f^A and the additional terms h^B, h^C have the same degree with respect to the chosen dilation, with weights $r_1 = 1$, $r_2 = 3$. □

The following theorem gives conditions for the stability analysis of the origin for system (1) and, when it can be applied, gives also a Lyapunov function in closed-form. The proof is omitted for space reasons.

Theorem 2: Assume that f is polynomial, with $f(0) = 0$, and that there exists a polynomial orbital symmetry g of f.

(1) Let the inverse integrating factor ω in (6) be irreducible. If the only solution of $\omega^2(x) = 0$ in a neighborhood of the origin is $x = 0$ and if $\operatorname{div}(f) \leq 0$ in a neighborhood of the origin, then the origin is stable for system (1). If the greatest invariant set contained in $\operatorname{div}(f)\,\omega^2 = 0$ is $x = 0$, then the origin is asymptotically stable.

(2) Let $\omega_i, i = 1, 2, ..., m$, $m \geq 1$, be the irreducible factors of the inverse integrating factor ω in (6), with λ_i being the corresponding characteristic polynomials. Let $\lambda := \begin{bmatrix} \lambda_1 & \lambda_2 & \dots & \lambda_m \end{bmatrix}^T$; if there exist $k \geq 1$ row vectors of positive integers $h_i = \begin{bmatrix} h_{i,1} & h_{i,2} & \dots & h_{i,m} \end{bmatrix}$, $i = 1, 2, ..., k$, such that $\tilde{\lambda}_i := h_i\lambda \leq 0$ in a neighborhood of the origin, and if $x = 0$ is the only solution of $\sum_{i=1}^k \tilde{\omega}_i^2 = 0$, with $\tilde{\omega}_i = \prod_{\ell=1}^m \omega_\ell^{h_{i,\ell}}$, then the origin is stable for system (1). If the greatest invariant set contained in $\sum_{i=1}^k \tilde{\lambda}_i\tilde{\omega}_i^2 = 0$ is $x = 0$, then the origin is asymptotically stable.

(3) Let $\omega_i, i = 1, 2, ..., m$, be the irreducible factors of the inverse integrating factor given in (6), with λ_i being the corresponding characteristic polynomials. Let $\lambda := \begin{bmatrix} \lambda_1 & \lambda_2 & \dots & \lambda_m \end{bmatrix}^T$; if there exists a row vector of positive integers $h = \begin{bmatrix} h_1 & h_2 & \dots & h_m \end{bmatrix}$ such that $\tilde{\lambda} := h\lambda \geq 0$ in a neighborhood of the origin, and $\left.\frac{\partial\tilde{\omega}}{\partial x}\right|_{x=0} \neq 0$, with $\tilde{\omega} = \prod_{\ell=1}^m \omega_\ell^{h_\ell}$, then the origin is not attractive. □

The following three examples illustrate the applicability of Theorem 2 based on the computation of symmetries.

Example 2: Let $f = \begin{bmatrix} -x_1x_2^2 \\ x_1^3 - \frac{1}{10}x_2^3 \end{bmatrix}$; f is homogeneous of degree -2 with respect to $g = \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T$, and the corresponding inverse integrating factor is $\omega = \omega_1\omega_2\omega_3$ with

$$\begin{aligned} \omega_1 &= x_1, \quad \omega_2 = x_1 + \sqrt[3]{\tfrac{9}{10}}x_2, \\ \omega_3 &= x_1^2 - \sqrt[3]{\tfrac{9}{10}}x_1x_2 + \sqrt[3]{\tfrac{81}{100}}x_2^2, \end{aligned}$$

and respective characteristic functions:

$$\begin{aligned} \lambda_1 &= -x_2^2, \quad \lambda_2 = \sqrt[3]{\tfrac{9}{10}}x_1^2 - \sqrt[3]{\tfrac{81}{100}}x_1x_2 - \tfrac{1}{10}x_2^2, \\ \lambda_3 &= -\sqrt[3]{\tfrac{9}{10}}x_1^2 + \sqrt[3]{\tfrac{81}{100}}x_1x_2 - \tfrac{1}{5}x_2^2. \end{aligned}$$

It can be seen that both λ_2 and λ_3 are not definite nor semidefinite, whereas both λ_1 and $\tilde{\lambda}_2 := \lambda_2 + \lambda_3 = -\frac{3}{10}x_2^2$ are negative semidefinite. Hence, the positive definite function:

$$V := \tfrac{1}{2}\omega_1^2 + \tfrac{1}{2}(\omega_2\omega_3)^2 = \tfrac{1}{2}x_1^2 + \tfrac{1}{2}\left(x_1^3 + \tfrac{9}{10}x_2^3\right)^2,$$

(obtained using statement (2) of Theorem 2, with $h_1 = [1, 0, 0]$ and $h_2 = [0, 1, 1]$) is such that

$$\dot{V} = -x_2^2x_1^2 - \tfrac{3}{10}x_2^2\left(x_1^3 + \tfrac{9}{10}x_2^3\right)^2.$$

Using KLS theorem, it is easy to prove the asymptotic stability of the origin. □

Example 3: Let f be in the PD-normal form, with linear part described by $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. Since the linear centralizer of A is spanned by $\{E, A\}$, being E the identity matrix, and since all the first integrals of $\dot{x} = Ax$ are of the form $G(x_1^2 + x_2^2)$, with G being an arbitrary function of the argument, then vector function f is given by $f = Ax + \psi x + \varphi Ax$, with ψ and φ being arbitrary functions of $x_1^2 + x_2^2$. Furthermore, $g = Ax$ is a symmetry of f and the corresponding inverse integrating factor is

$$\omega = \det\begin{bmatrix} x_2 + \psi x_1 + \varphi x_2 & x_2 \\ -x_1 + \psi x_2 - \varphi x_1 & -x_1 \end{bmatrix} = -\left(x_1^2 + x_2^2\right)\psi.$$

The semi-invariant $\omega_1 = x_1^2 + x_2^2$ has characteristic function $\lambda = 2\psi$. If $\psi(\xi)$ is assumed analytic at $\xi = 0$, then in a neighborhood of the origin we have $\psi(\xi) \approx a\xi^n$; if $a < 0$ and integer n is even, then, with the Lyapunov function $V = \frac{1}{2}\omega^2 = \frac{1}{2}\left(x_1^2 + x_2^2\right)^2$ having derivative $\dot{V} = 2\psi\omega^2 \approx 2a\left(x_1^2 + x_2^2\right)^{n+2}$, it is easy to see that the origin is asymptotically stable (exponentially if $n = 0$), independently of function φ. □

As for the instability of the origin, next theorem can be stated. Its proof is omitted for space reasons.

Theorem 3: Let $\omega_i(x_1, x_2)$ be a semi-invariant associated with system (1), such that $\omega_i(0, 0) = 0$. Assume that $\omega_i(x_1, x_2) = 0$ can be locally (around $(x_1, x_2) = 0$) rendered explicit with respect to one of the two variables, say x_1 without loss of generality; in particular, assume that there exists a function $\varphi_i(\cdot)$, $\varphi_i(0) = 0$, and an interval $\mathcal{I} = [0,\ \delta)$ (or, respectively, $\mathcal{I} = (-\delta,\ 0]$) such that $(\varphi_i(x_2), x_2) \in \mathcal{U}, \forall x_2 \in \mathcal{I}$ and $\omega_i(x_1, x_2) = 0 \Leftrightarrow x_1 = \varphi_i(x_2)$ for each $x_2 \in \mathcal{I}$. Consider the reduced system $\dot{x}_2 = h_i(x_2)$, with

$h_i(x_2) := f_2(\varphi_i(x_2), x_2)$. If $x_2 h_i(x_2) > 0$ for all $x_2 \in (0, \delta)$ (respectively, $x_2 \in (-\delta, 0)$), then the origin of system (1) is unstable; if $h_i(x_2) = 0$ for all $x_2 \in (0, \delta)$ (respectively, $x_2 \in (-\delta, 0)$), for a sufficiently small $\delta > 0$, then the origin of system (1) is not attractive. □

Theorem 3 generalizes what can be done by means of the center manifold theory, as shown in the following example, in which the center manifold coincides with the state space.

Example 4: Consider again the system S^C of Example 1. The equation $\omega_2^C = 3x_2 - 4x_1^3 = 0$ can be locally rendered explicit with respect to x_2, obtaining $x_2 = \varphi_2(x_1) = \frac{4}{3}x_1^3$; the corresponding reduced system is $\dot{x}_1 = h_2(x_1) = \frac{1}{3}x_1^3$. Since $x_1 h_2(x_1) = \frac{1}{3}x_1^4$ is positive for any $x_1 \neq 0$, then the origin of S^C is unstable.

Consider again system S^B of Example 1. The equation $\omega_2^B = x_2 - x_1^3 = 0$ can be locally rendered explicit with respect to x_2, obtaining $x_2 = \varphi_2(x_1) = x_1^3$; the corresponding reduced system is $\dot{x}_1 = h_2(x_1) = 0$. Since $h_2 = 0$ for all x_1, then the origin of S^B is not attractive. □

V. Conclusions

In this paper it has been shown how the semi-invariants of a planar system can be used for stability analysis of planar systems. The proposed technique is based on the use of (orbital) symmetries for the computation of semi-invariants, and on their use as elementary bricks for the explicit construction of Lyapunov functions. The main advantage of the techniques proposed in this paper is that they rely on the knowledge of geometric properties of the system, that can be related to its symmetries.

VI. References

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