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Convergence of Sampled-data Consensus Algorithms for Double-integrator Dynamics

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Abstract— **This paper studies convergence of two consensus algorithms for double-integrator dynamics with intermittent interaction in a sampled-data setting. The first algorithm guarantees that a team of vehicles reaches consensus on their positions with a zero final velocity while the second algorithm guarantees that a team of vehicles reaches consensus on their positions with a constant final velocity. We show conditions on the sampling period and the control gain such that consensus is reached using these two algorithms over, respectively, an undirected interaction topology and a directed interaction topology. In particular, necessary and sufficient conditions are shown in the case of undirected interaction while sufficient conditions are shown in the case of directed interaction. Consensus equilibria for both algorithms are also given.**

I. INTRODUCTION

Distributed multi-vehicle cooperative control has received significant attention in the control community in recent years. Consensus plays an important role in achieving distributed multi-vehicle cooperative control. The basic idea of consensus is that a team of vehicles reaches an agreement on a common value by negotiating with their neighbors. Consensus algorithms for single-integrator kinematics have been studied extensively in the literature (see [1] and references therein).

Taking into account the fact that equations of motion of a broad class of vehicles require a double-integrator dynamic model, consensus algorithms for double-integrator dynamics are studied in [2]–[9]. In particular, [2]–[4] derive conditions on the interaction topology and the control gains under which convergence is guaranteed. Refs. [5], [6] study formation keeping problems while [7]–[9] study flocking of multiple vehicle systems. All these algorithms are studied in a continuous-time setting.

In multi-vehicle cooperative control, vehicles may only be able to exchange information periodically but not continuously, which results in discrete-time or sampled-data formulation. Current discrete-time consensus algorithms are primarily studied for first-order kinematic models [10]–[12]. The algorithms are essentially distributed weighted averaging algorithms [13]–[15]. Few works study consensus algorithms for double-integrator dynamics in a sampled-data setting with a notable exception in [16], where a sampled-data algorithm is studied for double-integrator dynamics through average-energy-like Lyapunov functions. The analysis in [16] is limited to an undirected interaction topology. However, in cooperative control applications, information flow may often be directed, either due to heterogeneity, nonuniform communication powers, or sensing with a limited field of view. The case of directed interaction is much more challenging than that of undirected interaction.

In this paper, we study convergence of two sampled-data consensus algorithms for double-integrator dynamics. The first algorithm guarantees that a team of vehicles reaches consensus on their positions with a zero final velocity while the second algorithm guarantees that a team of vehicles reaches consensus on their positions with a constant final velocity. We show conditions on the sampling period and the control gain such that consensus is reached using these two algorithms over, respectively, an undirected interaction topology and a directed interaction topology. In particular, necessary and sufficient conditions are shown in the case of undirected interaction while sufficient conditions are shown in the case of directed interaction. Consensus equilibria for both algorithms are also given. In contrast to [16], our analysis is based on algebraic graph theory and matrix theory rather than a Lyapunov approach. Our results generalize the convergence conditions derived in [16].

II. BACKGROUND AND PRELIMINARIES

A. Graph Theory Notions

It is natural to model interaction among vehicles by directed or undirected graphs. Suppose that a team consists of n vehicles. A weighted graph $\mathcal{G}$ consists of a node set $\mathcal{V} = \{1, \ldots, n\}$, an edge set $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$, and a weighted adjacency matrix $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{n\times n}$. An edge (i, j) in a weighted directed graph denotes that vehicle j can obtain information from vehicle i, but not necessarily vice versa. In contrast, the pairs of nodes in a weighted undirected graph are unordered, where an edge (i, j) denotes that vehicles i and j can obtain information from one another. Weighted adjacency matrix $\mathcal{A}$ of a weighted directed graph is defined such that a_{ij} is a positive weight if $(j, i) \in \mathcal{E}$, while $a_{ij} = 0$ if $(j, i) \notin \mathcal{E}$. Weighted adjacency matrix $\mathcal{A}$ of a weighted undirected graph is defined analogously except that $a_{ij} = a_{ji}$, $\forall i \neq j$, since $(j, i) \in \mathcal{E}$ implies $(i, j) \in \mathcal{E}$.

A directed path is a sequence of edges in a directed graph of the form $(i_1, i_2), (i_2, i_3), \ldots$, where $i_j \in \mathcal{V}$. An undirected path in an undirected graph is defined analogously. A directed graph has a directed spanning tree if there exists at least one node having a directed path to all other nodes. An undirected graph is connected if there is an undirected path between every pair of distinct nodes.

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Let the (nonsymmetric) Laplacian matrix $\mathcal{L} = [\ell_{ij}] \in \mathbb{R}^{n\times n}$ associated with $\mathcal{A}$ be defined as [17] $\ell_{ii} = \sum_{j=1,j\neq i}^{n} a_{ij}$ and $\ell_{ij} = -a_{ij}$, $i \neq j$. For an undirected graph, $\mathcal{L}$ is symmetric positive semi-definite. However, $\mathcal{L}$ for a directed graph does not have this property. In both the undirected and directed cases, 0 is an eigenvalue of $\mathcal{L}$ with associated eigenvector $\mathbf{1}_n$, where $\mathbf{1}_n$ is the $n \times 1$ column vector of all ones.

B. Continuous-time Consensus Algorithms for Double-integrator Dynamics

Consider vehicles with double-integrator dynamics given by

$$\dot{r}_i = v_i, \qquad \dot{v}_i = u_i, \quad i = 1, \ldots, n, \tag{1}$$

where $r_i \in \mathbb{R}^m$ and $v_i \in \mathbb{R}^m$ are, respectively, the position and velocity of the ith vehicle, and $u_i \in \mathbb{R}^m$ is the control input.

A consensus algorithm for (1) is studied in [3], [18] as

$$u_i = -\sum_{j=1}^{n} a_{ij}(r_i - r_j) - \alpha v_i, \quad i = 1, \ldots, n, \tag{2}$$

where a_{ij} is the (i,j)th entry of weighted adjacency matrix $\mathcal{A}$ associated with graph $\mathcal{G}$ and α is a positive gain introducing absolute damping. Consensus is reached for (2) if for all $r_i(0)$ and $v_i(0)$, $r_i(t) \to r_j(t)$ and $v_i(t) \to 0$ as $t \to \infty$.

A consensus algorithm for (1) is studied in [2] as

$$u_i = -\sum_{j=1}^{n} a_{ij}[(r_i - r_j) + \alpha(v_i - v_j)], \quad i = 1, \ldots, n, \tag{3}$$

where a_{ij} is defined as in (2) and α is a positive gain introducing relative damping. Consensus is reached for (3) if for all $r_i(0)$ and $v_i(0)$, $r_i(t) \to r_j(t)$ and $v_i(t) \to v_j(t)$ as $t \to \infty$.

C. Sampled-data Consensus Algorithms for Double-integrator Dynamics

In a sampled-data setting, following [16], we let

$$u_i(t) = u_i[k], \quad kT \leq t \leq (k+1)T, \tag{4}$$

where k denotes the discrete-time index, T denotes the sampling period, and $u_i[k]$ is the control input at $t = kT$. Discretizing (1) with sampling period T, gives

$$\begin{aligned} r_i[k+1] &= r_i[k] + Tv_i[k] + \frac{T^2}{2}u_i[k] \\ v_i[k+1] &= v_i[k] + Tu_i[k], \end{aligned} \tag{5}$$

where $r_i[k]$ and $v_i[k]$ are the position and velocity of the ith vehicle at $t = kT$.

We study the following two algorithms

$$u_i[k] = -\sum_{j=1}^{n} a_{ij}(r_i[k] - r_j[k]) - \alpha v_i[k], \tag{6}$$

which corresponds to continuous-time algorithm (2) and

$$u_i[k] = -\sum_{j=1}^{n} a_{ij}[(r_i[k] - r_j[k]) + \alpha(v_i[k] - v_j[k])], \tag{7}$$

which corresponds to continuous-time algorithm (3). Note that [16] shows conditions for (7) over an undirected interaction topology through average-energy-like Lyapunov functions. Relying on algebraic graph theory and matrix theory, we will show necessary and sufficient conditions for convergence of both (6) and (7) over an undirected interaction topology and show sufficient conditions for convergence of both (6) and (7) over a directed interaction topology.

In the remainder of the paper, for simplicity, we suppose that $r_i \in \mathbb{R}$, $v_i \in \mathbb{R}$, and $u_i \in \mathbb{R}$. However, all results still hold for $r_i \in \mathbb{R}^m$, $v_i \in \mathbb{R}^m$, and $u_i \in \mathbb{R}^m$ by use of the properties of the Kronecker product.

III. Convergence Analysis of the Sampled-data Algorithm with Absolute Damping

In this section, we analyze algorithm (6) over, respectively, an undirected and a directed interaction topology. Before moving on, we need the following lemmas:

Lemma 3.1 (Schur's formula): Let $A, B, C, D \in \mathbb{R}^{n\times n}$. Let $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$. Then $\det(M) = \det(AD - BC)$, where $\det(\cdot)$ denotes the determinant of a matrix, if A, B, C, and D commute pairwise.

Lemma 3.2: Let $\mathcal{L}$ be the nonsymmetric Laplacian matrix (respectively, Laplacian matrix) associated with directed graph $\mathcal{G}$ (respectively, undirected graph $\mathcal{G}$). Then $\mathcal{L}$ has a simple zero eigenvalue and all other eigenvalues have positive real parts (respectively, are positive) if and only if $\mathcal{G}$ has a directed spanning tree (respectively, is connected). In addition, there exist $\mathbf{1}_n$ satisfying $\mathcal{L}\mathbf{1}_n = 0$ and $\mathbf{p} \in \mathbb{R}^n$ satisfying $\mathbf{p} \geq 0$, $\mathbf{p}^T\mathcal{L} = 0$, and $\mathbf{p}^T\mathbf{1} = 1$.[1]

Proof: See [19] for the case of undirected graphs and [12] for the case of directed graphs. ∎

Lemma 3.3: [20, Lemma 8.2.7 part(i), p. 498] Let $A \in \mathbb{R}^{n\times n}$ be given, let $\lambda \in \mathbb{C}$ be given, and suppose x and y are vectors such that (i) $Ax = \lambda x$, (ii) $A^T y = \lambda y$, and (iii) $x^T y = 1$. If $|\lambda| = \rho(A) > 0$, where $\rho(A)$ denotes the spectral radius of A, and λ is the only eigenvalue of A with modulus $\rho(A)$, then $\lim_{m\to\infty}(\lambda^{-1}A)^m \to xy^T$.

Using (6), (5) can be written in matrix form as

$$\begin{bmatrix} r[k+1] \\ v[k+1] \end{bmatrix} = \underbrace{\begin{bmatrix} I_n - \frac{T^2}{2}\mathcal{L} & (T - \frac{\alpha T^2}{2})I_n \\ -T\mathcal{L} & (1-\alpha T)I_n \end{bmatrix}}_{F} \begin{bmatrix} r[k] \\ v[k] \end{bmatrix}, \tag{8}$$

where $r = [r_1, \ldots, r_n]^T$, $v = [v_1, \ldots, v_n]^T$, and I_n denote the $n \times n$ identity matrix. To analyze (8), we first study the property of F. Note that the characteristic polynomial of F

[1]That is, $\mathbf{1}_n$ and $\mathbf{p}$ are, respectively, the right and left eigenvectors of $\mathcal{L}$ associated with the zero eigenvalue.

is given by

$$\begin{aligned}&\det(sI_{2n}-F)\\ =&\det\left(\begin{bmatrix} sI_n-(I_n-\frac{T^2}{2}\mathcal{L}) & -(T-\frac{\alpha T^2}{2})I_n\\ T\mathcal{L} & sI_n-(1-\alpha T)I_n\end{bmatrix}\right)\\ =&\det\big([sI_n-(I_n-\frac{T^2}{2}\mathcal{L})][sI_n-(1-\alpha T)I_n]\\ &-(T\mathcal{L}[-(T-\frac{\alpha T^2}{2})I_n])\big)\\ =&\det\left((s^2-2s+\alpha Ts+1-\alpha T)I_n+\frac{T^2}{2}(1+s)\mathcal{L}\right)\end{aligned}$$

where we have used Lemma 3.1 to obtain the second to the last equality.

Letting μ_i be the ith eigenvalue of $-\mathcal{L}$, we get $\det(sI_n+\mathcal{L}) = \prod_{i=1}^n(s-\mu_i)$. It thus follows that $\det(sI_{2n}-F) = \prod_{i=1}^n\left(s^2-2s+\alpha Ts+1-\alpha T-\frac{T^2}{2}(1+s)\mu_i\right)$. Therefore, the roots of $\det(sI_{2n}-F)=0$ (i.e., the eigenvalues of F) satisfy

$$s^2+(\alpha T-2-\frac{T^2}{2}\mu_i)s+1-\alpha T-\frac{T^2}{2}\mu_i=0. \tag{9}$$

Note that each eigenvalue of $-\mathcal{L}$, μ_i, corresponds to two eigenvalues of F, denoted by λ_{2i-1} and λ_{2i}.

Without loss of generality, let $\mu_1=0$. It follows from (9) that $\lambda_1=1$ and $\lambda_2=1-\alpha T$. Therefore, F has at least one eigenvalue equal to one. Let $[p^T,q^T]^T$, where $p,q\in\mathbb{R}^n$, be the right eigenvector of F associated with eigenvalue $\lambda_1=1$. It follows that $\begin{bmatrix} I_n-\frac{T^2}{2}\mathcal{L} & (T-\frac{\alpha T^2}{2})I_n\\ -T\mathcal{L} & (1-\alpha T)I_n\end{bmatrix}\begin{bmatrix}p\\q\end{bmatrix}=\begin{bmatrix}p\\q\end{bmatrix}$. After some manipulation, it follows from Lemma 3.2 that we can choose $p=\mathbf{1}_n$ and $q=\mathbf{0}_n$, where $\mathbf{0}_n$ is the $n\times 1$ column vector of all zeros. Similarly, it can be shown that $[\mathbf{p}^T,(\frac{1}{\alpha}-\frac{T}{2})\mathbf{p}^T]^T$ is a left eigenvector of F associated with eigenvalue $\lambda_1=1$.

Lemma 3.4: Using (6) for (5), $r_i[k]\to r_j[k]\to \mathbf{p}^Tr[0]+(\frac{1}{\alpha}-\frac{T}{2})\mathbf{p}^Tv[0]$ and $v_i[k]\to 0$ as $k\to\infty$ if and only if one is the unique eigenvalue of F with maximum modulus.

Proof: (Sufficiency.) Note that $x=[\mathbf{1}_n^T,\mathbf{0}_n^T]^T$ and $y=[\mathbf{p}^T,(\frac{1}{\alpha}-\frac{T}{2})\mathbf{p}^T]^T$ are, respectively, a right and left eigenvector of F associated with eigenvalue one. Also note that $x^Ty=1$. If one is the unique eigenvalue with maximum modulus, then it follows from Lemma 3.3 that $\lim_{k\to\infty}F^k \to \begin{bmatrix}\mathbf{1}_n\\ \mathbf{0}_n\end{bmatrix}[\mathbf{p}^T,(\frac{1}{\alpha}-\frac{T}{2})\mathbf{p}^T]$. Therefore, it follows that $\lim_{k\to\infty}\begin{bmatrix}r[k]\\v[k]\end{bmatrix}=\lim_{k\to\infty}F^k\begin{bmatrix}r[0]\\v[0]\end{bmatrix}=\begin{bmatrix}r[0]+(\frac{1}{\alpha}-\frac{T}{2})\mathbf{p}^Tv[0]\\ \mathbf{0}_n\end{bmatrix}$.

(Necessity.) Note that F can be written in Jordan canonical form as $F=PJP^{-1}$, where J is the Jordan block matrix. If $r_i[k]\to r_j[k]\to\mathbf{p}^Tr[0]+(\frac{1}{\alpha}-\frac{T}{2})\mathbf{p}^Tv[0]$ and $v_i[k]\to 0$ as $k\to\infty$, it follows that $\lim_{k\to\infty}F^k\to\begin{bmatrix}\mathbf{1}_n\\ \mathbf{0}_n\end{bmatrix}[\mathbf{p}^T,(\frac{1}{\alpha}-\frac{T}{2})\mathbf{p}^T]$, which has rank one. It thus follows that $\lim_{k\to\infty}J^k$ has rank one, which implies that all but one eigenvalue are within the unit circle. Noting that F has at least one eigenvalue equal to one, it follows that one is the unique eigenvalue of F with maximum modulus. ∎

A. Undirected Interaction

In this subsection, we show necessary and sufficient conditions on α and T such that consensus is reached using (6) over an undirected interaction topology. Note that all eigenvalues of $\mathcal{L}$ are real for undirected graphs.

Lemma 3.5: The polynomial

$$s^2+as+b=0, \tag{10}$$

where $a,b\in\mathbb{C}$, has all roots within the unit circle if and only if all roots of

$$(1+a+b)t^2+2(1-b)t+b-a+1=0 \tag{11}$$

are in the open left half plane (LHP).

Proof: By applying bilinear transformation $s=\frac{t+1}{t-1}$ [21], polynomial (10) can be rewritten as $(t+1)^2+a(t+1)(t-1)+b(t-1)^2=0$, which implies (11). Note that the bilinear transformation maps the open LHP one-to-one onto the interior of the unit circle. The lemma follows directly. ∎

Lemma 3.6: Suppose that undirected graph $\mathcal{G}$ is connected. All eigenvalues of F, where F is defined in (8), are within the unit circle except one eigenvalue equal to one if and only if α and T are chosen from the set

$$S_r=\bigcap_{\forall\mu_i\le 0}\{(\alpha,T)|-\frac{T^2}{2}\mu_i<\alpha T<2,\},^2 \tag{12}$$

where $\bigcap$ denotes the intersection of sets.

Proof: When undirected graph $\mathcal{G}$ is connected, it follows from Lemma 3.2 that $\mu_1=0$ and $\mu_i<0$, $i=2,\ldots,n$. Because $\mu_1=0$, it follows that $\lambda_1=1$ and $\lambda_2=1-\alpha T$. To ensure $|\lambda_2|<1$, it is required that $0<\alpha T<2$.

Let $a=\alpha T-2-\frac{T^2}{2}\mu_i$ and $b=1-\alpha T-\frac{T^2}{2}\mu_i$. It follows from Lemma 3.5 that for $\mu_i<0, i=2,\cdots,n$, the roots of (9) are within the unit circle if and only if all roots of

$$-T^2\mu_it^2+(T^2\mu_i+2\alpha T)t+4-2\alpha T=0 \tag{13}$$

are in the open LHP. Because $-T^2\mu_i>0$, the roots of (13) are always in the open LHP if and only if $T^2\mu_i+2\alpha T>0$ and $4-2\alpha T>0$, which implies that $-\frac{T^2}{2}\mu_i<\alpha T<2$, $i=2,\ldots,n$. Combining the above arguments proves the lemma. ∎

Theorem 3.1: Suppose that undirected graph $\mathcal{G}$ is connected. Let $\mathbf{p}$ be defined in Lemma 3.2. Using (6) for (5), $r_i[k]\to r_j[k]\to\mathbf{p}^Tr[0]+(\frac{1}{\alpha}-\frac{T}{2})\mathbf{p}^Tv[0]$ and $v_i[k]\to 0$ as $k\to\infty$ if and only if α and T are chosen from S_r, where S_r is defined by (12).

Proof: The statement follows directly from Lemmas 3.4 and 3.6. ∎

[2]Note that S_r is nonempty.

B. Directed Interaction

In this subsection, we show sufficient conditions on α and T such that consensus is reached using (6) over a directed interaction topology. Note that the eigenvalues of $\mathcal{L}$ may be complex for directed graphs, which makes the analysis more challenging.

Lemma 3.7: [22], [23] All the zeros of the complex polynomial $P(z) = z^n + \alpha_1 z^{n-1} + \ldots + \alpha_{n-1} z + \alpha_n$ satisfy $|z| \leq r_0$, where r_0 is the unique nonnegative solution of the equation $r^n - |\alpha_1| r^{n-1} - \ldots - |\alpha_{n-1}| r - |\alpha_n| = 0$. The bound r_0 is attained if $\alpha_i = -|\alpha_i|$.

Corollary 3.2: All roots of polynomial (10) are within the unit circle if $|a| + |b| < 1$. Moreover, if $|a+b| + |a-b| < 1$, all roots of (10) are still within the unit circle.

Proof: According to Lemma 3.7, the roots of (10) are within the unit circle if the unique nonnegative solution s_0 of $s^2 - |a|s - |b| = 0$ satisfies $s_0 < 1$. It is straightforward to show that $s_0 = \frac{|a| + \sqrt{|a|^2 + 4|b|}}{2}$. Therefore, the roots of (10) are within the unit circle if

$$|a| + \sqrt{|a|^2 + 4|b|} < 2. \tag{14}$$

We next discuss the condition under which (14) holds. If $b = 0$, then the statements of the corollary hold trivially. If $|b| \neq 0$, we have $\frac{(|a| + \sqrt{|a|^2+4|b|})(-|a| + \sqrt{|a|^2+4|b|})}{-|a| + \sqrt{|a|^2+4|b|}} < 2$. After some computation, it follows that condition (14) is equivalent to $|a| + |b| < 1$. Therefore, the first statement of the corollary holds. For the second statement, because $|a| + |b| \leq |a+b| + |a-b|$, if $|a+b| + |a-b| < 1$, then $|a| + |b| < 1$, which implies that the second statement of the corollary also holds. ∎

Lemma 3.8: Suppose that directed graph $\mathcal{G}$ has a directed spanning tree. Let $\mathrm{Re}(\cdot)$ and $\mathrm{Im}(\cdot)$ denote, respectively, the real and imaginary part of a number. There exist positive α and T such that $S_c \cap S_r$ is nonempty, where

$$S_c = \bigcap_{\forall \mathrm{Re}(\mu_i)<0 \text{ and } \mathrm{Im}(\mu_i)\neq 0} \{(\alpha, T) \,|\, |1+T^2\mu_i| + |3 - 2\alpha T| < 1\}, \tag{15}$$

and S_r is defined by (12). If α and T are chosen from $S_c \cap S_r$, then all eigenvalues of F are within the unit circle except one eigenvalue equal to one.

Proof: For the first statement, we let $\alpha T = \frac{3}{2}$. When $\mathrm{Re}(\mu_i) < 0$ and $\mathrm{Im}(\mu_i) \neq 0$, $|1 + T^2\mu_i| + |3 - 2\alpha T| < 1$ implies $|1 + T^2 \mu_i| < 1$ because $\alpha T = \frac{3}{2}$. It thus follows that $0 < T < \frac{\sqrt{-2\mathrm{Re}(\mu_i)}}{|\mu_i|}$, $\forall \mathrm{Re}(\mu_i) < 0$ and $\mathrm{Im}(\mu_i) \neq 0$. When $\mu_i \leq 0$, $-\frac{T^2}{2}\mu_i < \alpha T < 2$ can be simplified as $-T^2\mu_i < \frac{3}{2}$ because $\alpha T = \frac{3}{2}$. It thus follows that $0 < T < \sqrt{\frac{3}{-\mu_i}}$, $\forall \mu_i \leq 0$. Let $T_c = \bigcap_{\forall \mathrm{Re}(\mu_i)<0 \text{ and } \mathrm{Im}(\mu_i)\neq 0} \left\{T \,|\, 0 < T < \frac{\sqrt{-2\mathrm{Re}(\mu_i)}}{|\mu_i|}\right\}$ and $T_r = \bigcap_{\forall \mu_i \leq 0} \left\{T \,|\, 0 < T < \sqrt{\frac{3}{-\mu_i}}\right\}$.[3] It is straightforward to see that $T_c \cap T_r$ is nonempty. Recalling that $\alpha T = \frac{3}{2}$, it follows that $S_c \cap S_r$ is nonempty as well.

[3]When $\mu_i = 0$, $T > 0$ can be chosen arbitrarily.

For the second statement, note that if directed graph $\mathcal{G}$ has a directed spanning tree, then it follows from Lemma 3.2 that $\mu_1 = 0$ and $\mathrm{Re}(\mu_i) < 0$, $i = 2, \ldots, n$. Note that $\mu_1 = 0$ implies that $\lambda_1 = 1$ and $\lambda_2 = 1 - \alpha T$. To ensure that $|\lambda_2| < 1$, it is required that $0 < \alpha T < 2$. When $\mathrm{Re}(\mu_i) < 0$ and $\mathrm{Im}(\mu_i) \neq 0$, it follows from Corollary 3.2 that the roots of (9) are within the unit circle if $|1 + T^2\mu_i| + |3 - 2\alpha T| < 1$, where we have used the second statement of Corollary 3.2 by letting $a = \alpha T - 2 - \frac{T^2}{2}\mu_i$ and $b = 1 - \frac{T^2}{2}\mu_i - \alpha T$. When $\mu_i < 0$, it follows from the proof of Lemma 3.6 that the roots of (9) are within the unit circle if $-\frac{T^2}{2}\mu_i < \alpha T < 2$. Combining the above arguments proves the second statement. ∎

Theorem 3.3: Suppose that directed graph $\mathcal{G}$ has a directed spanning tree. Let $\mathbf{p}$ be defined in Lemma 3.2. Using (6) for (5), $r_i[k] \to r_j[k] \to \mathbf{p}^T r[0] + (\frac{1}{\alpha} - \frac{T}{2})\mathbf{p}^T v[0]$ and $v_i[k] \to 0$ as $k \to \infty$ if α and T are chosen from $S_c \cap S_r$, where S_c and S_r are defined by (15) and (12), respectively.

Proof: The statement follows directly from Lemma 3.4 and Lemma 3.8. ∎

IV. Convergence Analysis of the Sampled-data Algorithm with Relative Damping

In this section, we analyze algorithm (7) over, respectively, an undirected and an directed interaction topology.

Using (7), (5) can be written in matrix form as

$$\begin{bmatrix} r[k+1] \\ v[k+1] \end{bmatrix} = \underbrace{\begin{bmatrix} I_n - \frac{T^2}{2}\mathcal{L} & TI_n - \frac{T^2}{2}\mathcal{L} \\ -T\mathcal{L} & I_n - \alpha T \mathcal{L} \end{bmatrix}}_{G} \begin{bmatrix} r[k] \\ v[k] \end{bmatrix}. \tag{16}$$

A similar analysis to that for (8) shows that the roots of $\det(sI_{2n} - G) = 0$ (i.e., the eigenvalues of G) satisfy

$$s^2 - (2 + \alpha T\mu_i + \frac{1}{2}T^2\mu_i)s + 1 + \alpha T \mu_i - \frac{1}{2}T^2\mu_i = 0. \tag{17}$$

Similarly, each eigenvalue of $-\mathcal{L}$, μ_i, corresponds to two eigenvalues of G, denoted by ρ_{2i-1} and ρ_{2i}. Without loss of generality, let $\mu_1 = 0$, which implies that $\rho_1 = \rho_2 = 1$. Therefore, G has at least two eigenvalues equal to one.

Lemma 4.1: Using (7) for (5), $r_i[k] \to r_j[k] \to \mathbf{p}^T r[0] + kT\mathbf{p}^T v[0]$ and $v_i[k] \to v_j[k] \to \mathbf{p}^T v[0]$ for large k if and only if G has exactly two eigenvalues equal to one and all other eigenvalues have modulus smaller than one.

Proof: (Sufficiency.) Note from (17) that if G has exactly two eigenvalues equal to one (i.e., $\rho_1 = \rho_2 = 1$), then $-\mathcal{L}$ has exactly one eigenvalue equal to zero. Let $[p^T, q^T]^T$, where $p, q \in \mathbb{R}^n$, be the right eigenvector of G associated with eigenvalue one. It follows that $\begin{bmatrix} I_n - \frac{T^2}{2}\mathcal{L} & TI_n - \frac{T^2}{2}\mathcal{L} \\ -T\mathcal{L} & I_n - \alpha T\mathcal{L} \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix} = \begin{bmatrix} p \\ q \end{bmatrix}$. After some computation, it follows that eigenvalue one has geometric multiplicity equal to one even if it has algebraic multiplicity equal to two. It also follows from Lemma 3.2 that we can choose $p = \mathbf{1}_n$ and $q = \mathbf{0}_n$. In addition, a generalized right eigenvector associated with eigenvalue one can be chosen as $[\mathbf{0}_n^T, \frac{1}{T}\mathbf{1}_n^T]^T$. Similarly, it can be shown that $[\mathbf{0}_n^T, T\mathbf{p}_n^T]^T$ and $[\mathbf{p}^T, \mathbf{0}_n^T]^T$ are, respectively, a left eigenvector and generalized left eigenvector associated with eigenvalue one. Note that G can be written in Jordan canonical form as $G = PJP^{-1}$, where the columns of P, denoted by p_k, $k = 1, \ldots, 2n$, can be chosen to be the right eigenvectors or generalized right eigenvectors of G, the rows of P^{-1}, denoted by q_k^T, $k = 1, \ldots, 2n$, can be chosen to be the left eigenvectors or generalized left eigenvectors of G such that $p_k^T q_k = 1$ and $p_k^T q_\ell = 0$, $k \neq \ell$, and J is the Jordan block diagonal matrix with the eigenvalues of G being the diagonal entries. Note that $\rho_1 = \rho_2 = 1$ and $\mathrm{Re}(\rho_k) < 0$, $k = 3, \ldots, 2n$. Also note that we can choose $p_1 = [\mathbf{1}_n^T, \mathbf{0}_n^T]^T$, $p_2 = [\mathbf{0}_n^T, \frac{1}{T}\mathbf{1}_n^T]^T$, $q_1 = [\mathbf{p}^T, \mathbf{0}_n^T]^T$, and $q_2 = [\mathbf{0}_n^T, T\mathbf{p}_n^T]^T$. It follows that $G^k \to PJ^kP^{-1} \to \begin{bmatrix} \mathbf{1}_n & \mathbf{0}_n \\ \mathbf{0}_n & \frac{1}{T}\mathbf{1}_n \end{bmatrix} \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{p}^T & \mathbf{0}_n^T \\ \mathbf{0}_n^T & T\mathbf{p}^T \end{bmatrix} = \begin{bmatrix} \mathbf{1}_n\mathbf{p}^T & kT\mathbf{1}_n\mathbf{p}^T \\ \mathbf{0}_n & \mathbf{1}_n\mathbf{p}^T \end{bmatrix}$. Therefore, it follows that $r_i[k] \to r_j[k] \to \mathbf{p}^T r[0] + kT\mathbf{p}^T v[0]$ and $v_i[k] \to v_j[k] \to \mathbf{p}^T v[0]$ for large k.

(Necessity.) Note that G has at least two eigenvalues equal to one. If $r_i[k] \to r_j[k] \to \mathbf{p}^T r[0] + kT\mathbf{p}^T v[0]$ and $v_i[k] \to v_j[k] \to \mathbf{p}^T v[0]$ for large k, it follows that F^k has rank two for large t, which in turn implies that J^k has rank two for large k. It follows that G has exactly two eigenvalues equal to one and all other eigenvalues have modulus smaller than one. ■

A. Undirected Interaction

In this subsection, we show necessary and sufficient conditions on α and T such that consensus is reached using (7) over an undirected interaction topology.

Lemma 4.2: Suppose that undirected graph $\mathcal{G}$ is connected. All eigenvalues of G are within the unit circle except two eigenvalues equal to one if and only if α and T are chosen from the set

$$Q_r = \bigcap_{\forall \mu_i < 0} \{(\alpha, T) | \frac{T^2}{2} < \alpha T < -\frac{2}{\mu_i}\}. \tag{18}$$

[4]

Proof: Because undirected graph $\mathcal{G}$ is connected, it follows that $\mu_1 = 0$ and $\mu_i < 0$, $i = 2, \cdots, n$. Note that $\rho_1 = \rho_2 = 1$ because $\mu_1 = 0$. Let $a = -(2 + \alpha T\mu_i + \frac{1}{2}T^2\mu_i)$ and $b = 1 + \alpha T\mu_i - \frac{1}{2}T^2\mu_i$. It follows from Lemma 3.5 that for $\mu_i < 0$, $i = 2, \ldots, n$, the roots of (17) are within the unit circle if and only if all roots of

$$-T^2\mu_i t^2 + (T^2\mu_i - 2\alpha T\mu_i)t + 4 + 2\alpha T\mu_i = 0, \tag{19}$$

are in the open LHP. Because $-T^2\mu_i > 0$, the roots of (19) are always in the open LHP if and only if $4 + 2\alpha T\mu_i > 0$ and $T^2\mu_i - 2\alpha T\mu_i > 0$, which implies that $\frac{T^2}{2} < \alpha T < -\frac{2}{\mu_i}$, $i = 2, \ldots, n$. Combining the above arguments proves the lemma. ■

Theorem 4.1: Suppose that undirected graph $\mathcal{G}$ is connected. Let $\mathbf{p}$ be defined in Lemma 3.2. Using (7), $r_i[k] \to r_j[k] \to \mathbf{p}^T r[0] + kT\mathbf{p}^T v[0]$ and $v_i[k] \to v_j[k] \to \mathbf{p}^T v[0]$ for large k if and only if α and T are chosen from Q_r, where Q_r is defined by (18).

Proof: The statement follows directly from Lemmas 4.1 and 4.2. ■

B. Directed Interaction

In this subsection, we show sufficient conditions on α and T such that consensus is reached using (7) over a directed interaction topology. Note again that the eigenvalues of $\mathcal{L}$ may be complex for directed graphs, which makes the analysis more challenging.

Lemma 4.3: Suppose that $\mathrm{Re}(\mu_i) < 0$ and $\mathrm{Im}(\mu_i) \neq 0$. All roots of (17) are within the unit circle if $\frac{\alpha}{T} > \frac{1}{2}$ and $\alpha T < \phi(\mu_i)$, where $\phi(\mu_i) \triangleq -\frac{8\mathrm{Im}(\mu_i)^2}{|\mu_i|^4(T-2\alpha)^2} - \frac{2\mathrm{Re}(\mu_i)}{|\mu_i|}$.

Proof: As in the proof of Lemma 4.2, all roots of (17) are within the unit circle if and only if all roots of (19) are in the open LHP. Letting s_1 and s_2 denote the roots of (19), it follows that

$$s_1 + s_2 = 1 - 2\frac{\alpha}{T} \tag{20}$$

and

$$s_1 s_2 = -\frac{4}{\mu_i T^2} - 2\frac{\alpha}{T}. \tag{21}$$

Noting that (20) implies that $\mathrm{Im}(s_1) + \mathrm{Im}(s_2) = 0$, we define $s_1 = a_1 + jb$ and $s_2 = a_2 - jb$, where j is the imaginary unit. Note that s_1 and s_2 have negative real parts if and only if $a_1 + a_2 < 0$ and $a_1a_2 > 0$. Note from (20) that $a_1 + a_2 < 0$ is equivalent to $\frac{\alpha}{T} > \frac{1}{2}$. We next show conditions on α and T such that $a_1a_2 > 0$ holds. Substituting the definitions of s_1 and s_2 into (21), gives $a_1a_2 + b^2 + j(a_2 - a_1)b = -\frac{4}{\mu_i T^2} - 2\frac{\alpha}{T}$, which implies that

$$(a_2 - a_1)b = \frac{4\mathrm{Im}(\mu_i)}{|\mu_i|^2 T^2} \tag{22}$$

$$a_1a_2 + b^2 = \frac{-4\mathrm{Re}(\mu_i)}{|\mu_i|^2 T^2} - 2\frac{\alpha}{T}. \tag{23}$$

It follows from (22) that $b = \frac{4\mathrm{Im}(\mu_i)}{|\mu_i|^2 T^2 (a_2 - a_1)}$. Consider also the fact that $(a_2 - a_1)^2 = (a_2 + a_1)^2 - 4a_1a_2 = (1 - 2\frac{\alpha}{T})^2 - 4a_1a_2$. After some manipulation, (23) can be written as

$$4(a_1a_2)^2 + Aa_1a_2 - B = 0. \tag{24}$$

where $A \triangleq 4(\frac{4\mathrm{Re}(\mu_i)}{|\mu_i|^2T^2} + 2\frac{\alpha}{T}) - (1 - 2\frac{\alpha}{T})^2$ and $B \triangleq (\frac{4\mathrm{Re}(\mu_i)}{|\mu_i|^2T^2} + 2\frac{\alpha}{T})(1 - 2\frac{\alpha}{T})^2 + \frac{16\mathrm{Im}(\mu_i)^2}{|\mu_i|^4T^4}$. It follows that $A^2 + 16B = [4(\frac{4\mathrm{Re}(\mu_i)}{|\mu_i|^2T^2} + 2\frac{\alpha}{T}) + (1 - 2\frac{\alpha}{T})^2]^2 + \frac{16\mathrm{Im}(\mu_i)^2}{|\mu_i|^4T^4} \geq 0$, which implies that (24) has two real roots. Therefore, sufficient conditions for $a_1a_2 > 0$ are $B < 0$ and $A < 0$. Because $\frac{16\mathrm{Im}(\mu_i)^2}{|\mu_i|^4T^4} > 0$, if $B < 0$, then $4(\frac{4\mathrm{Re}(\mu_i)}{|\mu_i|^2T^2} + 2\frac{\alpha}{T}) < 0$, which implies $A < 0$ as well. Therefore, we only need to find conditions to guarantee $B < 0$. After some computation, it follows that $\alpha T < \phi(\mu_i)$ implies $B < 0$. Combining the previous arguments proves the lemma. ■

Lemma 4.4: Suppose that directed graph $\mathcal{G}$ has a directed spanning tree. There exist positive α and T such that $Q_c \cap Q_r$

[4]Note that Q_r is nonempty.

is nonempty, where

$$Q_c = \bigcap_{\forall \mathrm{Re}(\mu_i)<0 \text{ and } \mathrm{Im}(\mu_i)\neq 0} \left\{(\alpha, T) | \frac{1}{2} < \frac{\alpha}{T},\ \alpha T < \phi(\mu_i)\right\}, \tag{25}$$

where $\phi(\mu_i)$ is defined in Lemma 4.3 and Q_r is defined by (18). If α and T are chosen from $Q_r \cap Q_c$, then all eigenvalues of G are within the unit circle except two eigenvalues equal to one.

Proof: For the first statement, we let $\alpha > T > 0$. When $\mathrm{Re}(\mu_i) < 0$ and $\mathrm{Im}(\mu_i) \neq 0$, it follows that $\frac{\alpha}{T} > \frac{1}{2}$ holds apparently. Note that $\alpha > T$ implies $(T - 2\alpha)^2 > \alpha^2$. Therefore, a sufficient condition for $\alpha T < \phi(\mu_i)$ is

$$\alpha T < -\frac{8\mathrm{Im}(\mu_i)^2}{|\mu_i|^4\alpha^2} - \frac{2\mathrm{Re}(\mu_i)}{|\mu_i|^2}. \tag{26}$$

To ensure that there are feasible $\alpha > 0$ and $T > 0$ satisfying (26), we first need to ensure that the right side of (26) is positive, which requires $\alpha > \frac{2|\mathrm{Im}(\mu_i)|}{|\mu_i|\sqrt{-\mathrm{Re}(\mu_i)}}$. It also follows from (26) that $T < -\frac{8\mathrm{Im}(\mu_i)^2}{|\mu_i|^4\alpha^3} - \frac{2\mathrm{Re}(\mu_i)}{|\mu_i|^2\alpha}$, $\forall \mathrm{Re}(\mu_i) < 0$ and $\mathrm{Im}(\mu_i) \neq 0$. Therefore, (25) is ensured to be nonempty if α and T are chosen from, respectively, $\alpha_c = \bigcap_{\forall \mathrm{Re}(\mu_i)<0 \text{ and } \mathrm{Im}(\mu_i)\neq 0}\{\alpha | \alpha > \frac{2|\mathrm{Im}(\mu_i)|}{|\mu_i|\sqrt{-\mathrm{Re}(\mu_i)}}\}$ and $T_c = \bigcap_{\forall \mathrm{Re}(\mu_i)<0 \text{ and } \mathrm{Im}(\mu_i)\neq 0}\{T | T < -\frac{8\mathrm{Im}(\mu_i)^2}{|\mu_i|^4\alpha^3} - \frac{2\mathrm{Re}(\mu_i)}{|\mu_i|^2\alpha}$ and $0 < T < \alpha\}$. Note that (18) is ensured to be nonempty if α and T are chosen from, respectively, $\alpha_r = \{\alpha | \alpha > 0\}$ and $T_r = \bigcap_{\forall \mu_i < 0}\{T | 0 < T < 2\alpha$ and $T < -\frac{2}{\mu_i \alpha}\}$. It is straightforward to see that both $\alpha_c \cap \alpha_r$ and $T_c \cap T_r$ are nonempty. Combining the above arguments shows that $Q_c \cap Q_r$ is nonempty.

For the second statement, note that if directed graph $\mathcal{G}$ has a directed spanning tree, it follows from Lemma 3.2 that $\mu_1 = 0$ and $\mathrm{Re}(\mu_i) < 0$, $i = 2, \ldots, n$. Note that $\mu_1 = 0$ implies that $\rho_1 = 1$ and $\rho_2 = 1$. When $\mathrm{Re}(\mu_i) < 0$ and $\mathrm{Im}(\mu_i) \neq 0$, it follows from Lemma 4.3 that the roots of (17) are within unit circle if $\frac{\alpha}{T} > \frac{1}{2}$ and $\alpha T < \phi(\mu_i)$. When $\mu_i < 0$, it follows from Lemma 4.2 that the roots of (17) are within unit circle if $\frac{T^2}{2} < \alpha T < -\frac{2}{\mu_i}$. Combining the above arguments shows that all eigenvalues of G are within the unit circle except two eigenvalues equal to one if α and T are chosen from $Q_c \cap Q_r$. ∎

Theorem 4.2: Suppose that directed graph $\mathcal{G}$ has a directed spanning tree. Using (7), $r_i[k] \to r_j[k] \to \mathbf{p}^T r[0] + kT\mathbf{p}^T v[0]$ and $v_i[k] \to v_j[k] \to \mathbf{p}^T v[0]$ for large k if α and T are chosen from $Q_c \cap Q_r$, where Q_c and Q_r are defined in (25) and (18), respectively.

Proof: The proof follows directly from Lemma 4.2 and Lemma 4.4. ∎

V. Conclusion

We have studied the sampled-data consensus algorithms for double-integrator dynamics. Two sampled-date consensus algorithms with, respectively, absolute damping and relative damping have been studied over both undirected and directed interaction topologies. Necessary and sufficient conditions for convergence are given in the undirected case while sufficient conditions for convergence are given in the directed case. The final consensus equilibria for both algorithms have also been given.

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