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Decomposition of existence and stability analysis of periodic solutions of systems with impacts: application to bipedal walking robot

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Abstract— **The decomposition of the problem of existence and stability for fast periodic solutions of singularly perturbed nonlinear systems with the impact effects is considered. With this aim, theorem for existence and stability of fixed points for corresponding Poincaré sections is proved. These results are applied for the decomposition of the control design problem for bipedal robots with heavy torsos.**

Keywords. Impulse systems, singular perturbations, Poincaré sections, bipedal robot.

I. Introduction

The stability analysis of periodic nonlinear systems with the time of impulse depending on state variables is a hard task in many applications such as for walking gaits [10], [14], [13], [2]. The impulsive impact for the walking gait does not allow the use of standard methods for analysis of periodic solutions based on linearization. The use of Poincaré sections seems to be a good choice [1], [13], [2], [8]. Previous works have still proposed solutions for different robots as Raibert's one-legged hopper [9], [18], bipeds without torso [13], [12], [23], and five-link bipeds (with torso, legs with knees) [8], [16], [17], [21]. Ones of the more accomplished results [1], [13], [2], [21] consist in designing the control strategy (with, in particular, the finite time convergence property) such that the application of Poincaré sections method to three- and five-link is simplified: it yields an algorithm which allows, through the calculation of a continuous map from a subinterval of $I\!R$ to itself, to conclude on the stability of the walking gait. The problem of stability for slow periodic solutions of impulse systems could be decomposed using the central manifold technique [13]. The case of the heavy torso implies that the central integral manifold technique could not be used because the fast motions in this case are oscillating. The fast motions have an oscillatory behavior and then the central manifold technique could not be used. This last point implies that mathematical tools are needed allowing controllers design for the torso and the legs separately. The majority of existing results on impulse control are limited to open-loop control [19], [4], [20]. Recent publications ([6] and associated references) deal the active singularity. Then here passive impact is considered, *i.e.* the impulse time depends on state variables and there are no applied torques at the impact. For smooth singularly perturbed systems with fast periodic solutions, the averaging theorem has been proposed in [22] using Lyapunov methods. For systems with discontinuous right hand sides [15], the analogous of the averaging theorem is proved in terms of fixed points of the Poincaré sections. Singularly perturbed impulse systems were considered in [3] where value of impulses are computed at preprogrammed time moments. There exists another way to consider singularly perturbed systems with measurable fast motions based on averaging technique (see [11] and references there in as one of the recent one). But the cyclic motions are not considered. However decomposition existence and stability of periodic solutions for singularly perturbed systems with impacts is an important task. In this paper, dynamics of the biped walking with the heavy torso is rewritten in the form of singularly perturbed system with impacts. The problem of existence and stability of the fast periodic motions is reduced to the problem of existence and stability of a fixed point of Poincaré sections. A decomposition theorem is proved claiming that from the existence and exponential orbital stability of a periodic solution of the fast system and existence and exponential stability of equilibrium point of the averaging slow equations, one can conclude the existence and orbital exponential stability of the periodic solutions of the original singularly perturbed impulse system. Obtained result is applied on a biped with heavy torso. From a technical point of view for a biped it is better to place the actuator, the gear ratio boxes such as the center of mass is upper the hip joints to get the smallest inertial forces for the legs. Then biped with a heavy torso is not a particular unrealistic case.

The paper is organized as follows. Section II displays the bipedal robot. Section III presents the theoretical contribution of the paper, through the development of a stability analysis based on Poincaré sections of a nonlinear dynamic system written under singular perturbations formalism. Section IV presents the application of the previous contribution.

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II. Model of the bipedal robot

Models are displayed for a bipedal robot (Figure 1) which is a three-link biped with two identical legs without knee, a torso, and two actuators at the hips between each leg and the torso.

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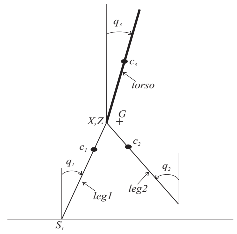

Fig. 1: Diagram of a three-link biped.

A. Dynamic model

Let us introduce $q_e = [q^t \ X \ Z]^t$ with (X,Z) the Cartesian coordinates of the actuated hips, and $q = [q_1 \ \ q_2 \ \ q_3]^t$ the three orientation angles of the legs and the torso for the biped without contact with the ground. For the three-link biped the dimension of q_e is 5. The stance (resp. swing) leg is leg 1 (resp. 2), and the angles are defined positive for the counter clockwise motion. Links are assumed massive and rigid, the joint are revolute and ideal; therefore, friction effects in the joints are neglected. Let $\Gamma = [\Gamma_1 \ \Gamma_2]^t$ denote the torques applied to the hips by the actuators, and $R = [R_1^t \ R_2^t]^t$ with $R_j = [R_{jN} \ R_{jT}]^t \ (j = 1,2)$ the forces applied to the j-leg tip following the tangential and horizontal axes to the ground.

B. General and reduced dynamic models

A general dynamic model is given by

$$\mathbf{D_e}\ddot{q}_e + \mathbf{H_e}\dot{q}_e + \mathbf{G_e} = \mathbf{B_e}\Gamma + \mathbf{J_R^t}R \tag{1}$$

$\mathbf{D_e}(q)$ (5×5) is the symmetric positive inertia matrix, $\mathbf{H_e}(q,\dot{q})$ (5×5) is the Coriolis and centrifugal effects matrix and $\mathbf{G_e}(q_e)$ (5×1) is the gravity effects vector. $\mathbf{B_e}$ (5×2) is a constant matrix composed of 1 and 0 and $\mathbf{J_R}(q_e)$ (5×4) is the Jacobian matrix converting the external forces into the corresponding joint torques.

Assume that during the swing phase of the motion, the stance leg is acting as a pivot i.e., there is no slipping of the swing leg. Then there are only three independent generalized coordinates in single support to define the configuration of the biped.In consequence, in single support the dynamics of the biped can be modeled by the following matrix equation:

$$\mathbf{D}\ddot{q} + \mathbf{H}\dot{q} + \mathbf{G} = \mathbf{B}\Gamma \tag{2}$$

Matrix $\mathbf{D}(q)$ (3×3) is the symmetric positive inertia, $\mathbf{H}(q,\dot{q})$ (3×3) is the Coriolis and centrifugal effects matrix, and $\mathbf{G}(q)$ (3×1) is the gravity effects vector. $\mathbf{B}$ (3×2) is a constant matrix composed of 1 and 0. Equation (2) can be written as

$$\begin{aligned}\dot{x} &= \begin{bmatrix} \dot{q} \\ \mathbf{D}^{-1}(-\mathbf{H}\dot{q} - \mathbf{G} + \mathbf{B}\Gamma) \end{bmatrix} \\ &= f(x) + g(x)\cdot\Gamma \end{aligned} \tag{3}$$

with $x = [q^t \ \dot{q}^t]^t$. The state space is taken such that $x \in \mathcal{X} \subset I\!R^{\,2(n-2)} = \{x \mid q \in \mathcal{M}\,,\ \dot{q} \in \mathcal{N}\}$, with $\mathcal{M} = (-\pi,\pi)^{\,n-2}$ and $\mathcal{N} = \{\dot{q} \in I\!R^n \mid |\dot{q}| < \dot{q}_M < \infty\}$.

C. Some considerations about dynamics of swing leg and stance leg with the torso

Let L and m be respectively the length and the mass of each leg. The three-link biped is assumed to have two identical legs. Let I be the moment of inertia of each leg, around the axis parallel to the joint axis and passing through the leg mass center, and r the position of the mass center of each leg from the hips. The torso is characterized by its mass m_3, its moment of inertia I_3 and the position from the hips of its center of mass r_3. If torso mass m_3 is much larger than legs mass m, element D_{11} of $\mathbf{D}$ can be approximated by m_3L^2. Let $\bar{\mathbf{D}}$ be the matrix equal to $\mathbf{D}$ with D_{11} chosen equal to m_3L^2. Taking into account the properties of the definite positive matrices, since $\mathbf{D}$ is definite positive, then $\bar{\mathbf{D}}$ is also definite positive and its inverse matrix exists. From structural analysis of $\bar{\mathbf{D}}^{\mathbf{-1}}$, it yields that term w_{22} is the single term depending on m_3^2 and m_3I_3. From (3), one concludes that for a three-link biped with a heavy trunk, the motions of the stance leg together with the torso are slow with respect to dynamics of the swing leg ones.

D. Passive impact model

The walking gait defined later will be such as the motion of the swing leg is not symmetric to the one of the stance leg. But the fact that there is no knees, when the two legs will be vertical we will consider that the swing leg touches the ground without impact. An impact will be taken only into account at the end of the single support, when the swing leg will reaches its final configuration. Let T denote the impact time. An impact occurs when state trajectories evolve in set $\mathcal{S}$ defined as

$$\mathcal{S} \quad = \quad \{x \in \mathcal{X} \mid q_2(T) = -q_1(0)\ \} \tag{4}$$

with $q_2(T)$ and $q_1(0)$ respectively the final and initial orientations of the swing and stance legs.

The impact is assumed to be a passive, i.e. without torques applied in the inter-link joints, absolutely inelastic and that the legs do not slip. Given these hypotheses, the ground reactions at T can be considered as impulsive forces acting on only the swing leg and defined by Dirac delta-functions $R_2 = I_{R_2}\delta(t-T)$ with $I_{R_2} = [I_{R_{2N}} \ I_{R_{2T}}]^t$ the vector of magnitudes of impulsive reaction [10]. Impact equations are obtained through the integration of (1) for the infinitesimal time. The torques supplied by the actuators at the joints and Coriolis and gravity forces have finite values: thus, they do not influence the impact. Consequently, the impact equations can be written as

$$\mathbf{D_e}(\dot{q}_e^+ - \dot{q}_e^-) = \mathbf{J_{R_2}^t} I_{R_2}, \tag{5}$$

q_e being the configuration of the biped at $t = T$ and $\dot{q}_e^-$ and $\dot{q}_e^+$ are the angular velocities just before and just after the impact. The swing leg after the impact becomes the supporting leg. Its tip velocity becomes zero after the impact

$$\mathbf{J_{R_2}^t}\dot{q}_e^+ = 0 \tag{6}$$

E. Nonlinear model all over the step

The overall biped model can be expressed as,

$$\begin{array}{rcll} \dot{x} & = & f(x)+g(x)\Gamma, & \text{for } x^- \notin \mathcal{S} \\ x^+ & = & \Delta[x^-], & \text{for } x^- \in \mathcal{S}. \end{array} \tag{7}$$

with $x^- = [q^{-t}\ \dot{q}^{-t}]^t$ and $x^+ = [q^{+t}\ \dot{q}^{+t}]^t$.

III. Decomposition Theorem

A. Preliminaries

Consider a nonlinear system with impact effects

$$\begin{array}{rcll} \dot{\zeta} & = & \bar{f}(\zeta), & \text{for } \zeta^- \notin \bar{\mathcal{S}} \\ \zeta^+ & = & \bar{\Delta}[\zeta^-], & \text{for } \zeta^- \in \bar{\mathcal{S}}. \end{array} \tag{8}$$

with $\zeta \in \mathcal{Z} \subset \mathbb{R}^{k+1}$ the state vector and $\bar{\Delta}$ the impact function. Let $y \in \Omega \subset \mathbb{R}$ (Ω is a closed compact set) be the scalar variable which characterizes that an impact occurs[1] such that $y = 0$. Then, at the impact, system (8) is evolving in

$$\bar{\mathcal{S}} \;=\; \{\zeta \in \mathcal{Z} \mid y(T) \;=\; 0\}. \tag{9}$$

Defining $\kappa \in \Omega^k \subset \mathbb{R}^k$ (Ω^k is a closed compact set), system (8) is rewritten as

$$\mathrm{d}\kappa/\mathrm{d}t \;=\; G(\kappa,y), \quad \mathrm{d}y/\mathrm{d}t \;=\; F(\kappa,y) \tag{10}$$

with initial conditions

$$\kappa(0) \;=\; \kappa^0, \quad y(0) = 0 \tag{11}$$

and $G : \mathbb{R}^k \times \mathbb{R} \to \mathbb{R}^k$, $F : \mathbb{R}^k \times \mathbb{R} \to \mathbb{R}$, $F, G \in C^2(\Omega^k \times \Omega)$, $\kappa^0 \in \Omega^k$. Denote that solutions of system (10) smoothly depend on initial conditions. Suppose that, for all $\kappa^0 \in \Omega^k$, there exists a time instant $T(\kappa^0)$ as the smallest positive root of the equation

$$y(T(\kappa^0)) \;=\; 0 \tag{12}$$

and, moreover,

$$\frac{\mathrm{d}y}{\mathrm{d}t}\left[T(\kappa^0)\right] \;=\; F(\kappa(T(\kappa^0)),0) \neq 0. \tag{13}$$

From implicit function theorem, one gets $T \in C^1(\Omega^k)$ and then T is a continuous function in $\kappa(0) = \kappa^0$. Define the solution of Cauchy problem (10)-(11) on $[0, T(\kappa^0))$. Suppose that the system has an impact at $T(\kappa^0)$ defined by

$$\kappa(T(\kappa^0)^+) \;=\; \bar{\Delta}_r\left[\kappa(T(\kappa^0)^-)\right] \tag{14}$$

with $\bar{\Delta}_r \in C^1(\Omega^k)$ being a "reduced" part of $\bar{\Delta}$. Then, define the Poincaré section P such that

$$P(\kappa^0) \;=\; \kappa\left(T(\kappa^0)^+\right) = \bar{\Delta}_r\left[\kappa(T(\kappa^0)^-)\right] \tag{15}$$

of the set Ω^k into itself. Remark that operator P is smooth on Ω^k as a composition of the smooth maps. Considering the solution of system (10) with the initial conditions

$$\kappa\left(T(\kappa^0)^+\right) = \bar{\Delta}_r\left[\kappa(T(\kappa^0)^-)\right], \quad y(0) = 0 \tag{16}$$

and applying the same process, one defines the solution of system (10)-(14) for all $t \geq T(\kappa^0)$, *i.e.* for all $t \in [0,\infty)$.

[1]For the previous walking biped, variable y is defined as $y = q_2(t) + q_1(0)$ with an impact occurring when $y = 0$.

Define a function $\{\kappa_0(t), y_0(t)\}$ as a T_0-periodic solution of the system if there exists a constant $T_0 > 0$ such that $\{\kappa_0(t+T_0), y_0(t+T_0)\} = \{\kappa_0(t), y_0(t)\}$ for all $t \in [0,\infty)$. Define the set $\mathcal{O} = \{\{\kappa_0(t), y_0(t)\}, t \geq 0\}$ a periodic orbit of system (10)-(14).

From [13], a periodic orbit is stable in Lyapunov sense if, for every $\varepsilon > 0$, there exists an open neighborhood $\mathcal{V}$ of $\mathcal{O}$ such that, for every $\kappa^0 \in \mathcal{V}$, for a solution $\{\kappa(t), y(t)\}$ of (10)-(14), $\mathrm{dist}(\{\kappa(t), y(t)\}, \mathcal{O}) < \varepsilon$ for all $t \geq 0$. $\mathcal{O}$ is *attractive* if there exists an open neighborhood $\mathcal{V}$ of $\mathcal{O}$ such that for every $\kappa^0 \in \mathcal{V}$, there exists a solution satisfying initial conditions $\kappa(0) = \kappa^0$ and $\lim_{t\to\infty} \mathrm{dist}(\{\kappa(t), y(t)\}, \mathcal{O}) = 0$. $\mathcal{O}$ is *asymptotically stable in the sense of Lyapunov* if it is both stable and attractive. In the sequel of the paper, the qualifier "in the sense of Lyapunov" will be systematically assumed if it is not explicitly made.

To prove the existence of an isolated periodic orbit $\mathcal{O}$, it is sufficient to show that the Poincaré section $P(\kappa^0)$ has an isolated fixed point $\kappa^* = P(\kappa^*)$, corresponding to $\mathcal{O}$. $\mathcal{O}$ is locally exponentially orbitally stable if

$$\left|\left|\frac{\partial P}{\partial \kappa^0}\right|\right|_{\kappa^0=\kappa^*} < 1. \tag{17}$$

B. Fast periodic oscillations in systems with impact

Consider now a nonlinear system with impact effects

$$\dot{\zeta} = \bar{f}(\zeta), \quad \text{for } \zeta^- \notin \bar{\mathcal{S}}, \quad \zeta^+ = \bar{\Delta}[\zeta^-], \quad \text{for } \zeta^- \in \bar{\mathcal{S}}. \tag{18}$$

with $\zeta \in \mathcal{Z} \subset \mathbb{R}^{m+k+1}$ the state vector. Suppose that its dynamics can be represented by the singularly perturbed system in the form

$$\mu\frac{\mathrm{d}\kappa}{\mathrm{d}t} = G(\kappa,y,z), \;\; \mu\frac{\mathrm{d}y}{\mathrm{d}t} = F(\kappa,y,z), \;\; \dot{z} = H(\kappa,y,z) \tag{19}$$

with $z \in \Omega^m \subset R^m$, Ω^m a compact set, $F, G, H \in C^2(\Omega^k \times \Omega \times \Omega^m)$, and $\mu > 0$ a small parameter. Rewrite system (19) in the "fast time" $\tau = t/\mu$ as

$$\frac{\mathrm{d}\kappa}{\mathrm{d}\tau} = G(\kappa,y,z), \;\; \frac{\mathrm{d}y}{\mathrm{d}\tau} = F(\kappa,y,z), \;\; \frac{\mathrm{d}z}{\mathrm{d}\tau} = \mu H(\kappa,y,z) \tag{20}$$

Consider the subsystem describing *only* fast motions in (20)

$$\begin{array}{rcl} \mathrm{d}\bar{\kappa}/\mathrm{d}\tau & = & G(\bar{\kappa},\bar{y},z) \\ \mathrm{d}\bar{y}/\mathrm{d}\tau & = & F(\bar{\kappa},\bar{y},z) \\ z & \in & \Omega^m \;\; \text{is a parameter} \end{array} \tag{21}$$

Denote $\{\kappa(\tau,\kappa^0,z^0,\mu),\; y(\tau,\kappa^0,z^0,\mu),\; z(\tau,\kappa^0,z^0,\mu)\}$ and $\{\bar{\kappa}(\tau,\kappa^0,z), \bar{y}(\tau,\kappa^0,z)\}$ the solutions of systems (20) and (21) respectively with initial conditions respectively

$$y(0) = \bar{y}(0) = 0, \;\; \kappa(0) = \kappa^0, \;\; z(0) = z^0, \;\; \bar{\kappa}(0) = \kappa^0. \tag{22}$$

Assume that, for all $\kappa^0 \in \Omega^k$, $z \in \Omega^m$, there exists a time instant $T(\kappa^0,z)$ as the smallest positive root of the equation

$$\bar{y}(T(\kappa^0,z),\kappa^0,z) \;=\; 0 \tag{23}$$

and moreover

$$\frac{\mathrm{d}\bar{y}}{\mathrm{d}\tau}\left[T(\kappa^0,z),\kappa^0,z\right] \quad \neq \quad 0. \tag{24}$$

As previously, note that, from implicit function theorem, one gets $T \in C^1(\Omega^k \times \Omega^m)$. Suppose that, at $T(\kappa^0,z)$, an impact occurs. It yields

$$\bar{\kappa}(T(\kappa^0,z)^+,\kappa^0,z) \quad = \quad \bar{\Delta}_r\left[\bar{\kappa}(T(\kappa^0,z)^-,\kappa^0,z)\right] \tag{25}$$

with $\bar{\Delta}_r \in C^1(\Omega^k)$ being a "reduced" part of $\bar{\Delta}$. Let the Poincaré section define as $\kappa^1 = P(\kappa^0,z) = \bar{\Delta}_r\left[\bar{\kappa}(T(\kappa^0,z)^-,\kappa^0,z)\right]$ of the set Ω^k into itself, generated by the system (21),(23) and (25).

C. Main theorem

Suppose that for all $z \in \bar{\Omega}^m$

A1. The Poincaré section $P(\kappa^0,z)$ has an isolated fixed point $\kappa^*(z) = P(\kappa^*(z),z)$, corresponding to the isolated $T_0(z)$-periodic solution $\{\bar{y}_0(\tau,z),\bar{\kappa}_0(\tau,z)\}$ for system (21)-(23)-(25) and

$$\mathrm{Det}\left[\frac{\partial P}{\partial \kappa_0}\right]_{\kappa^0=\kappa^*} \neq 0.$$

A2. $\left|\left|\dfrac{\partial P}{\partial \kappa^0}\right|\right|_{\kappa^0=\kappa^*} < 1.$

A3. An average system $\dfrac{\mathrm{d}\bar{z}}{\mathrm{d}t} = p(\bar{z})$ with

$$p(\bar{z}) \quad = \quad \frac{1}{T(\bar{z})}\int_0^{T(\bar{z})} H(\bar{y}_0(\tau,\bar{z}),\bar{\kappa}_0(\tau,\bar{z}),\bar{z})\mathrm{d}\tau \tag{26}$$

has an isolated asymptotically stable equilibrium point $\bar{z}^*$ such that $p(\bar{z}^*) = 0$ and the eigenvalues of the matrix $\dfrac{\mathrm{d}p}{\mathrm{d}\bar{z}}(\bar{z}^*)$, denoted as $\nu_j(\bar{z}^*)$ $(1 \le j \le m)$, are negative, *i.e.* $\nu_j(\bar{z}^*) < 0$.

Let us describe an impact in the systems (19) and (20). From implicit function theorem, it follows that there exists a small μ_0 such that, for all $\kappa^0 \in \Omega^k$, $z^0 \in \Omega^m$, $\mu \in [0,\mu_0]$, there exists such a time instant $T^\mu(\kappa^0,z^0,\mu)$ as a smallest positive root of the equation

$$y(T^\mu(\kappa^0,z^0,\mu),\kappa^0,z^0,\mu) \quad = \quad 0 \tag{27}$$

Suppose that an impact in the system (20) and consequently in the system (19) occurs. It yields

$$\kappa(T^\mu(\kappa^0,z^0,\mu)^+,\kappa^0,z^0,\mu) = \bar{\Delta}_r\left[\kappa(T^\mu(\kappa^0,z^0,\mu)^-,\kappa^0,z^0,\mu)\right]. \tag{28}$$

Theorem 1: Under conditions A1-A2-A3, system (19), (27), and (28) has a locally exponentially orbitally stable isolated periodic solution with the period $\mu(T(\bar{z}^*) + O(\mu))$.

IV. Application to a three-link biped

For the three-link biped, a cyclic reference walking gait is obtained assuming that the closed-loop system is perfectly tracking reference trajectories depending on q_1 for the two output variables q_2 and q_3. The fast part of the system is displayed by the swing leg whereas the slow part is represented by the stance leg and the trunk. From numerical results, the objective consists in showing the influence of the trunk mass on the orbital stability.

A. Description of the cyclic walking gait

The cyclic walking gait is designed on a horizontal plan for three-link biped (Figure 1) composed of single support phases and impacts. It is assumed that the closed-loop system is perfectly tracking reference trajectories by using a nonlinear decoupling controller [13] coupled to a finite time convergence control law [5]. When the swing leg impacts the ground, the stance leg which was previously in contact takes off. It yields that the two legs have a symmetrical role over a full step. The objective of the walking gait consists in transferring the robot during the swing phase from a given initial configuration to a given final one. In these boundary configurations, both legs are on the ground and define instantaneous configurations. Taking into account the unilateral constraints, there is an infinite number of solutions to design this cyclic walking gait for the biped. Since the biped has only two actuators for three degrees of freedom in single support, this is an underactuated system. Furthermore, numerical tests show that for a three-link biped during a "natural" walking gait, the evolution of the absolute orientation of the stance leg q_1 is monotonous [1], [13]. Then, in order to have the desired final configuration of the biped at the impact, the swing leg and torso reference trajectories are defined as functions of absolute orientation q_1 of the stance leg (see [2]) as ($i \in \{2,3\}$)

$$q_{i,ref} \quad = \quad a_{i0} + a_{i1}q_1 + a_{i2}q_1^2 + a_{i3}q_1^3 + a_{i4}q_1^4 \tag{29}$$

The five coefficients of $q_{i,ref}$ ($i \in \{2,3\}$) are computed from initial and final configurations, initial and final velocities and an intermediate configuration. The initial orientation $q_1(0)$, $q_2(0)$ and the final orientation $q_1(T)$, $q_2(T)$ of the two legs are such that $q_1(0) = -q_1(T)$, $q_1(T) = q_2(0)$ and $q_2(T) = q_1(0)$. The initial and final values of torso orientation are such that $q_3(0) = q_3(T)$. The final velocities of the biped are chosen such that $\dot{q}_1(T) = \dot{q}_2(T)$ and $\dot{q}_3(T) = 0$. Intermediate configurations of both legs are chosen such that $q_1 = q_2 = 0$ which leads to $a_{20} = 0$. The behavior of q_1 results from the dynamics of the biped. If variables q_2 and q_3 are exactly tracking their desired trajectories

$$q_2 = q_{2,ref}(q_1), \qquad q_3 = q_{3,ref}(q_1) \tag{30}$$

By applying the total angular momentum theorem in S_1 (see Figure 1), one gets a simplified dynamic model for the biped during the swing phase [1], [2]

$$\dot{\sigma}_{S_1} = -Mg\left[x_G(q_1) - x_{S_1}\right], \quad \dot{q}_1 = \frac{\sigma_{S_1}}{f_\sigma(q_1)} \tag{31}$$

with M the biped mass, g the gravity acceleration, $x_G(\alpha)$ and x_{S_1} the horizontal components of respectively the position of biped's mass center and the stance leg tip, σ_{S_1} the angular momentum.

B. Orbital stability of the walking gait

1) **Stability analysis of fast dynamics:** As q_1 has a monotonous behavior over one step, from (31), one gets [7]:

$$\frac{\mathrm{d}\sigma_{S_1}^2}{\mathrm{d}q_1} \quad = \quad -2Mg[x_G(q_1) - x_{S_1}]f_\sigma(\mathrm{d}q_1). \tag{32}$$

From (32), one gets

$$\begin{aligned} & \sigma_{S_1}^2(q_1(T)) - \sigma_{S_1}^2(q_1(0)) \\ = \quad & f_\sigma^2(q_1(T))\dot{q}_1^2(T) - f_\sigma^2(q_1(0))\dot{q}_1^2(0) \end{aligned} \tag{33}$$

A linear relation between the velocity $\dot{q}_1^-$ and the velocity $\dot{q}_1^+$ can be obtained [2]

$$\dot{q}_1^+ = J \cdot \dot{q}_1^- \quad \rightarrow \quad \dot{q}_1^- = J^{-1}\dot{q}_1^+ = \jmath\, \dot{q}_1^+ \tag{34}$$

Let $\mathcal{Q}_{n_s}$, the square velocity $(\dot{q}_1(0))^2$ (or $(\dot{q}_1^+)^2$) at the start of the half step number n_s. From (33)-(34), one gets the next value $\mathcal{Q}_{n_s+1}$, which is the initial velocity for the half step number n_s+1, which gives

$$f_\sigma^2(q_1(T))\jmath^2 \mathcal{Q}_{n_s+1} - f_\sigma^2(q_1(0))\mathcal{Q}_{n_s} = F_\sigma(q_1(T)) \tag{35}$$

Formula (35) gives the relation between values $\mathcal{Q}_{n_s}$ and $\mathcal{Q}_{n_s+1}$

$$\mathcal{Q}_{n_s+1} = P(\mathcal{Q}_{n_s}) = \mathcal{Q}_{n_s+1} = \left[\frac{F_\sigma(q_1(T)) + f_\sigma^2(q_1(0))\mathcal{Q}_{n_s}}{f_\sigma^2(q_1(T))\jmath^2}\right] \tag{36}$$

From (36), the Poincaré return map can be constructed in the plane $(\mathcal{Q}_{n_s}, \mathcal{Q}_{n_s+1})$. Let $\mathcal{Q}^*$ be the equilibrium (fixed) point of the transformation (36), *i.e.* $\mathcal{Q}^* = P(\mathcal{Q}^*)$. From (36), it yields

$$\mathcal{Q}^* = \left[\frac{F_\sigma(q_1(T))}{f_\sigma^2(q_1(T))\jmath^2 - f_\sigma^2(q_1(0))}\right] \tag{37}$$

Reference trajectories (29)) imply that the biped behavior depends on q_1 and $\dot{q}_1$. Then, the orbital stability problem is a one-dimension one. The equations (36) and (37) are scalar, which leads to contraction condition (17). Namely, in the current case, $\frac{\mathrm{d}P}{\mathrm{d}\mathcal{Q}_{n_s}}(\mathcal{Q}^*)$ is exactly $\frac{\mathrm{d}P}{\mathrm{d}\kappa^0}(\kappa^*)$ of (17), *i.e.*

$$\frac{\mathrm{d}P}{\mathrm{d}\kappa^0}(\kappa^*) = \frac{f_\sigma^2(q_1(0))}{f_\sigma^2(q_1(T))\jmath^2} = \frac{\left[\sigma_{S_2}^+(q_1(0))\right]^2}{\left[\sigma_{S_1}^-(q_1(T))\right]^2} \tag{38}$$

In the next section, for a given bipedal, it is proved that, for a trunk mass equals to 10kg, there exists a fixed point, and that the following inequality is verified in a neighborhood of the fixed point [7] and [8]:

$$\left|\frac{\sigma_{S_2}^+(q_1(0))}{\sigma_{S_1}^-(q_1(T))}\right| = \left|\sigma_{S_2}^+(q_1(0))\right| / \left|\sigma_{S_1}^-(q_1(T))\right| = \delta < 1 \tag{39}$$

From this, it means that Assumptions A1 and A2 are fulfilled: it yields that fast dynamics allow a stable periodic motion.

2) **Stability analysis of slow dynamics:** Denote $\bar{q}_2$ the average value of fast dynamics $q_2(q_1, q_3)$ with respect to slow dynamics. Then, given the reference trajectories previously described and from the point-of-view of slow dynamics, a control objective is stated through the output variables

$$y_1 \;=\; \bar{q}_2 - q_{2,ref}(q_1), \; y_2 \;=\; q_3 - q_{3,ref}(q_1) \tag{40}$$

There exists control law Γ ensuring that y_1 and y_2 converge to zero. Then, as $\bar{q}_2$ is stable and $y_1 \rightarrow 0$, given that $q_{2,ref}$ is a polynomial in q_1, it yields that q_1 is stable. From this latter point, by the same way applied to y_2, it yields that q_3 is stable. Assumption A3 is fulfilled.

C. Simulation results

Results are displayed in case of torso mass equals to 10 kg. An analysis of the influence of torso mass is made, which validates previous theory. The torso mass is acting not only on the fast dynamics but also the value of fixed point and step period. The variations of period and fixed point value are directly linked to the m_3 mass variations.

1) **Numerical parameters of the biped and simulations:** For the robot legs, which are identical, the physical parameters are $m_1 = m_2 = m = 0.4\ \ kg$, $I_1 = I_2 = I = 0.406\,kg{\cdot}m^2$, $r_1 = r_2 = r = 0.215\,m$ and $L_1 = L_2 = L = 0.68\,m$. For these simulations $m_3 = 10\,kg$. The duration T of the step is equal to $0.2310\ s$. Figure 2 shows that the behavior of q_1 is monotone. At the beginning of the step and at the impact, symmetrical orientation of both legs is such that $q_1(0) = q_2(T) = 0.2269\ rd$. Initial and final absolute orientations of the torso are $q_3(0) = q_3(T) = -0.1396\ rd$. Its intermediate orientation during the step is chosen to $0.2094\ rd$. The amplitude of the angular velocity of the swing leg are greater than the case of the stance leg (Figure 3. The founded initial angular velocities of both legs and torso equal: $\dot{q}_1(0) = -1.5623\ rd/s$, $\dot{q}_1(0) = -1.5623\ rd/s$ and $\dot{q}_3(0) = -0.8015\ rd/s$. The final velocities are chosen as $\dot{q}_1(T) = \dot{q}_2(T) = -1.8599\ rd/s$ and $\dot{q}_3(T) = 0.0\ rd/s$. Inequality (39) is fulfilled given that $\delta = 0.8377$.There exists an attraction basin w.r.t. $\dot{q}_1(0)$ such that the walking gait converges to the nominal cyclic gait. As a matter of fact, the Poincaré map $\mathcal{Q}_{n_s+1} = P(\mathcal{Q}_{n_s})$ numerically establishes that the biped converges to a stable periodic walking gait if the initial velocity $\dot{q}_1(0)$ is such that

$$\dot{q}_1(0) \quad \in \quad [-2.5534\ rd \cdot s^{-1}, -1.5492\ rd \cdot s^{-1}]$$

If $\dot{q}_1(0) < -2.5534\ rd \cdot s^{-1}$, the biped moves back to its initial configuration: the initial module of angular momentum around the contact point S_1 is not sufficiently large [7]. If $\dot{q}_1(0) > -1.5492\ rd \cdot s^{-1}$, the vertical component of the ground reaction in the supporting leg tends to zero which implies the biped is losing the contact with the ground. The fixed point $\mathcal{Q}^* = 2.4408\ (rd \cdot s^{-1})^2$ corresponds to the time square of the nominal initial velocity $\dot{q}_1(0)^*$.

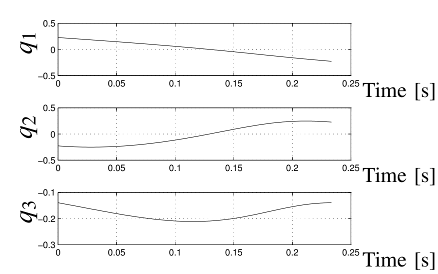

Fig. 2: Torso mass equals $10kg$: q_1, q_2, q_3 (rd) versus time (s).

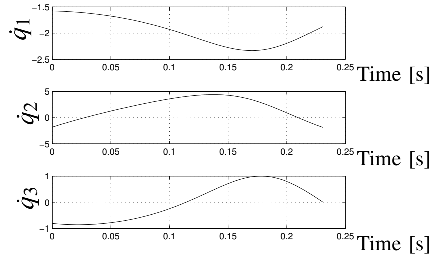

Fig. 3: Torso mass equals $10kg$: $\dot{q}_1$, $\dot{q}_2$, $\dot{q}_3$ (rd/s) versus time (s).

2) **Analysis of m_3-influence:** Suppose now that torso mass m_3 is very large (in the worst case, suppose that $m_3 = \infty$). From (3)-(20), it yields that parameter μ can be viewed as $1/m_3$. Simulations display that biped motions are stable for large masses; furthermore, the period of the walking gait T^μ as well as the fixed point of corresponding Poincaré map continuously depend on the parameter μ, as shown by Table I. As a matter of fact, state the reference point (first line of Table I) as the case when μ is the smallest ($\mu = 0,000005$ for $m_3 = 20000\ kg$). From numerical results displayed in Table I, it can be:

$$Q^* \approx Q^*_{m_3=2\ 10^5} + 4\mu, \quad \mu/100 \approx T^\mu - T_{m_3=2\ 10^5}$$

The deviation of the period and the fixed point of the Poincaré map smoothly depends on the small parameter μ.

$\mu = \frac{1}{m_3}$	T^μ (sec)	$Q^*(s^{-1})$	$Q^* - Q^*_{m_3=2\ 10^5}$ (s^{-1})	$T^\mu - T_{m_3=2\ 10^5}$
$5\ 10^{-6}$	0.22842089	2.882387	0	0
$5\ 10^{-5}$	0.22842138	2.882187	-0.0002	0.000000480
$5\ 10^{-4}$	0.22842623	2.880168	-0.0022	0.000005332
$5\ 10^{-3}$	0.22847908	2.859961	-0.0224	0.000058180
10^{-2}	0.22854679	2.837449	-0.0449	0.000125898

TABLE I: Deviation of the period and the fixed point w.r.t. m_3.

V. CONCLUSIONS

The decomposition of the problem of existence and stability for periodic solutions of singularly perturbed nonlinear systems with the impacts is considered. The decomposition theorem for the problem of existence and stability of the fixed points for corresponding Poincaré sections are given. These results are applied for the decomposition of the control design problem for a biped with a heavy torso. Its behavior can be described as a singularly perturbed system with the small parameter inverse to the mass of the torso and fast periodic oscillations of the swing leg and slow dynamics corresponding to the stance leg and torso. The simulations show that the periodic trajectory of the biped smoothly depends on the small parameter. This allows to conclude the control for the swing leg and for stance leg and torso could be designed separately.

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