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# On Cascades of Bilinear Systems and Generating Series of Weighted Petri Nets

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***Abstract*— It has been established in the literature that the cascade interconnection of two bilinear systems does not in general produce another bilinear system. The goals of this paper are two-fold. First, an alternative proof of the sufficient condition for preserving bilinearity under cascades due to Ferfera is presented which is much simpler than the original. Then it is shown that the well known correspondence between rational series and formal power series recognized by weighted finite-state automata can be generalized to produce a correspondence between the generating series of cascades of bilinear systems and a class of weighted Petri nets.**

## I. INTRODUCTION

Consider a bilinear state space system of the form

$$\begin{aligned}\dot{z}(t) &= Az(t) + \sum_{j=1}^{m} N_j z(t) u_j(t), \;\; z(0) = z_0 \\ y(t) &= Cz(t),\end{aligned}$$

where $z(t) \in \mathbb{R}^n$; $u_j(t) \in \mathbb{R}$; $y(t) \in \mathbb{R}^\ell$; and $A$, $N_j$ and $C$ are matrices of appropriate dimensions. It is easily verified that if two bilinear state space systems $(A_i, N_{\cdot,i}, C_i, z_{i,0})$, $i = 1, 2$ are interconnected in a cascade fashion, that is, if $m = \ell$ and one feeds the outputs of one system into the inputs of the other, then one possible state space realization for the input-output mapping $u_1 \mapsto y_2$ is

$$\dot{z}_1(t) = A_1 z_1(t) + \sum_{j=1}^{m} N_{j,1} z_1(t) u_{j,1}(t), \; z_1(0) = z_{1,0} \quad (1)$$

$$\dot{z}_2(t) = A_2 z_2(t) + \sum_{j=1}^{m} N_{j,2} z_2(t) (C_1 z_1(t))_j, \; z_2(0) = z_{2,0} \quad (2)$$

$$y_2(t) = C_2 z_2(t), \quad (3)$$

which is an affine-input nonlinear system $(f, g, h, z_0)$ having quadratic polynomial components [16]. (Here $(v)_j$ denotes the $j$-th component of $v \in \mathbb{R}^m$.) In 1972, Brockett asked under what conditions is bilinearity preserved under composition [3]. One trivial sufficient condition can be identified immediately from the state space system above: when a bilinear system is followed by a linear system. The composite system is bilinear since in this case $N_{j,2} = 0$, $j = 1, 2, \ldots, m$. But this condition is very restrictive and not necessary. In 1979, Ferfera provided in [6], [7] a much less restrictive sufficient condition using formal power series representations of the input-output mappings, namely, $F_{c_i} : u_i \mapsto y_i$, where $c_i$ is a generating series written in terms of a noncommutative alphabet $X = \{x_0, x_1, \ldots, x_m\}$ [9]–[11]. In this setting, system composition can be described by $F_{c_2} \circ F_{c_1} = F_{c_2 \circ c_1}$, where $c_2 \circ c_1$ denotes the composition product of two formal power series [6], [7], [14], [18]. Bilinearity is then equivalent to having a *rational* or *regular* generating series [1]. Ferfera introduced the notion of an *input-limited* rational series and showed that rationality is preserved under composition when an arbitrary rational series is followed by an input-limited rational series. (It is easily demonstrated that this condition is not necessary.) Therefore, the well known correspondence between rational series and formal power series recognized by weighted finite-state automata [17], [21], [22] is in general not applicable when cascades are introduced.

The goals of this paper are two-fold. First, the canonical counterexample that Ferfera introduced to demonstrate the loss of rationality under cascades will be significantly expanded upon. This then motivates an alternative proof of the theorem concerning the input-limited criterion which is much simpler than the original. In addition, this example provides an ideal illustration concerning the second goal of this paper, which is to show that the generating series of a cascade of two bilinear systems can always be put in correspondence with the generating series of a certain weighted Petri net. This result is an application of recent work by Foursov and Hespel [13] and promises to provide a more complete characterization of bilinear cascades when combined with their notion of multiset weighted grammars [12].

The paper is organized as follows. First some preliminaries are summarized in Section 2 to better frame the problems and introduce the notation. In the next section, the innovations concerning the composition product are presented. In Section 4, the connection between the generating series of cascaded bilinear systems and weighted Petri nets is developed. Directions for future research are suggested in the final section.

## II. PRELIMINARIES: RATIONAL SERIES AND FLIESS OPERATORS

A finite nonempty set of noncommuting symbols $X = \{x_0, x_1, \ldots, x_m\}$ is called an *alphabet*. Each element of $X$ is called a *letter*, and any finite string of letters from $X$, $\eta = x_{i_k} \cdots x_{i_1}$, is called a *word* over $X$. The *length* of $\eta$, $|\eta|$, is the number of letters in $\eta$. The set of all words with length $k$ will be denoted by $X^k$. The set of all words including the empty word, $\emptyset$, will be denoted

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by $X^*$. It forms a monoid under catenation. A *language* is any subset of $X^*$. Any mapping $c : X^* \to \mathbb{R}^\ell$ is called a *formal power series*. The value of $c$ at $\eta \in X^*$ is written as $(c, \eta)$. Typically, $c$ is represented as the formal sum $c = \sum_{\eta \in X^*}(c, \eta)\eta$. The collection of all formal power series over $X$ is denoted by $\mathbb{R}^\ell\langle\langle X\rangle\rangle$. It forms an $\mathbb{R}$-algebra under the Cauchy product. Given $c \in \mathbb{R}^\ell\langle\langle X\rangle\rangle$, the subset of $X^*$ defined by $\mathrm{supp}(c) = \{\eta : (c, \eta) \neq 0\}$ is called the *support* of $c$. The subset of $\mathbb{R}^\ell\langle\langle X\rangle\rangle$ consisting of all the series with finite support is denoted by $\mathbb{R}^\ell\langle X\rangle$, and its elements are called *polynomials*. A series $c \in \mathbb{R}\langle\langle X\rangle\rangle$ is called *proper* if $\emptyset \notin \mathrm{supp}(c)$ and *invertible* if there exists a series $c^{-1} \in \mathbb{R}\langle\langle X\rangle\rangle$ such that $cc^{-1} = c^{-1}c = 1$. In the event that $c$ is not proper, it is always possible to write $c = (c, \emptyset)(1 - c')$, where $c' \in \mathbb{R}\langle\langle X\rangle\rangle$ is proper. It then follows that

$$c^{-1} = \frac{1}{(c, \emptyset)}(1 - c')^{-1} = \frac{1}{(c, \emptyset)}(c')^*,$$

where $(c')^* := \sum_{i\geq 0}(c')^i$. It can be shown that $c$ is invertible if and only if $c$ is not proper. Now let $S$ be any subalgebra of the $\mathbb{R}$-algebra $\mathbb{R}\langle\langle X\rangle\rangle$. $S$ is said to be *rationally closed* when every invertible $c \in S$ has $c^{-1} \in S$. The *rational closure* of any set $E \subset \mathbb{R}\langle\langle X\rangle\rangle$ is the smallest rationally closed subalgebra of $\mathbb{R}\langle\langle X\rangle\rangle$ containing $E$.

*Definition 1:* [1] A series $c \in \mathbb{R}\langle\langle X\rangle\rangle$ is **rational** if it belongs to the rational closure of $\mathbb{R}\langle X\rangle$.

Thus, a given rational series can be obtained from a finite set of polynomials by performing a finite number of additions, scalar products, catenation products and inversions (or star operations), the so called *rational operations*. The following definitions and theorem provide another characterization of rational series which can be used to establish the precise connection between rational series and series recognized by weighted finite-state automaton [22].

*Definition 2:* A **linear representation** of a series $c \in \mathbb{R}\langle\langle X\rangle\rangle$ is any triple $(\mu, \gamma, \lambda)$, where $\mu : X^* \to \mathbb{R}^{n\times n}$ is a monoid morphism, $\gamma,\ \lambda^T \in \mathbb{R}^{n\times 1}$, and $(c, \eta) = \lambda\mu(\eta)\gamma$ for all $\eta \in X^*$.

*Definition 3:* A series is called **recognizable** if it has a linear representation.

*Theorem 1:* [22] A formal power series is rational if and only if it is recognizable.

For each $c \in \mathbb{R}^\ell\langle\langle X\rangle\rangle$, one can formally associate a causal $m$-input, $\ell$-output operator, $F_c$, in the following manner. Let $\mathfrak{p} \geq 1$ and $t_0 < t_1$ be given. For a measurable function $u : [t_0, t_1] \to \mathbb{R}^m$, define $\|u\|_\mathfrak{p} = \max\{\|u_i\|_\mathfrak{p} : 1 \leq i \leq m\}$, where $\|u_i\|_\mathfrak{p}$ is the usual $L_\mathfrak{p}$-norm for a measurable real-valued function, $u_i$, defined on $[t_0, t_1]$. Let $L^m_\mathfrak{p}[t_0, t_1]$ denote the set of all measurable functions defined on $[t_0, t_1]$ having a finite $\|\cdot\|_\mathfrak{p}$ norm and $B^m_\mathfrak{p}(R)[t_0, t_1] := \{u \in L^m_\mathfrak{p}[t_0, t_1] : \|u\|_\mathfrak{p} \leq R\}$. Define recursively for each $\eta \in X^*$ the mapping $E_\eta : L^m_1[t_0, t_1] \to \mathcal{C}[t_0, t_1]$ by setting $E_\emptyset[u] \equiv 1$, and

$$E_{x_i\bar{\eta}}[u](t, t_0) = \int_{t_0}^{t} u_i(\tau)E_{\bar{\eta}}[u](\tau, t_0)\, d\tau,$$

where $x_i \in X$, $\bar{\eta} \in X^*$, and $u_0(t) \equiv 1$. The input-output operator corresponding to $c$ is then

$$F_c[u](t) = \sum_{\eta\in X^*}(c, \eta)\, E_\eta[u](t, t_0),$$

which is referred to as a *Fliess operator* [9]–[11], [14]–[16], [18]. When there exist real numbers $K, M > 0$ such that $|(c, \eta)| \leq KM^{|\eta|}|\eta|!$ for all $\eta \in X^*$, where $|z| := \max\{|z_1|, |z_2|, \ldots, |z_\ell|\}$ when $z \in \mathbb{R}^\ell$, then $F_c$ constitutes a well-defined operator from $B^m_\mathfrak{p}(R)[t_0,\, t_0 + T]$ into $B^\ell_\mathfrak{q}(S)[t_0,\, t_0 + T]$ for sufficiently small $R, S, T > 0$, where the numbers $\mathfrak{p}, \mathfrak{q} \in [1, \infty]$ are conjugate exponents, i.e. $1/\mathfrak{p}+1/\mathfrak{q} = 1$ [15]. Such a power series $c$ is said to be *locally convergent*. $F_c$ will be referred to as a *rational operator* whenever $c$ is rational. It can be easily shown via Theorem 1 that every rational series is locally convergent. In fact, they always respect the more strict growth condition $|(c, \eta)| \leq KM^{|\eta|}$ for all $\eta \in X^*$. Given any linear representation $(\mu, \gamma, \lambda)$ of $c$, it follows that

$$c = \sum_{k=0}^{\infty}\sum_{i_1,\ldots,i_k=0}^{m}(\lambda N_{i_k}\cdots N_{i_1}\gamma)\, x_{i_k}\cdots x_{i_1},$$

where $N_i = \mu(x_i)$. Thus, the corresponding rational operator is realized by the bilinear realization

$$\dot{z}(t) = N_0 z(t) + \sum_{i=1}^{m} N_i z(t)\, u_i(t), \quad z(t_0) = \gamma \quad (4)$$

$$y(t) = \lambda z(t) \quad (5)$$

in the sense that (4) has a well-defined solution $\Phi(t, t_0, \gamma, u)$ on at least some interval $[t_0, t_1]$ for every $u \in B^m_\mathfrak{p}(R)[t_0, t_1]$ with $\mathfrak{p} \geq 1$ and $R > 0$ sufficiently small, and

$$F_c[u](t) = \lambda\Phi(t, t_0, \gamma, u), \quad \forall t \in [t_0, t_1].$$

## III. Cascaded Systems

The cascade of two Fliess operators $F_c$ and $F_d$, where $c \in \mathbb{R}^\ell\langle\langle X\rangle\rangle$ and $d \in \mathbb{R}^m\langle\langle X\rangle\rangle$, can be described in terms of the composition product defined below.

*Definition 4:* [6], [7] For any $\eta \in X^*$ and series $d \in \mathbb{R}^m\langle\langle X\rangle\rangle$, the **composition** of $\eta$ with $d$ is defined in a recursive manner by

$$\eta \circ d = \begin{cases} \eta & : \ |\eta|_{x_i} = 0,\ \forall\ i \neq 0 \\ x_0^{k+1}[d_i \sqcup\!\sqcup (\eta' \circ d)] & : \ \eta = x_0^k x_i \eta',\ k \in \mathbb{N}, \\ & \quad\ i \neq 0,\ \eta' \in X^*, \end{cases}$$

where $\sqcup\!\sqcup$ denotes the shuffle product on $\mathbb{R}\langle\langle X\rangle\rangle$ [1, p. 20], $|\eta|_{x_i}$ is the number of times the letter $x_i$ appears in $\eta$, and $d_i : \xi \mapsto (d, \xi)_i$ with $(d, \xi)_i$ being the $i$-th component of the coefficient $(d, \xi)$. The **composition** of any $c \in \mathbb{R}^\ell\langle\langle X\rangle\rangle$ with $d$ is

$$c \circ d = \sum_{\eta\in X^*}(c, \eta)\eta \circ d.$$

*Theorem 2:* [6], [14] Let $c \in \mathbb{R}^\ell\langle\langle X\rangle\rangle$ and $d \in \mathbb{R}^m\langle\langle X\rangle\rangle$. The composition $F_c \circ F_d$ has generating series $c \circ d$, i.e., $F_c \circ F_d = F_{c\circ d}$. In addition, if $c$ and $d$ are locally convergent then $c \circ d$ is also locally convergent.

The following example is due to Ferfera [6], [7]. It shows that the composition product does not preserve rationality. The approach taken here, however, is distinct from the existing analysis in that it can be completely generalized as demonstrated at the end of this section.

*Example 1:* Suppose $X = \{x_0, x_1\}$ and consider the rational series $c = (1 - x_1)^{-1} = x_1^*$. The claim is that $c$ composed with itself is not rational. The main goal is to show that

$$(c \circ c, x_0^{k_0} x_1^{k_1}) = (k_0)^{k_1}, \;\; k_0 \geq 0, \;\; k_1 \geq 0,$$

or equivalently,

$$(x_1^{-k_1} x_0^{-k_0}(c \circ c), \emptyset) = (k_0)^{k_1}. \tag{6}$$

Here the left-shift operator $\xi^{-1}(\cdot)$ is defined for any $\xi \in X^*$ by

$$\begin{aligned} \xi^{-1} \;&:\; X^* \to \mathbb{R}\langle X \rangle \\ &:\; \eta \mapsto \begin{cases} \eta' & : \;\; \eta = \xi\eta' \\ 0 & : \;\; \text{otherwise}, \end{cases} \end{aligned}$$

and $\xi^{-1}(c) := \sum_{\eta \in X^*} (c, \eta)\, \xi^{-1}(\eta)$. (The left-shift $(x_0^k)^{-1}(\cdot)$ is denoted by $x_0^{-k}(\cdot)$.) The claim is trivial when $k_0 = k_1 = 0$ provided that $0^0 := 1$. If $k_0 = 1$ and $k_1 = 0$, observe that

$$\begin{aligned} x_0^{-1}(c \circ c) &= \underbrace{x_0^{-1}(c)}_{0} \circ c + c \sqcup\!\sqcup (\underbrace{x_1^{-1}(c)}_{c} \circ c) \\ &= c \sqcup\!\sqcup (c \circ c). \end{aligned}$$

The intermediate claim then is that

$$x_0^{-k_0}(c \circ c) = c^{\sqcup\!\sqcup\, k_0} \sqcup\!\sqcup (c \circ c), \;\; k_0 \geq 1.$$

If the identity above holds up to some fixed $k_0 \geq 1$ then

$$\begin{aligned} &x_0^{-k_0-1}(c \circ c) \\ &= x_0^{-1}(c^{\sqcup\!\sqcup\, k_0} \sqcup\!\sqcup (c \circ c)) \\ &= x_0^{-1}(c^{\sqcup\!\sqcup\, k_0}) \sqcup\!\sqcup (c \circ c) + c^{\sqcup\!\sqcup\, k_0} \sqcup\!\sqcup x_0^{-1}(c \circ c) \\ &= \Big[k_0 c^{\sqcup\!\sqcup\, (k_0-1)} \sqcup\!\sqcup \underbrace{x_0^{-1}(c)}_{0}\Big] \sqcup\!\sqcup (c \circ c) \\ &\quad + c^{\sqcup\!\sqcup\, k_0} \sqcup\!\sqcup (c \sqcup\!\sqcup (c \circ c)) \\ &= c^{\sqcup\!\sqcup\, (k_0+1)} \sqcup\!\sqcup (c \circ c). \end{aligned}$$

The identities

$$\begin{aligned} x^{-1}(c^{\sqcup\!\sqcup\, k}) &= k c^{\sqcup\!\sqcup\, (k-1)} \sqcup\!\sqcup x^{-1}(c) \\ x_0^{-1}(c \circ d) &= x_0^{-1}(c) \circ d + \sum_{i=1}^{m} d_i \sqcup\!\sqcup [x_i^{-1}(c) \circ d] \\ x_i^{-1}(c \circ d) &= 0, \;\; i = 1, 2, \ldots, m \end{aligned}$$

have been employed above. Hence, the intermediate identity in question holds for $k_0 \geq 0$. Next observe that

$$\begin{aligned} &x_1^{-1} x_0^{-k_0}(c \circ c) \\ &= x_1^{-1}(c^{\sqcup\!\sqcup\, k_0} \sqcup\!\sqcup (c \circ c)) \\ &= x_1^{-1}(c^{\sqcup\!\sqcup\, k_0}) \sqcup\!\sqcup (c \circ c) + c^{\sqcup\!\sqcup\, k_0} \sqcup\!\sqcup \underbrace{x_1^{-1}(c \circ c)}_{0} \\ &= k_0 c^{\sqcup\!\sqcup\, (k_0-1)} \sqcup\!\sqcup \underbrace{x_1^{-1}(c)}_{c} \sqcup\!\sqcup (c \circ c) \\ &= k_0 c^{\sqcup\!\sqcup\, k_0} \sqcup\!\sqcup (c \circ c). \end{aligned}$$

The second intermediate claim is that

$$x_1^{-k_1} x_0^{-k_0}(c \circ c) = (k_0)^{k_1} c^{\sqcup\!\sqcup\, k_0} \sqcup\!\sqcup (c \circ c).$$

If this is the case up to some fixed $k_1 \geq 1$ then

$$\begin{aligned} x_1^{-k_1-1} x_0^{-k_0}(c \circ c) &= x_1^{-1}((k_0)^{k_1} c^{\sqcup\!\sqcup\, k_0} \sqcup\!\sqcup (c \circ c)) \\ &= (k_0)^{k_1} \Big[x_1^{-1}(c^{\sqcup\!\sqcup\, k_0}) \sqcup\!\sqcup (c \circ c) + \\ &\qquad c^{\sqcup\!\sqcup\, k_0} \sqcup\!\sqcup \underbrace{x_1^{-1}(c \circ c)}_{0}\Big] \\ &= (k_0)^{k_1} \left[k_0 c^{\sqcup\!\sqcup\, k_0} \sqcup\!\sqcup (c \circ c)\right] \\ &= (k_0)^{k_1+1} c^{\sqcup\!\sqcup\, k_0} \sqcup\!\sqcup (c \circ c). \end{aligned}$$

Hence, this claim holds for all $k_1, k_0 \geq 0$. To validate (6), simply compare the constant coefficients in the above identity:

$$\begin{aligned} (x_1^{-k_1} x_0^{-k_0}(c \circ c), \emptyset) &= ((k_0)^{k_1} c^{\sqcup\!\sqcup\, k_0} \sqcup\!\sqcup (c \circ c), \emptyset) \\ (c \circ c, x_0^{k_0} x_1^{k_1}) &= (k_0)^{k_1}. \end{aligned}$$

Setting $k_0 = k_1$ reduces the expression to

$$(c \circ c, x_0^k x_1^k) = k^k, \;\; k \geq 0.$$

The key observation is that these coefficients of $c \circ c$ are growing faster than any sequence of coefficients from a rational series can possibly grow, that is, at a rate exceeding $KM^{|\eta|}$ for any real numbers $K, M > 0$. Therefore, the series $c \circ c$ can not be rational. □

The next definition and theorem describe Ferfera's sufficient condition for preserving rationality under composition. The original proof of this result, which appears only in [6], is a complex argument relying extensively on the theory of rational transductions [2], [8], [19]. A re-interpretation of this approach appeared in [5]. In this paper, a much simpler and shorter proof is presented. It employs only basic results concerning rational series and was ultimately motivated by the calculations presented in the previous example.

*Definition 5:* A series $c \in \mathbb{R}\langle\langle X \rangle\rangle$ is **limited relative to** $\boldsymbol{x_i}$ if there exists an integer $\mathcal{N}_i \geq 0$ such that

$$\sup_{\eta \in \mathrm{supp}(c)} |\eta|_{x_i} = \mathcal{N}_i < \infty.$$

If $c$ is limited relative to $x_i$ for every $i = 1, \ldots, m$ then c is said to be **input-limited**. In such cases, let $\mathcal{N}_c := \max_i \mathcal{N}_i$. A series $c \in \mathbb{R}^\ell\langle\langle X \rangle\rangle$ is input-limited if each component series, $c_j$, is input-limited for $j = 1, \ldots, \ell$. In this case, $\mathcal{N}_c := \max_j \mathcal{N}_{c_j}$.

*Theorem 3:* [6], [7] Let $c \in \mathbb{R}^\ell\langle\langle X \rangle\rangle$ and $d \in \mathbb{R}^m\langle\langle X \rangle\rangle$ be two rational series. If $c$ is input-limited then the series $c \circ d$ is rational.

The proof presented here relies on the following lemma.

*Lemma 1:* Let $c \in \mathbb{R}^{\ell}\langle\langle X\rangle\rangle$ be a rational series with a linear representation $(\mu,\gamma,\lambda)$. Let $N_i := \mu(x_i) \in \mathbb{R}^{n\times n}$, $i = 0,1,\ldots,m$. Then for any $d \in \mathbb{R}^m\langle\langle X\rangle\rangle$ it follows that

$$c \circ d = \sum_{\eta\in\hat{X}} \lambda D_\eta((N_0x_0)^*)\gamma,$$

where $\hat{X} := \{x_1,x_2,\ldots,x_m\}$, and $D_\eta$ is the monoid morphism defined by

$$\begin{aligned} D_{x_i} &: \mathbb{R}^{n\times n}\langle\langle X\rangle\rangle \to \mathbb{R}^{n\times n}\langle\langle X\rangle\rangle \\ &: E \mapsto x_0(N_0x_0)^*N_i(d_i \sqcup\!\sqcup E). \end{aligned}$$

(Here $D_\emptyset$ is taken to be the identity map, and the shuffle product above is defined componentwise.)

*Proof:* Without lost of generality, assume $\ell = 1$. Observe from the definition of the composition product that

$$\begin{aligned}
c \circ d &\\
= &\sum_{k\ge 0}\sum_{i_1,\ldots,i_k=1}^{m}\sum_{n_0,\ldots,n_k\ge 0} \lambda N_0^{n_k}N_{i_k}N_0^{n_{k-1}}N_{i_{k-1}}\cdots \\
&N_0^{n_1}N_{i_1}N_0^{n_0}\gamma\cdot x_0^{n_k}x_{i_k}x_0^{n_{k-1}}x_{i_{k-1}}\cdots x_0^{n_1}x_{i_1}x_0^{n_0}\circ d \\
= &\sum_{k\ge 0}\sum_{i_1,\ldots,i_k=1}^{m}\sum_{n_0,\ldots,n_k\ge 0} \lambda N_0^{n_k}N_{i_k}N_0^{n_{k-1}}N_{i_{k-1}}\cdots \\
&N_0^{n_1}N_{i_1}N_0^{n_0}\gamma\cdot x_0^{n_k+1}[d_{i_k}\sqcup\!\sqcup [x_0^{n_{k-1}+1}[d_{i_{k-1}}\sqcup\!\sqcup \cdots \\
&x_0^{n_1+1}[d_{i_1}\sqcup\!\sqcup x_0^{n_0}]\cdots]]] \\
= &\sum_{k\ge 0}\sum_{i_1,\ldots,i_k=1}^{m} \lambda x_0\left(\sum_{n_k\ge 0}(N_0x_0)^{n_k}\right)N_{i_k}\Bigg[d_{i_k}\sqcup\!\sqcup\cdot \\
&\Bigg[x_0\left(\sum_{n_{k-1}\ge 0}(N_0x_0)^{n_{k-1}}\right)N_{i_{k-1}}\Bigg[d_{i_{k-1}}\sqcup\!\sqcup\ \cdots \\
&x_0\left(\sum_{n_1\ge 0}(N_0x_0)^{n_1}\right)N_{i_1}\Bigg[d_{i_1}\sqcup\!\sqcup\cdot \\
&\left(\sum_{n_0\ge 0}(N_0x_0)^{n_0}\right)\Bigg]\cdots\Bigg]\Bigg]\Bigg]\gamma \\
= &\sum_{k\ge 0}\sum_{i_1,\ldots,i_k=1}^{m} \lambda x_0(N_0x_0)^*N_{i_k}\left[d_{i_k}\sqcup\!\sqcup\cdot\right. \\
&x_0\left[(N_0x_0)^*N_{i_{k-1}}\left[d_{i_{k-1}}\sqcup\!\sqcup\cdots\right.\right. \\
&\left.\left.\left. x_0(N_0x_0)^*N_{i_1}\ [d_{i_1}\sqcup\!\sqcup(N_0x_0)^*]\cdots\right]\right]\right]\gamma \\
= &\sum_{k\ge 0}\sum_{x_{i_k}\cdots x_{i_1}\in\hat{X}^k} \lambda D_{x_{i_k}}D_{x_{i_{k-1}}}\cdots D_{x_{i_1}}((N_0x_0)^*)\gamma \\
= &\sum_{\eta\in\hat{X}^*}\lambda D_\eta((N_0x_0)^*)\gamma,
\end{aligned}$$

and the lemma is proved. ■

*Proof of Theorem 3:* Since $c$ is input-limited, it follows from Lemma 1 that

$$c\circ d = \sum_{k=0}^{N_c}\sum_{\eta\in\hat{X}^k}\lambda D_\eta((N_0x_0)^*)\gamma.$$

Clearly each operator $D_\eta$ is mapping a rational series to another rational series as it involves only a finite number of rational operations (including the shuffle product [1]). Therefore, for any integer $k \ge 0$ the formal power series

$$\sum_{\eta\in\hat{X}^k}\lambda D_\eta((N_0x_0)^*)\gamma$$

is again rational since the summation is finite. Thus, $c\circ d$ must be rational. ■

*Example 2:* Reconsider the series $c\circ c$, where $c = x_1^*$ as in Example 1. The nested inductive argument used there can be directly extended to establish the identity

$$\begin{aligned}(c\circ c, x_0^{k_0}x_1^{k_1}\cdots x_0^{k_{l-1}}x_1^{k_l}) &= (k_0)^{k_1}(k_0+k_2)^{k_3}\cdots \\ &\quad (k_0+k_2+\cdots+k_{l-1})^{k_l} \quad (7)\end{aligned}$$

for all $l \ge 0$ and $k_i \ge 0$, $i = 0,1,\ldots,l$. In which case,

$$\begin{aligned}&(c\circ c, x_0^{n_0}x_1x_0^{n_1}x_1\cdots x_0^{n_{j-1}}x_1x_0^{n_j}) \\ &\quad = n_0(n_0+n_1)\cdots(n_0+n_1+\cdots+n_{j-1}) \quad (8)\end{aligned}$$

$$(c\circ c, x_1^{m_0}x_0x_1^{m_1}\cdots x_0x_1^{m_k}) = 0^{m_0}1^{m_1}2^{m_2}\cdots k^{m_k} \quad (9)$$

for all $j \ge 0$ and $n_i \ge 0$, $i = 0,1,\ldots,j$; and all $k \ge 0$ and $m_i \ge 0$, $i = 0,1,\ldots,k$. Using identity (9), observe that

$$\begin{aligned}
c\circ c &\\
= &\sum_{m_0\ge 0}(c\circ c, x_1^{m_0})\,x_1^{m_0} + \\
&\sum_{k\ge 1}\sum_{m_0,\ldots,m_k\ge 0}(c\circ c, x_1^{m_0}x_0x_1^{m_1}\cdots x_0x_1^{m_k})\cdot \\
&x_1^{m_0}x_0x_1^{m_1}\cdots x_0x_1^{m_k} \\
= &\ 1+\sum_{k\ge 1}\sum_{m_1,\ldots,m_k\ge 0}1^{m_1}2^{m_2}\cdots k^{m_k} \\
&x_0x_1^{m_1}x_0x_1^{m_2}\cdots x_0x_1^{m_k} \\
= &\ 1+\sum_{k\ge 1}x_0\left(\sum_{m_1\ge 0}x_1^{m_1}\right)x_0\left(\sum_{m_1\ge 0}(2x_1)^{m_2}\right)\cdots \\
&x_0\left(\sum_{m_k\ge 0}(kx_1)^{m_k}\right) \\
= &\ 1+\sum_{k\ge 1}x_0x_1^*x_0(2x_1)^*\cdots x_0(kx_1)^*. \qquad (10)
\end{aligned}$$

Alternatively, observe that $x_1^*$ has a linear representation with $N_0 = 0$, $N_1 = 1$ and $\lambda = \gamma = 1$. Thus, $D_{x_1}: e \to x_0(x_1^*\sqcup\!\sqcup e)$, and from Lemma 1

$$\begin{aligned}
c\circ c &= \sum_{\eta\in\hat{X}^*}\lambda D_\eta((N_0x_0)^*)\gamma \\
&= \sum_{k\ge 0}D_{x_1^k}(1) \\
&= 1+\sum_{k\ge 1}x_0(x_1^*\sqcup\!\sqcup(x_0(x_1^*\sqcup\!\sqcup(\cdots x_0(x_1^*\sqcup\!\sqcup 1)\cdots)))) \\
&= 1+\sum_{k\ge 1}x_0x_1^*x_0(2x_1)^*\cdots x_0(kx_1)^*,
\end{aligned}$$

which is consistent with (10). Clearly, if the first argument in $c \circ c$ is truncated, then the resulting series composition produces a rational series as expected from Theorem 3. □

## IV. Bilinear Cascades and Generating Series of Weighted Petri Nets

A wide variety of Petri net definitions appear in the literature [4], [20]. The focus here is on a class of marked Petri nets as described in [13].

*Definition 6:* A **marked Petri net** $(P, T, A, W, M_0)$ is a weighted bipartite graph, where

$P = \{z_1, z_2, \ldots, z_n\}$ is the set of places
$T = \{u_0, u_1, \ldots, u_m\}$ is the set of transitions
$A \subseteq (P \times T) \cup (T \times P)$ is the set of arcs from places to transitions and from transitions to places
$W : A \to \mathbb{N}$ is the arc weight function
$M_0 \in \mathbb{N}^n$ is an initial marking of the places.

*Definition 7:* A **weighted Petri net** is a marked Petri net $(P, T, A, W, M_0)$ with a transition weight function $K : T \to \mathbb{R}$.

The transition labels in $T$ need not be unique, but that generalization is not pursued here. With any weighted Petri net, one can associate a generating series in $\mathbb{R}\langle\langle X\rangle\rangle$. This is analogous to the way in which rational series are generated from weighted finite-state automata [17], [21], [22].

*Definition 8:* The **generating series** for a weighted Petri net $(P, T, A, W, M_0, K)$ is defined to be $c_P \in \mathbb{R}\langle\langle X\rangle\rangle$, where

$$\begin{aligned}(c_P, x_{j_1} x_{j_2} \cdots x_{j_r}) \quad &= \quad v_1^{k_1} v_2^{k_2} \cdots v_n^{k_n} \cdot \\ &\qquad K_1(u_{j_1}) K_2(u_{j_2}) \cdots K_r(u_{j_r}) \quad (11)\end{aligned}$$

if $(u_{j_1}, u_{j_2}, \ldots, u_{j_r})$ is an admissible firing sequence and the resulting terminal marking is $(k_1, k_2, \ldots, k_n) \in \mathbb{N}^n$. The fixed real number $v_i$ above denotes the *value* of a token in place $z_i$. The real numbers $K_l(u_{j_l})$ are computed according to the expression

$$K_l(u_{j_l}) = \binom{\tilde{k}_1}{w_1}\binom{\tilde{k}_2}{w_2} \cdots \binom{\tilde{k}_s}{w_s} K(u_{j_l}),$$

where the enabled transition $u_{j_l}$ has $s$ inputs with arc weights $w_1, w_2, \ldots, w_s$, and $\tilde{k}_1, \tilde{k}_2, \ldots, \tilde{k}_s$ denotes the number of tokens in each place to which these inputs are connected at the instant before $u_{j_{l+1}}$ fires. If a firing sequence is not admissible then the corresponding series coefficient is zero. For the empty word let $(c_P, \emptyset) = v_1^{k_1} v_2^{k_2} \cdots v_n^{k_n}$.

The importance of weighted Petri nets in connection with nonlinear dynamical systems can be explained in the context of the following definition and theorem. For brevity, only single-output systems are considered.

*Definition 9:* A **polynomial state space system** $(f, g, h, z_0)$ has the property that $f$, $g$ and $h$ have only polynomial components. Without loss of generality, it is assumed that $h(z) = z_1^{k_1} z_2^{k_2} \cdots z_n^{k_n}$, where $n$ denotes the number of system states.

*Theorem 4:* [13] The generating series for a polynomial system $(f, g, h, z_0)$ is equivalent to the generating series of a weighted Petri net $(P, T, A, W, M_0)$, where $A$ and $W$ have the property that each transition has exactly one input and its arc weight is 1. Specifically,

- The places of the Petri net correspond to the states of the dynamical system.
- The transitions of the Petri net corresponding to the inputs of the dynamical system, or equivalently, the letters of the alphabet $X$.
- Each term in the summand on the right-hand side of the equation for $\dot{z}_i$, i.e., $K(u_j) u_j z_1^{w_1} z_2^{w_2} \cdots z_n^{w_n}$, corresponds to a transition labeled $u_j$ with transition weight $K(u_j)$ and having a single input from place $z_i$ with arc weight 1 and outputs to places $z_s$ with arc weights $w_s$ for $s = 1, 2, \ldots, n$.
- The initial marking $M_0 = (k_{1,0}, k_{2,0}, \ldots, k_{n,0})$, where the output function is $h(z) = z_1^{k_{1,0}} z_2^{k_{2,0}} \cdots\ z_n^{k_{n,0}}$.
- The value $v_i$ of a token at place $z_i$ is taken to be the initial state $z_{i,0}$. Therefore, equation (11) becomes

$$\begin{aligned}(c_P, x_{j_1} x_{j_2} \cdots x_{j_r}) \quad &= \quad z_{1,0}^{k_1} z_{2,0}^{k_2} \cdots z_{n,0}^{k_n} \cdot \\ &\qquad K_1(u_{j_1}) K_2(u_{j_2}) \cdots K_r(u_{j_r}). \quad (12)\end{aligned}$$

This theorem provides a graph theoretic interpretation of the usual process of computing the generating series for a dynamical system via iterated Lie derivatives of the output function $h$ with respect to the vector fields $f$ and $g$ when all the component functions are polynomial [16]. The following lemma makes the essential link between a class of weighted Petri nets and cascade connections of bilinear systems.

*Lemma 2:* The cascade connection of any two single-input, single output bilinear state space systems is a quadratic polynomial state space system.

*Proof:* The claim is immediately evident from (1)-(3). Introducing the additional state $\tilde{z} = C_2 z_2$ yields an input-output equivalent polynomial system with output $y_2 = \tilde{h}(\tilde{z}) = \tilde{z}$. ■

The following theorem is an immediate consequence of this lemma, Theorem 1, Theorem 3 and Theorem 4. It is illustrated using Ferfera's example from the previous section.

*Theorem 5:* Let $X = \{x_0, x_1\}$ and $c, d \in \mathbb{R}\langle\langle X\rangle\rangle$ be two rational series. Then $c \circ d$ is equivalent to the generating series of a weighted Petri net corresponding to a quadratic polynomial state space system. If $c$ is input-limited, it is also equivalent to the generating series of a weighted finite-state automaton.

*Example 3:* Reconsider the series in Examples 1 and 2. It is easily verified that $x_1^*$ is realized by

$$\begin{aligned}\dot{z} \quad &= \quad z u_1, \quad z_0 = 1 \\ y \quad &= \quad z.\end{aligned}$$

The corresponding weighted Petri net is shown in Figure 1. In this case, $P = \{z\}$, $T = \{u_0, u_1\}$, $A$ and $W$ are evident from the diagram, $M_0 = 1$, $K(u_0) = 0$, $K(u_1) = 1$ and $v = z_0 = 1$. (The transition for $u_0$ is not shown in this case since its weight is zero.) The admissible firing sequences are of the form $(u_1^r) := (\underbrace{u_1, u_1, \ldots, u_1}_{r \text{ times}})$, $r \geq 0$. In which case, $K_i(u_1) = 1$ for all $i \geq 0$. From equation (12) then the generating series for the Petri net is $c_P = 1 + x_1 + x_1^2 +$

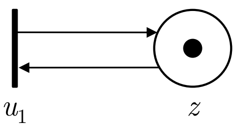


Fig. 1. The weighted Petri net for $x_1^*$ with initial marking.

$\cdots = x_1^*$ as expected. For the cascade connection $x_1^* \circ x_1^*$, a corresponding polynomial state space system is clearly

$$\begin{aligned} \dot{z}_1 &= z_1 u_1, \quad z_{1,0} = 1 \\ \dot{z}_2 &= z_1 z_2, \quad z_{2,0} = 1 \\ y &= z_2. \end{aligned}$$

The associated weighted Petri net is shown in Figure 2. Here

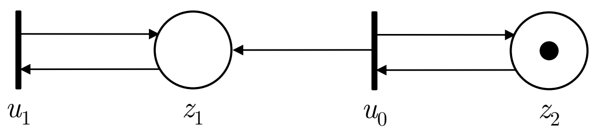


Fig. 2. The weighted Petri net for $x_1^* \circ x_1^*$ with initial marking.

$P = \{z_1, z_2\}$, $T = \{u_0, u_1\}$, $A$ and $W$ are shown, $M_0 = (0,1)$, $K(u_i) = 1$ and $v_i$=1 for $i = 1, 2$. Clearly, any firing sequence of the form $(u_1, u_{i_2}, u_{i_3}, \ldots)$ is *not* admissible, so the coefficient $(c_P, x_1\eta) = 0$ for any $\eta \in X^*$. All other firing sequences are admissible. This is consistent with the coefficients of $x_1^* \circ x_1^*$ as determined by equation (7). Consider the firing sequence $(u_0^{k_0})$. The resulting marking is shown in Figure 3, and it easily verified that $K_i(u_0) = 1$ for $i = 1, 2, \ldots, k_0$. This marking will not change if $u_1$ is then

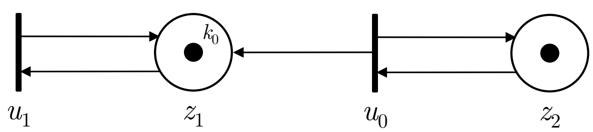


Fig. 3. The weighted Petri net for $x_1^* \circ x_1^*$ after the firing sequence $(u_0^{k_0})$ occurs.

fired $k_1$ times, but in this case

$$K_{k_0+i}(u_1) = \binom{k_0}{1} K(u_1), \quad i = 1, 2, \ldots, k_1.$$

Thus,

$$\begin{aligned} (c_P, x_0^{k_0} x_1^{k_1}) &= K_1(u_0) K_2(u_0) \cdots K_{k_0}(u_0) \cdot \\ &\quad K_{k_0+1}(u_1) K_{k_0+2}(u_1) \cdots K_{k_0+k_1}(u_1) \\ &= \binom{k_0}{1}\binom{k_0}{1} \cdots \binom{k_0}{1} \\ &= (k_0)^{k_1}. \end{aligned}$$

Continuing the process of firing $u_0$ in succession followed by firing $u_1$ in succession will directly generate the coefficients of $x_1^* \circ x_1^*$ as given in equation (7). □

## V. Future Research

The connection between bilinear cascades could be further investigated in several ways. First, it may be possible to refine the description of the subclass of weighted Petri nets that corresponding bilinear cascades. This analysis, coupled with Petri net simulation software, might provide an efficient automated method for computing the generating series of such interconnections. Other system interconnections, e.g., feedback connections, can also be explored in this context. Finally, it may be possible to classify the underlying Petri net language involved in bilinear cascades, perhaps using the multiset weighted grammars that are described in [12].

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