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FROM FEEDBACK TO CASCADE-INTERCONNECTED SYSTEMS: BREAKING THE LOOP

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Abstract— **The purpose of this paper is to observe that a feedback interconnection is equivalent to a *cascaded* interconnection "if you twist your eyes". We establish conditions under which a feedback interconnected (time-invariant or non-autonomous) system can be regarded as a cascaded *time-varying* system. The "technique" finds motivation in numerous particular applications, notably, in output feedback (observer-based) control where two subsystems are feedback interconnected and it results desirable to analyze the system as a *cascade*. Indeed, cascaded-based design allows for considerably simple controllers that exploit the physical structure of the system and sufficient conditions for stability of cascaded systems are often easier to apply than finding strict Lyapunov functions. As an illustration of how to use our main results we revisit a recently published article on a problem that, to the best of our knowledge, remains open for many years: to establish uniform global asymptotic stability of the closed-loop system for robot manipulators via (dynamic) position feedback.**

I. INTRODUCTION

A. Problem formulation

Consider the feedback-interconnected system

$$\Sigma_1 \quad : \quad \dot{x}_1 = f_1(t, x_1) + g(t, x_1, x_2) \qquad \text{(1a)}$$

$$\Sigma_2 \quad : \quad \dot{x}_2 = f_2(t, x_1, x_2) \qquad \text{(1b)}$$

where, for $i \in \{1, 2\}$, all functions are Lipschitz with respect to $x_i \in \mathbb{R}^{n_i}$ uniformly in t hence, for any pair of initial conditions, solutions are locally defined. We denote the initial conditions by $(t_\circ, x_\circ) \in \mathbb{R} \times \mathbb{R}^{n_1+n_2}$ where $x_\circ := [x_{1\circ}^\top, x_{2\circ}^\top]^\top$ and the solutions

$$x(t, t_\circ, x_\circ) := \begin{pmatrix} x_1(t, t_\circ, x_\circ) \\ x_2(t, t_\circ, x_\circ) \end{pmatrix}.$$

The problem we are interested in is to establish sufficient conditions for uniform global asymptotic stability bypassing the often difficult to apply Lyapunov's direct method. Our approach is to regard the feedback-interconnected system (Σ_1, Σ_2) as a cascade –*cf.* Fig. (1) To that end, we regard Eq. (1b) as an equation of the state x_2 where f_2 depends on the *parameter* x_1. Correspondingly, the parameterized solutions of Σ_2 are denoted by $x_2(t, t_\circ, x_{2\circ}, x_1)$. When the parameter x_1 takes the values generated by the solution $x_1(\cdot)$ of (1a) we obtain the *cascaded* system

$$\Sigma_1' \quad : \quad \dot{\xi}_1 = f_1(t, \xi_1) + g(t, \xi_1, \xi_2)\xi_2 \qquad \text{(2a)}$$

$$\Sigma_2' \quad : \quad \dot{\xi}_2 = f_2(t, x_1(t), \xi_2) \qquad \text{(2b)}$$

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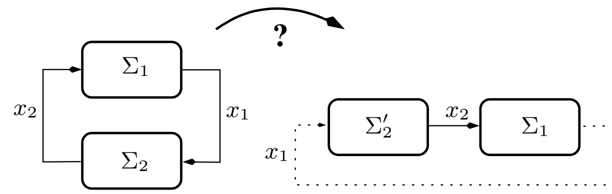

Fig. 1. A feedback interconnection is equivalent to a cascaded system "*if you twist your eyes*"

where we have used the compact notation $x_1(t)$ to denote a solution of (1a) with initial conditions $(t_\circ', x_\circ)$. That is, for each pair $(t_\circ', x_\circ)$ we define $\tilde{f}_2(t, \xi_2) := f_2(t, x_1(t, t_\circ', x_\circ), \xi_2)$ and rewrite (2b) as

$$\Sigma_2' \quad : \quad \dot{\xi}_2 = \tilde{f}_2(t, \xi_2). \qquad (3)$$

The function $\tilde{f}_2$ is parameterized by the initial conditions $(t_\circ', x_\circ)$ and, specifically, by $x_{1\circ}$. By the regularity properties of f_i and g it follows that the solutions $\xi(t, t_\circ', \xi_\circ)$ with $\xi_\circ = (\xi_{1\circ}, \xi_{2\circ})$ are locally defined from the initial time $\max\{t_\circ, t_\circ'\}$. Furthermore, for each pair of initial conditions $(t_\circ', \xi_\circ) = (t_\circ, x_\circ)$ the solutions of the *cascaded* system (2a), (3) *i.e.*, $\xi(t, t_\circ, x_\circ)$, coincide for all $t \geq t_\circ$ with those of the feedback-interconnected system (1) *i.e.*, $\xi(t, t_\circ, x_\circ) \equiv x(t, t_\circ, x_\circ)$.

Such observation is useful in cases when analyzing the dynamics of (2) is an easier task than studying (1); for instance, we can conclude that $x = 0$ is a UGAS equilibrium for (1) if $\xi = 0$ is UGAS for (2) however, we emphasize that 'U' in 'UGAS for (2)' stands for 'uniformity with respect to the initial conditions and the parameter $x_{1\circ}$. The interest of this is that to study the stability of (2a), (3) –hence of (1)– we can rely on theorems for cascaded systems –*cf.* [1] and references therein (such as [2]), as opposed to classical Lyapunov theory which is stymied by the difficulty of finding a strict Lyapunov function.

Generally speaking, under the following hypotheses:

Assumption 1 $x_1 = 0$ is a UGAS equilibrium for

$$\dot{x}_1 = f_1(t, x_1) \qquad (4)$$

Assumption 2 $x_2 = 0$ is a UGAS equilibrium for

$$\dot{x}_2 = \tilde{f}_2(t, x_2); \qquad (5)$$

to establish UGAS of the cascade (2a), (3) it is sufficient and necessary that the solutions are uniformly globally bounded. The necessity is obvious (as UGAS ⇒ UGB), sufficiency was established for autonomous systems in [3], [4] and for time-varying systems in [5].

In the context of interest here, the function $\tilde{f}_2$ (resp. f_2) depends on the trajectories $x_1(t)$ (resp. on the parameter x_1) therefore, properties such as stability and attractivity (rate of convergence) of the null solution $\xi_2 = 0$ of (3) depend on $x_{1\circ}$. Furthermore, for the solutions of (3) to make any sense the trajectories $x_1(t)$ must exist at least locally. While local existence may be implied by common appropriate regularity assumptions (Lipschitz) forward completeness is not obvious in general. To illustrate this point, consider the system

$$\dot{x}_1 := -x_1 + 2x_1^2 \operatorname{sat}(x_2) \quad (6a)$$

$$\dot{x}_2 := -[1 + x_1^2]x_2 \,. \quad (6b)$$

where sat$(\cdot)$ denotes a smooth saturation function with saturation level, one (*e.g.*, $\tanh(\cdot)$). Assumption 1 holds and, defining $V(x_2(t)) := 0.5x_2(t)^2$ yields $\dot{V}(x_2(t)) \leq -2V(x_2(t))$ hence,

$$|x_2(t)| \leq \mathrm{e}^{-(t-t_\circ)} \,.$$

One is tempted to conclude that Assumption 2 also holds however, strictly speaking, this bound holds only on the interval of existence of $x_1(\cdot)$ which is finite for certain choices of initial conditions. All we can say is that "as long as solutions exist, $x_2(t)$ tends to zero".

Consequently, theorems on stability of cascades do not apply *directly* on systems of the form (2a), (3). In this note we take a step back (with respect to existing literature on UGAS of cascades) in the stability analysis of cascades and lay conditions for forward completeness and uniform boundedness of the solutions of (2) – hence of (1). Under these conditions one may apply theorems on stability of cascaded systems to analyze the qualitative behavior of (1).

B. A motivating example

The example of Eq. (6) illustrates the type of results we seek; for further motivation we emphasize now a stability problem that naturally appears in the context of observer-based certainty-equivalence output feedback control –*cf.* *e.g.*, [6] and which may be treated (under appropriate conditions) using theorems on cascaded systems. Consider the system

$$\dot{x} = A(y)x + B(y)u \quad (7)$$

$$y = h(x) \quad (8)$$

with control $u \in \mathbb{R}^n$, unmeasurable state $x \in \mathbb{R}^n$ and measurable output $y = h(x)$; for simplicity let all functions be smooth. The control goal is to design an observer-based certainty equivalence controller to drive the state to zero. Furthermore, it is desired to establish uniform global asymptotic stability of the closed-loop system, under the condition that $u = u^*(x, y)$ renders the origin of

$$\dot{x} = A(y)x + B(y)u^*(x, y) \quad (9)$$

UGAS. Let $\hat{x}$ denote an estimate of x and $\bar{x} := \hat{x} - x$. The system in closed loop with the certainty-equivalence control $u^*(\hat{x}, y)$ is written as

$$\dot{x} = f_1(t, x) + g(t, x, \bar{x}) \quad (10)$$

where

$$\begin{aligned} f_1(t,x) &:= A(h(x))x + B(h(x))u^*(x, h(x)) \\ g(t,x,\bar{x}) &:= B(h(x))\,[\,\hat{u}^*(x,\bar{x}) - u^*(x, h(x))\,] \quad (11) \\ \hat{u}^*(x,\bar{x}) &:= u^*(\hat{x}, h(x)) = u^*(\bar{x} + x, h(x)) \,. \end{aligned}$$

Clearly, (10) is of the form (1a) with $x_1 := x$ and $x_2 := \bar{x}$. On the other hand, it is fairly standard to design an observer for (7) by defining

$$\dot{\hat{x}} = A(y)\hat{x} + B(y)u - L(y)[h(\hat{x}) - h(x)] \quad (12)$$

where L is the observer gain. Subtracting (7) from (12) we obtain

$$\dot{\bar{x}} = A(y)\bar{x} - L(y)[h(\hat{x}) - h(x)] \quad (13)$$

which may be written as (1b) with

$$f_2(t, x_1, x_2) := A(h(x_1))x_2 - L(h(x_1))[h(x_2 + x_1) - h(x_1)] \quad (14)$$

To establish stability of the closed-loop system (10), (11), (14) relying on classical Lyapunov theory may become a considerably difficult task and often leads to unnecessarily conservative conditions on design gains or restrictions on the domain of attraction. Instead, one may follow the reasoning explained in the previous section *i.e.*, to regard the feedback interconnected system (10), (14) as a cascade. To that end, we write

$$\left\{ \begin{array}{l} \dot{x} = A(y)x + B(y)u^*(x,y) + B(y)\,[\,\hat{u}^*(x,\bar{x}) - u^*(x,y)\,] \\ y := h(x) \end{array} \right. \quad (15a)$$

$$\dot{\bar{x}} = A(y(t))\bar{x} - L(y(t))[h(\bar{x} + x(t)) - y(t)] \,. \quad (15b)$$

That is, Eq. (15b) is of the form (3). In general, even if we dispose of $V_2(t, x_2)$ that satisfies the usual assumptions of Lyapunov's direct method, as illustrated for system (6), strictly speaking, the conclusion is valid only on the interval of existence of $x(\cdot)$ hence of $y(t)$ and, respectively, of the right-hand side of (15b). To circumvent this obstacle a common assumption in observer-design theory is to suppose that state-trajectories are *bounded*; while this seems a fairly reasonable assumption in *observer* design, it is too restrictive in the context of output feedback certainty-equivalence *control*. Indeed, the supposedly bounded trajectories $h(x(t))$ are indirectly generated by $\bar{x}(t)$ and vice-versa. The main results of this paper allow, but are not limited to, avoiding the restrictive boundedness assumption.

Remark 1 In the example above, we have assumed that the system is linear in the unmeasured states. Generally speaking, a typical technical obstacle in the problem of observer design *alone* is the presence of high order nonlinearities which depend on the unmeasured states. In such cases, one may rely on other types of restrictions such as sector bounds (see *e.g.*, [7], [8] for passivity-based designs), Lipschitz continuity (see, *e.g.*, [9] for an early high-gain design and [10], [11] for more recent references). Other approaches consist in finding coordinate transformations such that the system dynamics becomes linear in the unmeasured variables (see, *e.g.*, [12], [13]).

II. Main results

A. Forward completeness

In our *analysis* we consider system (1b) as a *parameterized* with parameter $x_1 \in \mathbb{R}^{n_1}$. Then, the solutions of (1b) with initial conditions $(t_\circ, x_{2\circ}) \in \mathbb{R} \times \mathbb{R}^{n_2}$ are denoted for each fixed x_1, by $x_2(t, t_\circ, x_{2\circ}, x_1)$.

Assumption 3 Let $t_\circ \geq 0$ and $t_{\max} \in [t_\circ, \infty]$. Assume that the solutions of (1) are defined on $[t_\circ, t_{\max})$. More precisely, there exist class $\mathcal{K}$ functions σ_{11}, σ_{12}, σ_{21}, σ_{22} and positive constants c_{11}, c_{22} such that for all $t \in [t_\circ, t_{\max})$

$$|x_1(t, t_\circ, x_\circ)| \leq \sigma_{11}(t - t_\circ) + \sigma_{12}(|x_\circ|) + c_{12} \quad (16a)$$
$$|x_2(t, t_\circ, x_{2\circ}, x_1)| \leq \sigma_{21}(t - t_\circ) + \sigma_{22}(|x_{2\circ}|) + c_{22} \quad (16b)$$

and $\sigma_{i1}(t - t_\circ) \to \infty$ as $t \to t_{\max}$.

The functions σ above are independent of the initial conditions and, moreover, σ_{2i} are independent of the "parameter" x_1. In the particular case when $t_{\max} = +\infty$, Assumption 3 becomes a condition of forward completeness, uniform in the initial conditions. Furthermore, for system (1b) it is assumed that forward completeness holds uniformly for any value of x_1 (the divergence rate is independent of x_1). Following [14] we see that this holds if and only if there exists a positive definite radially unbounded function $V_2(t, x_2)$ such that, for all $x_1 \in \mathbb{R}^{n_1}$,

$$\dot{V}(t, x_2) := \frac{\partial V_2}{\partial t} + \frac{\partial V_2}{\partial x_2} f_2(t, x_1, x_2) \leq V(t, x_2)\,.$$

Finding such a function V_2 is a hard task in general however, walking down the lines of [15], one can also impose the following condition with obvious changes. For the sequel the assumption given below is stated for system Σ_1.

Assumption 4 There exists a $\mathcal{C}^1$ positive definite radially unbounded function $\widetilde{V} : \mathbb{R}\times\mathbb{R}^{n_1} \to \mathbb{R}_{\geq 0}$, $\alpha_1 \in \mathcal{K}_\infty$ and continuous non-decreasing functions α_4, $\alpha_4' : \mathbb{R}_{\geq 0} \times \mathbb{R} \to \mathbb{R}_{\geq 0}$ such that

$$\widetilde{V}(t, x_1) \geq \alpha_1(|x_1|)$$

and defining,

$$\dot{\widetilde{V}}_{(1a)}(t, x_1) := \frac{\partial \widetilde{V}}{\partial t} + \frac{\partial \widetilde{V}}{\partial x_1}\left[f_1(t, x_1) + g(t, x_1, x_2)\right]$$

we have

$$\dot{\widetilde{V}}_{(1a)}(t, x_1) \leq \alpha_4(|x_1|)\alpha_4'(|x_2|)\,; \quad (17a)$$
$$\int_a^\infty \frac{d\tilde{v}}{\alpha_4(\alpha_1^{-1}(\tilde{v}))} = \infty \quad (17b)$$

Lemma 1 *Let Assumptions 3 and 4 hold. Then, the solutions are defined for all $t \geq t_\circ$; more precisely, (16) holds with $t_{max} = +\infty$.*

Proof Define $\tilde{v}(t) := \widetilde{V}(t, x_1(t))$ and let

$$x_2^* := \sup_{t\in[t_\circ, t_{\max})} \{|x_2(t)|\}\,. \quad (18)$$

On the interval of existence we have, from (17a),

$$\dot{\tilde{v}}(t) \leq \alpha_4(|x_1(t)|)\alpha_4'(x_2^*)\,. \quad (19)$$

Without loss of generality let $\tilde{v}(t_\circ) > 0$; observe that $\lim_{t\to t_{\max}} \tilde{v}(t) = +\infty$ and integrate on both sides of (19) from $t_\circ \to t_{\max}$ to obtain

$$\int_{\tilde{v}(t_\circ)}^\infty \frac{d\tilde{v}}{\alpha_4\left(\alpha_1^{-1}(\tilde{v})\right)} = \alpha_4'(x_2^*)\int_{t_\circ}^{t_{\max}} dt\,.$$

By assumption, the first integral is infinite; hence $t_{\max} = +\infty$. ■

B. Boundedness of solutions

According with [5, Lemma 2] uniform boundedness of solutions is a sufficient condition (necessity is evident) for UGAS of time-varying nonlinear cascades satisfying Assumptions 1 and 2. In this section we present sufficient conditions for uniform global boundedness based on conditions which are, to some extent, refinements of Assumptions 1 and 2 for system (1).

Assumption 5 We dispose of a $\mathcal{C}^1$ function $V : \mathbb{R}\times\mathbb{R}^{n_1} \to \mathbb{R}_{\geq 0}$, α_1, $\alpha_2 \in \mathcal{K}_\infty$ and a positive semidefinite function W such that

$$\alpha_1(|x_1|) \leq V(t, x_1) \leq \alpha_2(|x_1|) \quad (20a)$$
$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x_1} f_1(t, x_1) \leq -W(x_1) \quad (20b)$$

for all $t \in [t_\circ, t_{\max})$ and all $x_1 \in \mathbb{R}^{n_1}$.

Note that for $t_{\max} = +\infty$ Assumption 1 implies the existence of V and W positive definite such that Assumption 5 holds.

Assumption 6 There exists $\gamma \in \mathcal{KL}$ such that the solutions $x_2(t, t_\circ, x_{2\circ}, x_1)$ of the parameterized system (1b) satisfy

$$|x_2(t, t_\circ, x_{2\circ}, x_1)| \leq \gamma(|x_{2\circ}|) \qquad \forall\, t \in [t_\circ, t_{\max}) \quad (21)$$

with γ independent of the initial conditions and the *parameter* x_1.

Assumption 6 holds, for instance, if there exists a positive definite radially unbounded function $V_2(t, x_2)$ such that, defining $v_2(t) := V_2(t, x_2(t))$ we have

$$|x_2(t)| \geq \eta \quad \Longrightarrow \quad \dot{v}_2(t) \leq 0$$

for all $x_1 \in \mathbb{R}^{n_1}$.

We reconsider the main theorems on boundedness (hence on stability) from [5]. The first theorem is reminiscent of the case when the "function f_1 in (1a) majorates the function g".

Theorem 1 *Consider the system (1) under the following hypotheses.*

(i) *Assumption 5 holds;*
(ii) *Assumption 6 holds;*
(iii) *for each $x_2 \in \mathbb{R}^{n_2}$ and all $t \geq 0$*

$$\frac{\partial V(t, x_1)}{\partial x_1} g(t, x_1, x_2) = o(W(x_1))\,.$$

Then, the solutions of (1) are uniformly globally bounded.

Proof Let $t_{\max}$ determine the maximal interval of existence of $x_1(\cdot)$. By assumption –*cf.* item (iii), for any $\varepsilon > 0$ there exists $\eta_\varepsilon > 0$ such that

$$|x_1| \geq \eta_\varepsilon \quad \Rightarrow \quad \left|\frac{\partial V(t, x_1)}{\partial x_1} g(t, x_1, x_2)\right| < \varepsilon W(x_1)$$

For any r let $c^* := \gamma(r)$ where γ is generated by Assumption 6. Fix r arbitrarily and let ε above be such that $\varepsilon < 1/c^*$; define η_ε accordingly. Define

$$L_g V := \frac{\partial V(t, x_1)}{\partial x_1} g(t, x_1, x_2)$$
$$[L_g V](t) := \frac{\partial V(t, x_1(t))}{\partial x_1} g(t, x_1(t), x_2(t))\,.$$

From Assumption 5 we have

$$\dot{V}_{(1a)}(t, x_1) \leq -W(x_1) + \left|\frac{\partial V(t, x_1)}{\partial x_1} g(t, x_1, x_2)\right|\,.$$

so, defining $v(t) := V(t, x_1(t))$ and $w(t) := W(x_1(t))$ we have, on the interval of existence of solutions,

$$\dot{v}(t) \leq -w(t) + [L_g V](t) . \qquad (22)$$

Assume that the solutions grow unboundedly, moreover, assume that as $t \to t_{\max}$ we have $|x_1(t)| \to \infty$. Hence, there exists $t^*_{\max} \in [0, t_{\max} - t_\circ)$ such that $|x_1(t_\circ + t^*_{\max})| = \eta_\varepsilon$ and

$$\dot{v}(t) \leq -(1-\varepsilon) w(t) \quad \forall\, t \in [t_\circ + t^*_{\max}, t_{\max}) .$$

By continuity, the inequality above holds also at $t = t_{\max}$. The argument can be repeated for any $t_{\max} \geq t_\circ$ and any $r > 0$; it follows that $t_{\max}$ may be infinite and the trajectories $x_1(t)$ exist for all $t \geq t_\circ + t^*_{\max}$ all $t_\circ \geq 0$ and all $t^*_{\max} \geq 0$. Moreover, $\dot{v}(t) \leq 0$ for all $t \geq t_\circ + t^*_{\max}$. From this and (20a) it follows that the trajectories $x_1(t)$ of (1a) are uniformly globally bounded. The result follows considering, in addition, (21). ■

We now make a statement for the case when the functions f_1 and g are of the same order with respect to x_1 for each fixed x_2, uniformly in t.

Assumption 7 There exists $\beta \in \mathcal{KL}$ such that the solutions $x_2(t, t_\circ, x_{2\circ}, x_1)$ of (5) satisfy

$$|x_2(t, t_\circ, x_{2\circ}, x_1)| \leq \beta(|x_{2\circ}|, t - t_\circ) \qquad \forall\, t \in [t_\circ, t_{\max}). \quad (23)$$

Assumption 7 holds, for instance, if there exists a positive definite radially unbounded function $V_2(t, x_2)$ such that, defining $v_2(t) := V_2(t, x_2(t))$ we have on the interval of existence,

$$\dot{v}_2(t) \leq -\alpha_3(|x_2(t)|)\alpha'_3(|x_1(t)|)$$

where $\alpha_3 \in \mathcal{K}$ and there exists $c > 0$ such that $\alpha'_3 - c \in \mathcal{K}$. Then, Assumption 7 holds is a consequence of the comparison Lemma –*cf.* [16] and the bound (16a).

Theorem 2 *Consider system (1) under the following conditions:*

(i) *Assumptions 4, 5 and 7 hold;*
(ii) *there exist α_5, $\alpha'_5 \in \mathcal{K}$ such that*

$$|[L_g V]| \leq \alpha_5(|x_1|)\alpha'_5(|x_2|)$$

and for each $r > 0$ there exist λ_r, $\eta_r > 0$ such that

$$t \geq 0, \quad |x_1| \geq \eta_r \quad \Longrightarrow \quad \alpha_5(|x_1|) \leq \lambda_r W(x_1)$$

Then, the solutions of (1) are uniformly globally bounded.

Proof Assumption 7 implies (16b). From this and Assumption 4 it follows, by Lemma 1 that the system is uniformly forward complete that is, Assumption 3 holds with $t_{\max} = +\infty$. Fix $r > 0$ arbitrarily and let it generate λ_r, η_r by item (ii). From Assumption 7 and forward completeness, there exists $t_\eta \geq 0$ such that

$$\alpha'_5 \circ \beta(|x_{2\circ}|, t_\circ + t_\eta) \leq \frac{1}{\lambda_r}; \qquad \beta(|x_{2\circ}|, t_\circ + t_\eta) \leq r$$

hence,

$$|[L_g V](t)| \leq \frac{\alpha_5(|x_1(t)|)}{\lambda_r} \qquad \forall\, t \geq t_\circ + t_\eta .$$

Invoking forward completeness and assuming that $|x_1(t)| \to \infty$ as $t \to \infty$ it follows that there exists t'_η such that $\left|x_1(t_\circ + t'_\eta)\right| \geq \eta_r$. From (22) we obtain $\dot{v}_{(1)}(t) \leq 0$ for all $t \geq t_\circ + t'_\eta$. Integrating and using (20a) the result follows. ■

The last theorem of this section parallels [5, Theorem 5]; we consider the case when the function f_1 is majorized by g with respect to x_1.

Theorem 3 *Let*

(i) *Assumptions 4, 5 and 6 hold;*
(ii) *there exist α_5, α'_5 and $\varphi \in \mathcal{K}$ such that*

$$\alpha'_4(s) \leq \alpha_5(s)\alpha'_5(s) ,$$

$$\int_{t_\circ}^{\infty} \alpha'_5(|x_2(t, t_\circ, x_{2\circ}, x_1)|)dt \leq \varphi(|x_{2\circ}|) \quad \forall\, x_1 \in \mathbb{R}^{n_1},\, t \geq t_\circ \geq 0 .$$

Then, the solutions of (1) are uniformly globally bounded.

Proof We follow similar lines as for the proof of [5, Theorem 5]. Firstly, uniform forward completeness follows as for Theorem 2 above. For UGB, consider the function $\widetilde{V}$ from Assumption 4 and the function γ from Assumption 6. Define

$$V_{\text{new}}(t, x_1) := \int_a^{\max\{\widetilde{V}(t,x_1), a\}} \frac{dv}{\alpha_4(\alpha^{-1}(v))}$$

which is radially unbounded, in view of (17b) and properness of $\widetilde{V}$. Correspondingly, for all $x_{2\circ}$ such that $|x_{2\circ}| < r$, we define $v_{\text{new}}(t) := V_{\text{new}}(t, x_1(t), r)$. The upper-right Dini derivative of v_{new} yields

$$D^+ v_{\text{new}} \leq \frac{\dot{\tilde{v}}}{\alpha_4(\alpha^{-1}(\tilde{v}))} .$$

From (17a) and item (ii) we have

$$\dot{\tilde{v}}(t) \leq \alpha_4(\alpha^{-1}(\tilde{v}))\alpha_5(|x_2(t)|)\alpha'_5(|x_2(t)|) .$$

From Assumption 6 and for any $r > 0$ we have

$$\dot{\tilde{v}}(t) \leq \alpha_4(\alpha^{-1}(\tilde{v}))\alpha_5(\gamma(r))\alpha'_5(|x_2(t)|) .$$

hence, $D^+ v_{\text{new}} \leq \alpha_5(\gamma(r))\alpha'_5(|x_2(t)|)$ for all $t \geq t_\circ$. Integrating the latter on both sides, from $t_\circ$ to ∞ and using item (ii) of the theorem we obtain $v_{\text{new}}(t) \leq \alpha_5(\gamma(r))\varphi(|x_{2\circ}|) \leq \alpha_5(\gamma(r))\varphi(\gamma(r))$ for any $r > 0$, $t_\circ \geq 0$ and all $t \geq t_\circ$. UGB of $x_1(t)$ follows using the last bound with $r = |x_{2\circ}|$ and recalling that V_{new} is radially unbounded. ■

C. Stability

Now, we make the following statements on uniform global asymptotic stability of $(x_1, x_2) = (0, 0)$ for system (1) under the relaxed Assumption 7 and which follow as corollaries of the previous theorems.

Proposition 1 *For system (1a) let Assumptions 1, 5 and item (iii) of Theorem 1 hold. For system (1b) let Assumption 7 hold. Then, the origin of (1) is UGAS.*

Proof Assumption 7 implies Assumption 6. UGB follows from Theorem 1. Since the system is forward complete, by Assumption 7 the origin of (1b) is UGAS, uniformly in the parameter x_1. Setting $x_1 = x_1(t, t_\circ, x_\circ)$ for any pair of initial conditions and following the discussion below (3) we have that the solutions of (1) coincide with those of (2) for appropriate choices of initial conditions. UGAS follows in view of Assumption 1 by invoking [5, Lemma 2]. ■

The proofs of the following statements follow similar lines:

Proposition 2 *Under Assumption 1 and the conditions of Theorem 2 the origin of (1) is UGAS.*

Proposition 3 *Under Assumptions 1, 7 and the conditions of Theorem 3 the origin of (1) is UGAS.*

III. APPLICATION: SEPARATION PRINCIPLE FOR EL SYSTEMS

For illustration we revisit a stability and control problem of output feedback control that has remained open for many years. It consists in establishing uniform global asymptotic stability for a robot manipulator in closed loop with a position-feedback dynamic controller. Our main purpose here is to comment the main result from [17]; to that end we need to introduce some notation and the formal statement of the problem.

Consider Euler-Lagrange systems *i.e.*,

$$D(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) = \tau \tag{24}$$

where, in our notation, the matrix $C(q,\dot{q})$ contains the terms corresponding to centrifugal and Coriolis effects, and the vector $g(q)$ corresponds to potential forces (*e.g.* gravitational). It is assumed that all functions are smooth, uniformly bounded and with bounded partial derivatives[2]

Problem 1. Let $t \mapsto q^*$ be a smooth bounded function with bounded derivatives which is given as a *reference* trajectory. Assume that only q is available for measurement. Define the tracking errors:

$$\tilde{q} := q - q^*; \qquad \dot{\tilde{q}} := \dot{q} - \dot{q}^* .$$

It is required to design a dynamic output-feedback controller

$$\tau = (t,\zeta,q) \tag{25a}$$
$$\dot{\zeta} = f_\zeta(t,\zeta,q) \tag{25b}$$

such that the origin of the closed-loop system, (24)–(25) *i.e.*, $(\zeta, \dot{\tilde{q}}, \tilde{q}) = (0,0,0)$ is UGAS.

Controller of [17]. In that reference a certainty-equivalence controller and an observer are proposed. Besides some implementation obstacles that we shall not discuss here, the stability proof which appeals to theorems for cascaded systems but skips any argument about boundedness or forward completeness of the system. We present the main result from [17] in a summarized fashion and in the context of the present paper, thereby filling in a gap seemingly overlooked in the latter reference.

State feedback controller. Assuming that both $\dot{q}$ and q are measured the control input is given by

$$\tau = g(q) + D(q)[\ddot{q}^* + \dot{\alpha}] + C(q,\dot{q})[\dot{q}^* + \alpha] - K_2 z_2 - H z_1 \tag{26a}$$
$$z_1 := [\tilde{q}^\top,\ \nu]; \quad z_2 := \dot{\tilde{q}} - \alpha; \quad \alpha := -K_1 z_1 \tag{26b}$$
$$\dot{\nu} = \tilde{q}\,. \tag{26c}$$

According to [17], $K_2 = K_2^\top > 0$ and K_1 is chosen such that $A - BK_1$ is Hurwitz and the pair $(A-BK_1, B)$ is strictly positive real. Hence, for any $Q = Q^\top > 0$ there exists $P = P^\top > 0$ such that $P(A-BK_1) + (A-BK_1)^\top P = -Q$; the matrix H is fixed to be $H^\top := PB$. To write the closed-loop system under the state feedback we introduce the following notation:

$$z_1 := [z_{11}^\top\ z_{12}^\top]^\top; \quad x_1 := [z_1^\top,\ z_2^\top] \tag{27a}$$
$$q(t,x_1) := z_{12} + q^*(t), \quad \dot{q}(t,x_1) := z_2 - K_1 z_1 + \dot{q}^*(t) \tag{27b}$$

[2]For matrix-valued functions, let us say that these assumptions hold on each of their elements.

So we can write the closed-loop system in the form (4) with

$$f_1(t,x_1) := \begin{bmatrix} (A - BK_1)z_1 + Bz_2 \\ -D(q(t,x_1))^{-1}\left[C(q(t,x_1),\dot{q}(t,x_1)) + K_2 z_2 + H z_1\right] \end{bmatrix} \tag{28}$$

As shown in [17] the total time derivative of

$$V(t,x_1) := \frac{1}{2}\left(z_1^\top P z_1 + z_2^\top D(q(t,x_1)) z_2 \right) \tag{29}$$

yields

$$\dot{V}(t,x_1) = -\frac{1}{2}\left(z_1^\top Q z_1 + z_2^\top K_2 z_2 \right) \tag{30}$$

where the usual properties of Euler-Lagrange systems have been used: boundedness of $D(\cdot)$ from above and below; skew-symmetry of $\dot{D}(\cdot) - 2C(\cdot,\cdot)$; besides strict real-positivity of $(A-BK_1, B)$ and the stability of $(A-BK_1)$. Hence, Assumption (5) holds and, moreover with W in (20b) positive definite and radially unbounded and $t_{\max} = +\infty$. Yet, in *general*, $t_{\max}$ may be finite when the observer and certainty-equivalence control is used. To see further, let us recall the observer from [17].

The observer and the estimation error dynamics. The observer is given by

$$\dot{\hat{q}} = \hat{q}_2 + h_1(\tilde{q},\chi) + K_{o2}^{-1}K_{o1}\tilde{q} \tag{31a}$$
$$\dot{\hat{q}}_2 = D(q)^{-1}\left[\, \tau + h_2(t,\tilde{q},\chi) + K_{o2}\tilde{q} \right. \tag{31b}$$
$$\left. -C(q, \hat{q}_2 + h_1(\tilde{q},\chi))[\hat{q}_2 + h_1(\tilde{q},\chi)] \,\right] . \tag{31c}$$

We do not make explicit all definitions of the symbols above for space constraints; for this, the reader is invited to see the original reference [17] however, we make the following fitting observations about the observer (31). The functions h_i are chosen so that the observer asymptotically tracks the unmeasured state trajectories. In [17] it is mentioned that $t \mapsto \chi$ is a "*bounded* function to be determined"; as a matter of fact, $\chi(t)$ is required to be solution of the differential equation:

$$\dot{\chi} = -k_\chi(\chi - \cosh(\hat{\xi}^\top\hat{\xi})) + 2\sinh(\hat{\xi}^\top\hat{\xi})\hat{\xi}^\top\dot{\hat{\xi}} \tag{32}$$

where

$$\hat{\xi} := \hat{q}_2 + h_1(\tilde{q},\chi)\,.$$

For the purposes of the main remarks we want to make here, it is convenient to observe that

$$\hat{\xi} = -K_{o2}^{-1}K_{o1}\tilde{q} + (-\tilde{\xi} + K_{o2}^{-1}K_{o1}\tilde{q} + (\dot{q}^*(t) + z_2 - K_1 z_1))\,. \tag{33}$$

The estimation errors are

$$x_2 := [\tilde{q}^\top,\ \tilde{\xi}^\top,\ e_\chi^\top]^\top$$

and the estimation error dynamics, following [17], is

$$\dot{\tilde{q}} = \tilde{\xi} - K_{o2}^{-1}K_{o1}\tilde{q} \tag{34a}$$
$$\dot{\tilde{\xi}} = D(q)^{-1}\left[-K_{o2}\tilde{q} - [C(q,\dot{q}) + C(q,\hat{\xi})]\tilde{\xi} \right] - \bar{k}_{o1}\tilde{\xi} - \gamma\cosh\left(\left|\hat{\xi}\right|^2\right)\tilde{\xi} - \gamma\tilde{\xi}e_\chi \tag{34b}$$
$$\dot{e}_\chi = -k_\chi e_\chi \tag{34c}$$

where we have kept the notation of [17]; the symbols γ, $\bar{k}_\chi$, $\bar{k}_{o1}$ are positive design parameters; the variables q, $\dot{q}$ and $\hat{\xi}$ which appear in (34) are functions of time and the tracking errors x_1 –*cf.* Eqs. (27b) and (33). Hence, the system (34) is of the form (1b).

In [17] it is shown that for the estimation error dynamics the function

$$V_2(t,x_1,x_2) := \frac{1}{2}\left(\tilde{\xi}^\top D(q(t,x_1))\tilde{\xi} + \tilde{q}^\top K_{o2}\tilde{q}\right)$$

satisfies

$$\begin{aligned}\dot{V}_2(t,x_1,x_2) \le & -\alpha_1 |\tilde{q}|^2 - \alpha_2 \left|\tilde{\xi}\right|^2 \\ & - \left(\gamma d_m \cosh\left(\left|\hat{\xi}(t,x_1,x_2)\right|\right) - C_B \left|\hat{\xi}(t,x_1,x_2)\right|\right)\left|\tilde{\xi}\right|^2 .\end{aligned} \tag{35}$$

We stress the following: the function V_2 is considered here as a function of the *state* x_2, time t and the *parameter* x_1. While it is clear that V_2 also depends on x_1 it is bounded from above and below by functions of $|x_2|$ only. Even though $\hat{\xi}$ is function of x_1 and t, the term in brackets is always negative, uniformly in t, x_1 and x_2. We conclude that $\dot{V}_2$ is negative definite however, one cannot conclude UGAS of $x_2 = 0$ since (35) is valid only on the interval of existence of solutions but we may conclude that Assumption 7 holds.

Now, assuming that $\chi(t)$, defined as a solution of (32), is bounded is an abusive (and *unnecessary*) assumption when we consider the overall certainty-equivalence controller (34) and (26) with the state estimates *i.e.*,

$$\hat{\tau} \;:=\; g(q) + D(q)[\ddot{q}^* + \dot{\hat{\alpha}}] + C(q,\dot{\hat{q}})[\dot{q}^* + \alpha] - K_2\hat{z}_2 - Hz_1 \tag{36a}$$

$$\dot{\hat{\alpha}} \;:=\; -K_1[\bar{q},^\top \;\dot{\hat{q}}^\top - \dot{q}^*(t)^\top]^\top ; \quad \hat{z}_2 := \dot{\hat{q}} - \dot{q}^* - \alpha\,. \tag{36b}$$

As shown in [17] the closed-loop system with the certainty-equivalence controller (36) yields

$$\dot{x}_1 = f_1(t,x_1) + g(t,x_1,x_2)$$

where f_1 is given by (28) and

$$\begin{aligned} & g(t,x_1,x_2) := \\ & \quad \big(-K_{12} + D^{-1}(q(t,x_1))[C(q(t,x_1),\dot{q}^*(t) - K_1 z_1) - K_2]\big) L x_2 \end{aligned} \tag{37}$$

where all matrices $K_{(\cdot)}$ and L are constant design parameters of appropriate dimensions. The Coriolis matrix $C(x,y)$ is globally Lipschitz in y uniformly in x and the induced norm $|D(\cdot)|$ is uniformly bounded from above and below. From this, we obtain that $g(t,x_1,x_2) = \mathcal{O}(|x_1|)$ for each x_2, uniformly in t. A simple calculation shows that V defined in (29) and g above satisfy item (ii) of Theorem 2. Item (i) of the theorem also holds: Assumption 4 holds with $\alpha_4(s) \propto \alpha_1(s) \propto s^2$ hence $\alpha^4 \circ \alpha_1(s) \propto s$; Assumption 5 holds in view of (30). By Theorem 2 the solutions are uniformly globally bounded.

Finally, by Proposition 2 and in view of (29), (30) –the SPR assumption, the origin of the closed-loop system is UGAS. As a matter of fact, since all Lyapunov functions are "quadratic" one may conclude uniform global exponential stability. This claim is made in [17, Proposition 3].

IV. CONCLUSION

Stability analysis via Lyapunov's direct method is challenging in general for nonlinear time-varying systems. Besides, while the existence of a Lyapunov function is a well-known necessary condition for UGAS, sufficiency is at the basis of methods which (when they apply) often lead to unnecessarily restrictive assumptions or complex nonlinear controllers (universal formula, backstepping, *etc.*) A non-systematic alternative is that of cascades-based analysis and design. We have established conditions under which feedback interconnected systems may be regarded as cascades. As an illustration of the utility of our main results we have revisited a recent work on output feedback control of mechanical systems.

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