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Linearization of Hamiltonian systems through state immersion

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***Abstract*— This paper deals with the linearization of nonlinear Hamiltonian systems through state immersion: under the existence of a suitable symmetry (not necessarily Hamiltonian) of the Hamiltonian system, this is achieved first by linearizing the symmetry through a canonical diffeomorphism, which simultaneously brings the Hamiltonian system into a homogeneous form of degree 0 with respect to a certain dilation and, secondly, by immersing such homogeneous Hamiltonian system into a linear one. This has some direct consequences: the computation in closed form of the flow of the Hamiltonian system and the computation in closed form of the semi-invariants of the system.**

I. INTRODUCTION

The concept of (orbital) symmetry of a differential equation was introduced by S. Lie [1]-[2] in the second half of the 19-th century, as an attempt of generalizing the theory of Galois, and it was primarily used for the solution in closed form of differential equations admitting given orbital symmetries: modern reference on the subject can be found in many books, among which we mention [3]-[7].

In this paper, the linearization of nonlinear Hamiltonian systems through state immersion is considered; under the assumption that there exists a symmetry of the Hamiltonian system that can be linearized through a canonical diffeomorphism, the Hamiltonian system is first transformed into a homogeneous form of degree 0 with respect to a certain dilation and, secondly, immersed into a linear system. This allows one to compute in closed form the flow and the semi-invariants of the system. This a generalization of the linearization of nonlinear systems through diffeomorphism, which dates back to Poincaré (see , e.g., [8]).

The paper is organized as follows. Preliminaries are given in Sections II and III; first, the case of planar Hamiltonian systems in considered in Section IV and then extended to the general case in Section V. Finally, some possible future researches are discussed in Section VI.

II. PRELIMINARIES

Assume that $x \in \mathbb{R}^{2n}$, with $x^T = \begin{bmatrix} q^T & p^T \end{bmatrix}$, $q, p \in \mathbb{R}^n$. The entries of x, q, p are denoted by x_i, q_i and p_i, respectively. In the following we will assume, if not otherwise specified, that all the functions are analytic in some open and connected domain $\mathcal{U}$ of $\mathbb{R}^{2n}$: this implies that the differential equations under consideration have unique maximal solution in some $\mathcal{V} \subseteq \mathbb{R} \times \mathbb{R}^{2n}$, including $\{0\} \times \mathcal{U}$. Given two vector functions $f(x), g(x) : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ and a scalar function $h(x) : \mathbb{R}^{2n} \to \mathbb{R}$, the *Lie derivative* $L_f h \in \mathbb{R}$ of h by f is $L_f h := \frac{\partial h}{\partial x} f$, where $\frac{\partial h}{\partial x}$ is the gradient of h; the *Lie derivative* $L_f g \in \mathbb{R}^{2n}$ of g by f is the vector having $L_f g_i$ as i-th entry, with g_i being the i-th entry of g, i.e., $L_f g := \frac{\partial g}{\partial x} f$, where $\frac{\partial g}{\partial x}$ is the Jacobian of g; the *Lie bracket* $[f, g] \in \mathbb{R}^{2n}$ of f by g is $[f, g] := \frac{\partial g}{\partial x} f - \frac{\partial f}{\partial x} g = L_f g - L_g f$, where $\frac{\partial g}{\partial x}$ and $\frac{\partial f}{\partial x}$ are the Jacobians of g and f, respectively. For u and v scalar functions of x, the *Poisson bracket* of u by v is

$$\{u, v\} := \sum_{i=1}^{n} \frac{\partial u}{\partial q_i} \frac{\partial v}{\partial p_i} - \frac{\partial u}{\partial p_i} \frac{\partial v}{\partial q_i} : \qquad (1)$$

properties of the Poisson bracket thus defined are $\{u, v\} = -\{v, u\}$, $\{u, vz\} = \{u, v\}z + \{u, z\}v$, $\{u, \{v, z\}\} + \{v, \{z, x\}\} + \{z, \{x, v\}\} = 0$, $\{u, v + z\} = \{u, v\} + \{u, z\}$, where u, v, z are arbitrary functions of x. The local coordinates q and p are *canonical* with respect to the chosen Poisson bracket in the sense that

$$\begin{aligned} \{q_i, p_j\} &= \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j \end{cases}, \\ \{q_i, q_j\} &= 0, \{p_i, p_j\} = 0, \forall i, j. \end{aligned}$$

Given an Hamiltonian (scalar) function $H(x)$, the Hamiltonian system associated with H is described by

$$\dot{x} = f(x), \qquad f(x) := S \left(\frac{\partial H}{\partial x} \right)^T, \qquad (2)$$

with $S = \begin{bmatrix} 0 & I_{n \times n} \\ -I_{n \times n} & 0 \end{bmatrix}$. For any scalar u, we have the identity $L_f u = \{u, H\}$, with f given in (2): then, system (2) can be rewritten componentwise as follows:

$$\dot{x}_i = \{x_i, H\}, \qquad i = 1, ..., 2n.$$

There is a strong relation between a Lie bracket and a Poisson bracket (see Lemma 12.9 of [9]): if f (respectively, g) is the vector field characterizing the Hamiltonian system associated with the Hamiltonian function H (respectively, K), then $[f, g]$ is the vector field characterizing the Hamiltonian system associated with the Hamiltonian function $\{H, K\}$.

A *first integral* of system (2) is a scalar function $I(x)$ such that $L_f I = \{I, H\} = 0$; if I is a constant, then the first integral is said to be *trivial*, *non-trivial* otherwise. Clearly, from $\{H, H\} = 0$, we have that H is a first integral of the associated Hamiltonian system.

Remark 1: Assume $n = 1$. Then, $\{u, v\} = \det \left(\frac{\partial}{\partial x} \begin{bmatrix} u \\ v \end{bmatrix} \right)$, which shows that if $\{u, v\} = 1$,

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then $y_1 = u$ and $y_2 = v$ qualify as a diffeomorphism and, in particular, y_1 and y_2 are canonical (this is called a canonical diffeomorphism). Given the function u, if v_0 is any function such that $\{u, v_0\} = 1$, then all the functions v such that $\{u, v\} = 1$ are parameterized by $v = v_0 + C(u)$, where C is an arbitrary function of the argument. As a matter of fact, from $\{u, v_0\} = 1$ and $\{u, v\} = 1$, we have $\{u, v - v_0\} = 0$: the statement is proven by noticing that the ker of the function $\{u, \cdot\}$ is given by arbitrary functions of u.

Consider another system described by

$$\frac{dx}{d\tau} = g(x); \tag{3}$$

denote by $x(t) = \Phi_f(t, x_0)$ and $x(\tau) = \Phi_g(\tau, x_0)$, respectively, the solutions of (2) and (3) from the initial condition x_0. The flow $y = \Phi_g(\tau, x)$ is a *symmetry* (respectively, an *orbital symmetry*) of system (2) and system (3) is its *infinitesimal generator* if $[f, g] = 0$ (respectively, $[f, g] = \mu f$), with μ being a scalar analytic function. With a little abuse of notation, g is also called a symmetry (respectively, an orbital symmetry) of f; g is said to be *non-trivial* if it is not colinear with f, *trivial* otherwise (any f is a trivial symmetry of itself).

As well known, the flow $y = \Phi_g(\tau, x)$ of system (3) qualifies as a one-parameter group of transformations; for any admissible τ, an arbitrary orbit of system (2) is mapped into the same orbit by $y = \Phi_g(\tau, x)$ if and only if g is an orbital symmetry of f (in addition, the time-parameterization along the orbit is also preserved by $y = \Phi_g(\tau, x)$ if and only if g is a symmetry of f).

There is a strong relation between a symmetry g and a first integral of f if, in addition to f, also g is Hamiltonian (in such a a case, g is called a Noether symmetry: see Remark 12.51 of [9]): let f (respectively, g) be the Hamiltonian system associated with the Hamiltonian function H (respectively, K); hence, $\{K, H\} = c$, for some real c, if and only if $[f, g] = 0$ (if $\{K, H\} = c$, then $K - ct$ is a time-varying first integral of the Hamiltonian system).

Example 1: Let $H = \frac{1}{4}x_1^4 + \frac{1}{2}x_3^2 + \frac{1}{2}x_4^2$, with the associated $f = \begin{bmatrix} x_3 & x_4 & -x_1^3 & 0 \end{bmatrix}^T$. Clearly, $K = x_4$ satisfies $\{K, H\} = 0$, whence the associated Hamiltonian system $g = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}^T$ is a symmetry of f, $[f, g] = 0$.

Given a vector of integers $r = \begin{bmatrix} r_1 & \dots & r_{2n} \end{bmatrix}^T$ (the r_i's are called *integer weights*), an *integer dilation* $\delta_\varepsilon^r x$ is defined as $\delta_\varepsilon^r x := \begin{bmatrix} \varepsilon^{r_1} x_1 & \dots & \varepsilon^{r_{2n}} x_{2n} \end{bmatrix}^T$, for any scalar real $\varepsilon \neq 0$. A scalar function $h(x) : \mathbb{R}^{2n} \to \mathbb{R}$ is homogeneous of degree $m \in \mathbb{Z}$, with respect to the integer dilation $\delta_\varepsilon^r x$, if (see Sections 1.1 and 1.2 of [10]; see also [11], [12] and Section 5.3 of [13]) $h(\delta_\varepsilon^r x) = \varepsilon^m h(x)$, whenever defined. A vector function $f := \begin{bmatrix} f_1 & \dots & f_{2n} \end{bmatrix}^T : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is homogeneous of degree $m \in \mathbb{Z}$, with respect to the integer dilation $\delta_\varepsilon^r x$, if f_i is homogeneous of degree $r_i - m$ with respect to the integer dilation $\delta_\varepsilon^r x$, $i = 1, ..., 2n$ (see Sections 1.1 and 1.2 of [10]; see also [11], [12] and Section 5.3 of [13]), namely if $f_i(\varepsilon^{r_1} x_1, ..., \varepsilon^{r_{2n}} x_{2n}) = \varepsilon^{r_i - m} f_i(x_1, ..., x_{2n}), i = 1, ..., 2n$. Let $g = \begin{bmatrix} r_1 x_1 & \dots & r_{2n} x_{2n} \end{bmatrix}^T$; if f is homogeneous of degree m with respect to $\delta_\varepsilon^r x$ (with the integer m being possibly negative), then g is an orbital symmetry of f (a symmetry if $m = 0$): in particular, $[f, g] = mf$ (see [14], [15]). If $[f, g] = mf$ for some g (not necessarily of the form $g = \begin{bmatrix} r_1 x_1 & \dots & r_{2n} x_{2n} \end{bmatrix}^T$), with $m \in \mathbb{Z}$, then f is said to be homogeneous of degree m with respect to g (see [14]). Notice that a linear f is homogeneous of degree 0 with respect to the standard dilation $\delta_\varepsilon^r x$, with $r = \begin{bmatrix} 1 & \dots & 1 \end{bmatrix}^T$: a homogeneous system of degree 0 with respect to a given symmetry can be seen as an "extension" of a linear system (see [16]).

III. Problem definition

The aim of this paper is to find a state immersion $y_e = T(x)$, with $T(\cdot) : \mathbb{R}^{2n} \to \mathbb{R}^m$ and $m \geq 2n$, such that system (2) expressed in the new coordinates is linear $\dot{y}_e = A_e y_e$ and the rank of $\frac{\partial T}{\partial x}$ is full in an open and connected subset of $\mathbb{R}^{2n}$. This has a direct consequence in the computation in closed form of the semi-invariants of the original Hamiltonian system, according to the following definition (a general reference to semi-invariants can be found in [17], [18]).

Let u^T be a real (respectively, complex) left eigenvector of matrix A_e with a real (respectively, complex) eigenvalue λ, $u^T A_e = \lambda u^T$; then, $\omega(y_e) = u^T y_e$ (respectively, $\omega(y_e) = \left(u^{*T} y_e\right)\left(u^T y_e\right)$) is *a linear* (respectively, *quadratic*) *semi-invariant* of the extended linear system with constant characteristic function, namely

$$\begin{aligned} \dot{\omega} &= L_{A_e y_e}\omega = \lambda\omega, \quad \text{if } \lambda \in \mathbb{R}, \\ \dot{\omega} &= L_{A_e y_e}\omega = 2\mathrm{Re}\left(\lambda\right)\omega, \quad \text{if } \lambda \notin \mathbb{R}. \end{aligned}$$

Since $\omega(t) = e^{\lambda t}\omega(0)$ (respectively, $\omega(t) = e^{2\mathrm{Re}(\lambda)t}\omega(0)$), then

$$\omega(t) = 0, \forall t \in \mathbb{R}, \quad \Longleftrightarrow \quad \omega(0) = 0,$$

whence the set of points $y_e \in \mathbb{R}^m$ such that $\omega(y_e) = 0$ is A_e-invariant. Then, going back to the original coordinates, if $\hat{\omega}(x) := \omega(T(x))$, then $\hat{\omega}(x)$ is a semi-invariant for the original nonlinear system (2), for which we have

$$\begin{aligned} \dot{\hat{\omega}} &= L_f\hat{\omega} = \lambda\hat{\omega}, \quad \text{if } \lambda \in \mathbb{R}, \\ \dot{\hat{\omega}} &= L_f\hat{\omega} = 2\mathrm{Re}\left(\lambda\right)\hat{\omega}, \quad \text{if } \lambda \notin \mathbb{R}. \end{aligned}$$

Hence, the set of points $x \in \mathbb{R}^{2n}$ such that $\hat{\omega}(x) = 0$ is invariant for the nonlinear system (2). In particular, if $\mathrm{Re}(\lambda) = 0$, then $\hat{\omega}(x)$ is a first integral of (2).

IV. Planar Hamiltonian systems

In this section we will assume that $n = 1$.

Theorem 1: Consider the following Hamiltonian function $H(x) = K(h)$, where $h(x_1, x_2) \in \mathbb{R}^2$ is an arbitrary function of (x_1, x_2), analytic in a neighborhood of $x = 0$, with $h(0, 0) = 0$ and such that $\{h_1, h_2\} = 1$. Assume that $K(h)$ is polynomial and homogeneous of degree $k = r_1 + r_2$ with respect to $\delta_\varepsilon^r h$, with $r = \begin{bmatrix} r_1 & r_2 \end{bmatrix}^T$, $r_1, r_2 \in \mathbb{Z}$, $r_2, r_1 \geq 0$. Consider the Hamiltonian system described by

$$f = \begin{bmatrix} \frac{\partial H}{\partial x_2} \\ -\frac{\partial H}{\partial x_1} \end{bmatrix} = \begin{bmatrix} \frac{\partial K}{\partial h}\frac{\partial h}{\partial x_2} \\ -\frac{\partial K}{\partial h}\frac{\partial h}{\partial x_1} \end{bmatrix}.$$

Then,

(1) $y = h(x)$ qualifies as a canonical diffeomorphism;

(2) $g = \left(\frac{\partial h}{\partial x}\right)^{-1} \begin{bmatrix} r_1 h_1 & r_2 h_2 \end{bmatrix}^T$ is a (not necessarily Hamiltonian) symmetry of f;

(3) g can be linearized by $y = h$, thus obtaining in the new coordinates $\tilde{g} = \begin{bmatrix} r_1 y_1 & r_2 y_2 \end{bmatrix}^T$;

(4) since $\tilde{f} = \frac{\partial h}{\partial x} f\big|_{x=h^{-1}(y)}$ is analytic at $y = 0$ and homogeneous of degree 0 with respect to the standard dilation $\tilde{g}(y) = \begin{bmatrix} r_1 y_1 & r_2 y_2 \end{bmatrix}^T$, then $\tilde{f}$ can be rendered linear by a finite-dimensional state immersion;

(5) if $r_1 = r_2 = 1$, then $\tilde{f}$ is linear.

Proof: Since h is analytic in a neighborhood of $x = 0$ and $\det\left(\frac{\partial h}{\partial x}\right) = 1$, then $y = h(x)$ qualifies as a diffeomorphism in a neighborhood of $x = 0$, thus proving Statement (1). In the y-coordinates, we have $\tilde{g} = \frac{\partial h}{\partial x} g\big|_{x=h^{-1}(y)} = \begin{bmatrix} r_1 h_1 & r_2 h_2 \end{bmatrix}^T\Big|_{x=h^{-1}(y)} = \begin{bmatrix} r_1 y_1 & r_2 y_2 \end{bmatrix}^T$, thus proving Statement (3). Due to the assumption $\{h_1, h_2\} = 1$, $y_1 = h_1$ and $y_2 = h_2$ qualify as canonical coordinates (see [19]): in these new coordinates, since h_1 and h_2 are time-independent, the Hamiltonian function $H(x)$ takes the form $K(y)$, and the Hamiltonian system takes the form:

$$\tilde{f} = \begin{bmatrix} \frac{\partial K}{\partial y_2} \\ -\frac{\partial K}{\partial y_1} \end{bmatrix}.$$

Then, clearly, $\frac{\partial K}{\partial y_2}$ is polynomial and homogeneous of degree $k - r_2 = r_1$ and $-\frac{\partial K}{\partial y_1}$ is polynomial and homogeneous of degree $k - r_1 = r_2$, whence $\tilde{f}$ is polynomial and homogeneous of degree 0 with respect to $\delta_\varepsilon^r y$, whence $\tilde{g} = \begin{bmatrix} r_1 y_1 & r_2 y_2 \end{bmatrix}^T$ is a symmetry of $\tilde{f}$. Due to the invariance of the Lie bracket with respect to diffeomorphisms, then $g = \left(\frac{\partial h}{\partial x}\right)^{-1} \begin{bmatrix} r_1 h_1 & r_2 h_2 \end{bmatrix}^T$ is a symmetry of f, thus proving (2). Since if $\tilde{f}$ is polynomial and homogeneous of degree 0 with respect to $\delta_\varepsilon^r y$, with $r = \begin{bmatrix} 1 & 1 \end{bmatrix}^T$, then $\tilde{f}$ is linear, thus proving (5). If $\tilde{f}$ is polynomial and homogeneous of degree 0 with respect to $\delta_\varepsilon^r y$, then $\tilde{f}_i$ is the sum of some monomials $m_j^{r_i} = y_1^{j_1} y_2^{j_2}$ homogeneous with respect to $\delta_\varepsilon^r y$ of degree r_i; if $u(y)$ is homogeneous with respect to $\delta_\varepsilon^r y$ of a certain degree, then $L_{\tilde{f}} u = \{u, K\}$ is homogeneous with respect to $\delta_\varepsilon^r y$ of the same degree, which shows that $\tilde{f}$ can be linearized by taking as state variables all the monomials $m_j^{r_i}$, $i = 1, 2$, thus proving (4). ■

Remark 2: The use of monomials as additional state variables is a step of the classical Carleman linearization (see [20], where such a procedure is used for obtaining a bilinear approximation of a nonlinear control system); the drawback of the Carleman linearization is that the resulting linear system is, in general, infinite dimensional, and only finite dimensional approximation can be obtained. Theorem 1 gives conditions under which a given nonlinear Hamiltonian system, when expressed in suitable coordinates, can be effectively linearized (without approximations) by the Carleman technique.

Example 2: Take $h = \begin{bmatrix} x_1 + x_2^2 & x_2 \end{bmatrix}^T$, $K(h) = \frac{1}{2} h^T B h$ and $B = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$, which yields the Hamiltonian system described by

$$f = \begin{bmatrix} 2x_2^3 + 6x_2^2 + (2x_1 + 3)\, x_2 + 2x_1 \\ -x_1 - x_2^2 - 2x_2 \end{bmatrix};$$

a symmetry g of f is $g = \begin{bmatrix} x_1 - x_2^2 & x_2 \end{bmatrix}^T$ (notice that such a symmetry is not Hamiltonian, because $\mathrm{div}(g) = 2 \neq 0$). It is easy to check that g can be linearized by $y = \begin{bmatrix} x_1 + x_2^2 & x_2 \end{bmatrix}^T$, obtaining $\tilde{g} = \begin{bmatrix} y_1 & y_2 \end{bmatrix}^T$; by the same diffeomorphism, we have $\tilde{f} = \begin{bmatrix} 2y_1 + 3y_2 & -y_1 - 2y_2 \end{bmatrix}^T$. Clearly, $H = \frac{1}{2} h^T B h$ is a first integral associated with f, $L_f H = 0$. Matrix SB, i.e., the dynamic matrix of the linearization of f, admits the following pairs of real (left eigenvector, eigenvalue) $\left(u_1^T = \begin{bmatrix} 1 & 1 \end{bmatrix}, \lambda_1 = 1\right)$ and $\left(u_2^T = \begin{bmatrix} 1 & 3 \end{bmatrix}, \lambda_2 = -1\right)$, which yield the following two polynomial semi-invariants $\omega_1 = u_1^T h = x_1 + x_2^2 + x_2$ and $\omega_2 = u_2^T h = x_1 + x_2^2 + 3x_2$ for the original system: as a matter of fact, $L_f \omega_1 = \omega_1$ and $L_f \omega_2 = -\omega_2$; actually, notice that $H = \frac{1}{2}\omega_1\omega_2$, according to the fact that $L_f H = \frac{1}{2}\left(\omega_2 L_f \omega_1 + \omega_1 L_f \omega_2\right) = \frac{1}{2}\left(\omega_1\omega_2 - \omega_1\omega_2\right) = 0$.

Example 3: Take $h = \begin{bmatrix} x_1 + \frac{1}{2}x_2^2 & x_1 + x_2 + \frac{1}{2}x_2^2 \end{bmatrix}^T$ and $H(h) = a h_1 h_2 + \frac{b}{3} h_1^3$; it is easy to see that H is homogeneous of degree 3 with respect to $\delta_\varepsilon^r h$, with $r = \begin{bmatrix} 1 & 2 \end{bmatrix}^T$, and that $\{h_1, h_2\} = \det \begin{bmatrix} 1 & x_2 \\ 1 & 1 + x_2 \end{bmatrix} = 1$. The corresponding Hamiltonian system is described by $f = \begin{bmatrix} f_1 & f_2 \end{bmatrix}^T$, with $f_1 = \frac{1}{4} b x_2^5 + (a + b x_1)\, x_2^3 + \frac{3}{2} a x_2^2 + \left(2a x_1 + b x_1^2\right) x_2 + a x_1$ and $f_2 = -\frac{1}{4} b x_2^4 + (-a - b x_1)\, x_2^2 - a x_2 - 2a x_1 - b x_1^2$; a symmetry g of f is then (notice that such a symmetry is not Hamiltonian, because $\mathrm{div}(g) = 3 \neq 0$)

$$g = \begin{bmatrix} x_1 - \frac{3}{2}x_2^2 - x_2 x_1 - \frac{1}{2}x_2^3 \\ x_1 + \frac{1}{2}x_2^2 + 2x_2 \end{bmatrix}.$$

With the diffeomorphism $y_1 = x_1 + \frac{1}{2}x_2^2$, $y_2 = x_1 + x_2 + \frac{1}{2}x_2^2$, we obtain

$$\tilde{g} = \begin{bmatrix} y_1 \\ 2y_2 \end{bmatrix}, \tilde{f} = \begin{bmatrix} a y_1 \\ -a y_2 - b y_1^2 \end{bmatrix}.$$

Clearly, $\tilde{f}$ can be immersed into a linear system with the position $y_3 = y_1^2$, thus obtaining the linear system

$$\begin{aligned} \dot{y}_1 &= a y_1, \\ \dot{y}_2 &= -a y_2 - b y_3, \\ \dot{y}_3 &= 2a y_3. \end{aligned}$$

The flow of the above system is

$$\Phi_{A_e y_e}(t, y_e) = e^{A_e t} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} e^{ta} y_1 \\ e^{-ta} y_2 - \frac{1}{3} b \frac{e^{2ta} - e^{-ta}}{a} y_3 \\ e^{2ta} y_3 \end{bmatrix},$$

which, taking into account that $y_3 = y_1^2$, yields the following flow of the system $\dot{y} = \tilde{f}(y)$:

$$\Phi_{\tilde{f}}(t, y) = \begin{bmatrix} e^{ta} y_1 \\ e^{-ta} y_2 - \frac{1}{3} b \frac{e^{2ta} - e^{-ta}}{a} y_1^2 \end{bmatrix},$$

whence, taking into account that $x_1 = y_1 - \frac{1}{2} y_2^2 + y_2 y_1 - \frac{1}{2} y_1^2, x_2 = y_2 - y_1$, one can compute the flow of the original Hamiltonian system (such an expression is omitted).

Such results can be easily extended also when some dissipation is present in the Hamiltonian system, as explained in the following. Consider an Hamiltonian function $H = \frac{1}{2} h^T B h$, with the entries h_1 and h_2 of h satisfying $\{h_1, h_2\} = 1$, and the corresponding Hamiltonian system described by $f = \left(\frac{\partial h}{\partial x}\right)^{-1} S B h$, with $S = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$; a symmetry of f is $g = \left(\frac{\partial h}{\partial x}\right)^{-1} h$. Dissipative effects can be easily taken into account by substituting matrix S with matrix $S_d = \begin{bmatrix} 0 & 1 \\ -1 & -d \end{bmatrix}$, with d being a real constant: $f_d = \left(\frac{\partial h}{\partial x}\right)^{-1} S_d B h$. Since the entries of $y = h$ qualify as canonical coordinates, both f_d and g can be linearized by $y = h$, $\tilde{f}_d = \frac{\partial h}{\partial x} f\big|_{x=h^{-1}(y)} = S_d B y$ and $\tilde{g} = \frac{\partial h}{\partial x} g\big|_{x=h^{-1}(y)} = y$; since $\tilde{g}$ is a symmetry of $\tilde{f}$, then g is a symmetry of f_d.

Example 4: Take $B = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$. The semi-invariants associated with the system are $\omega_1 = h_1 \left(\frac{1}{2} d - \frac{1}{2}\sqrt{d^2+4}\right) + h_2$ and $\omega_2 = h_1 \left(\frac{1}{2} d + \frac{1}{2}\sqrt{d^2+4}\right) + h_2$ with respective (constant) characteristic functions $\lambda_1 = \frac{1}{2} d + \frac{1}{2}\sqrt{d^2+4}$ and $\lambda_2 = \frac{1}{2} d - \frac{1}{2}\sqrt{d^2+4}$: the origin of the Hamiltonian system is clearly unstable for all values of d (λ_1 is a positive function of d, and λ_2 is a negative function of d).

Remark 3: A simple extension of the case considered in Statement (5) of Theorem 1 is $H = \frac{1}{2} \sum_{i=1}^n h_i^2$, with $h_i(x_1, x_2) \in \mathbb{R}$ being arbitrary functions of (x_1, x_2), analytic in a neighborhood of $x = 0$, with $h_i(0,0) = 0$. Assume that $\{h_i, h_j\} = c_{i,j}$, for some constants $c_{i,j}$, and that $rank\left(\frac{\partial h}{\partial x}\big|_{x=0}\right) = 2$. Consider the Hamiltonian system

$$f = \begin{bmatrix} \frac{\partial H}{\partial x_2} \\ -\frac{\partial H}{\partial x_1} \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^n h_i \frac{\partial h_i}{\partial x_2} \\ -\sum_{i=1}^n h_i \frac{\partial h_i}{\partial x_1} \end{bmatrix}.$$

Define the state immersion $y_k = h_k$, $k = 1, 2, .., n$; clearly, in these coordinates the system is linear:

$$\begin{aligned} \dot{y}_k &= \frac{\partial h_k}{\partial x_1} \sum_{i=1}^n h_i \frac{\partial h_i}{\partial x_2} - \frac{\partial h_k}{\partial x_2} \sum_{i=1}^n h_i \frac{\partial h_i}{\partial x_1} \\ &= \sum_{i=1}^n \{h_k, h_i\} h_i \\ &= \sum_{i=1}^n c_{k,i} h_i, \end{aligned}$$

where $\{h_k, h_k\} = 0$. Actually this is an apparent extension, because such a case can always be reduced to the case considered in Statement (5) of Theorem 1. As a matter of fact, if three functions u, v, z are such that $\{u, v\} = a$, $\{u, z\} = b$ and $\{v, z\} = c$, with a, b, c being arbitrary real constants (not all equal to 0), then one of the three functions can be expressed as a linear combination of the other two.This implies, by the assumption $\{h_i, h_j\} = c_{i,j}$ and by a possible reordering of the functions h_i, that the quadratic function $H = \frac{1}{2} \sum_{i=1}^n h_i^2$ can be rewritten as $H = \frac{1}{2} \begin{bmatrix} h_1 & h_2 \end{bmatrix} B \begin{bmatrix} h_1 \\ h_2 \end{bmatrix}$, for some real matrix B.

The rationale behind Theorem 1 is simple: given a Hamiltonian system, find one of its symmetries such that there exists a diffeomorphism linearizing the symmetry and simultaneously transforming the Hamiltonian system into a polynomial form, homogeneous of degree 0 with respect to a certain integer dilation. Then, it is relevant to compute all the symmetries of a Hamiltonian system.

Consider the Hamiltonian function $H(x)$ and the corresponding Hamiltonian system described by $f(x) = \begin{bmatrix} \frac{\partial H}{\partial x_2} & -\frac{\partial H}{\partial x_1} \end{bmatrix}^T$. Let $K(x)$ be a function such that $\{K, H\} = 1$: since $\{K, H\} = \det \begin{bmatrix} \frac{\partial K}{\partial x} \\ \frac{\partial H}{\partial x} \end{bmatrix}$, then the condition $\{K, H\} = 1$ can hold only about a regular point x_0 of f, $f(x_0) \neq 0$ (namely, such that $\frac{\partial H}{\partial x}\big|_{x=x_0} \neq 0$), according to the Frobenius theorem (see Theorem 1.4.1 of [21]). In the canonical coordinates $y_1 = K$ and $y_2 = H$, the Hamiltonian function takes the form $H = y_2$, and the Hamiltonian system expressed in these coordinates is straightened, $\tilde{f}(y) = \begin{bmatrix} \frac{\partial H}{\partial y_2} & -\frac{\partial H}{\partial y_1} \end{bmatrix}^T = \begin{bmatrix} 1 & 0 \end{bmatrix}^T$, according to the general property that $\dot{u} = \{u, H\}$, with u being an arbitrary function of x. All the symmetries of $\tilde{f}(y)$ are parameterized by $\tilde{g}(y) = \begin{bmatrix} C_0(y_2) & C_1(y_2) \end{bmatrix}^T$, where $C_i(y_2)$ is an arbitrary function of y_2, whence (by the invariance of the Lie Bracket to diffeomorphisms) all the symmetries g of f are parameterized by

$$g = \left(\frac{\partial}{\partial x} \begin{bmatrix} K \\ H \end{bmatrix}\right)^{-1} \begin{bmatrix} C_0(H) & C_1(H) \end{bmatrix}^T.$$

Example 5: Consider the Hamiltonian function $H(x) = a x_1 x_2 + \frac{b}{3} x_1^3$, with $a \neq 0$, and the corresponding Hamiltonian system described by $f = \begin{bmatrix} a x_1 & -\left(a x_2 + b x_1^2\right) \end{bmatrix}^T$. Define $K(x) = \frac{1}{a} \log x_1$, for which $\{K,H\} = \det\begin{bmatrix} \frac{1}{ax_1} & 0 \\ ax_2+bx_1^2 & ax_1 \end{bmatrix} = 1$. Then all the symmetries g of f are parameterized by $g = \begin{bmatrix} ax_1C_0 & \left(-ax_2-bx_1^2\right)C_0+\frac{1}{ax_1}C_1 \end{bmatrix}^T$, with C_0 and C_1 being arbitrary functions of H. In particular, taking $C_0 = \frac{1}{a}$ and $C_1 = 3H = 3\left(ax_1x_2+\frac{b}{3}x_1^3\right)$, we have $g = \begin{bmatrix} x_1 & 2x_2 \end{bmatrix}$, according to the fact that f is homogeneous of degree 0 with respect to the integer dilation with weight $r_1 = 1$ and $r_2 = 2$.

It is also relevant, given a vector function g, to compute all the Hamiltonian systems having g as symmetry.

Assume the existence of two functions J_0 and J_1 such that $L_gJ_0 = 1$, $L_gJ_1 = 0$ and $\{J_0, J_1\} = 1$. This means that $y_1 = J_0$, $y_2 = J_1$ qualify as canonical coordinates with respect to the given Poisson bracket and, in particular, in these coordinates g is straightened: $\tilde{g}(y) = \begin{bmatrix} 1 & 0 \end{bmatrix}^T$. If f is a Hamiltonian system having g as a symmetry, since y_1 and y_2 are canonical, then the system $\dot{x} = f(x)$ transformed into the y-coordinates $\dot{y} = \tilde{f}(y)$ is still Hamiltonian and has $\tilde{g}(y)$ as a symmetry. All $\tilde{f}$ having $\tilde{g}$ as a symmetry are parameterized by $\tilde{f} = \begin{bmatrix} C_0(y_2) & C_1(y_2) \end{bmatrix}^T$, with C_0 and C_1 being arbitrary functions of y_2; if $\tilde{f}$ is Hamiltonian, then it must be area preserving (namely, $\mathrm{div}\left(\tilde{f}\right) = 0$): then, all Hamiltonian $\tilde{f}$ having $\tilde{g}$ as a symmetry are parameterized by $\tilde{f} = \begin{bmatrix} C_0(y_2) & C_1 \end{bmatrix}^T$, with C_0 being an arbitrary function of y_2 and C_1 being constant, with respective Hamiltonian function $K(y_1, y_2) = \int C_0(y_2)dy_2 - C_1y_1$ (clearly, $\int C_0(y_2)dy_2$ is an arbitrary function of y_2). Going back to the original coordinates x, all Hamiltonian f having g as a symmetry are parameterized by

$$f = \left(\frac{\partial}{\partial x}\begin{bmatrix} J_0 \\ J_1 \end{bmatrix}\right)^{-1}\begin{bmatrix} C_0(J_1) \\ C_1 \end{bmatrix},$$

with respective Hamiltonian function $H(x) = K(J_0, J_1) = \int C_0(y_2)dy_2\big|_{y_2=J_1} - C_1J_0$.

Example 6: Consider $g = \begin{bmatrix} 1+x_2 & -1 \end{bmatrix}^T$. Clearly, $J_0 = x_1 + \frac{1}{2}x_2^2$ and $J_1 = x_1 + x_2 + \frac{1}{2}x_2^2$ satisfy

$$\begin{aligned} L_gJ_0 &= \begin{bmatrix} 1 & x_2 \end{bmatrix}\begin{bmatrix} 1+x_2 \\ -1 \end{bmatrix} = 1, \\ L_gJ_1 &= \begin{bmatrix} 1 & 1+x_2 \end{bmatrix}\begin{bmatrix} 1+x_2 \\ -1 \end{bmatrix} = 0, \\ \{J_0, J_1\} &= \det\begin{bmatrix} 1 & x_2 \\ 1 & 1+x_2 \end{bmatrix} = 1. \end{aligned}$$

Then, all the Hamiltonian systems having g as a symmetry are parameterized by the Hamiltonian function $H = C_2\left(x_1+x_2+\frac{1}{2}x_2^2\right) - \left(x_1+\frac{1}{2}x_2^2\right)C_1$, where $C_2(y_2) = \int C_0(y_2)dy_2$ is an arbitrary function of y_2 and C_1 is a constant. For instance, if we take $C_2(y_2) = \frac{1}{2}y_2^2$ and $C_1 = 1$, we have the Hamiltonian function $H = \frac{1}{2}\left(x_1+x_2+\frac{1}{2}x_2^2\right)^2 - \left(x_1+\frac{1}{2}x_2^2\right)$, with the respective Hamiltonian system described by

$$f = \begin{bmatrix} x_1 + x_1x_2 + \frac{3}{2}x_2^2 + \frac{1}{2}x_2^3 \\ -\left(x_1 + x_2 + \frac{1}{2}x_2^2 - 1\right) \end{bmatrix}.$$

V. The general case

The proof of the following theorem is omitted because similar to the one of Theorem 1.

Theorem 2: Consider the following Hamiltonian function $H(x) = K(h)$, where $h \in \mathbb{R}^{2n}$ is an arbitrary function of $x = \begin{bmatrix} q^T & p^T \end{bmatrix}^T$, analytic in a neighborhood of $x = 0$, $h(0,0) = 0$. Assume that $K(h)$ is polynomial and homogeneous of degree k with respect to $\delta_\varepsilon^r h$, with $r = \begin{bmatrix} r_1 & \dots & r_n & r_{n+1} & \dots & r_{2n} \end{bmatrix}^T$, $r_i \in \mathbb{Z}$, $r_i \geq 0$, $r_i + r_{i+n} = k$, $i = 1,...,n$. Consider in addition the Hamiltonian system associated with H and described by (2); assume that the functions h_i satisfy the following conditions:

$$\begin{aligned} \{h_i, h_j\} &= 0, \{h_{i+n}, h_{j+n}\} = 0, \forall i,j \in \{1,...,n\}, \\ \{h_i, h_{j+n}\} &= \begin{cases} 0 & \text{if} \quad i \neq j \\ 1 & \text{if} \quad i = j \end{cases} \quad \forall i,j \in \{1,...,n\}. \end{aligned}$$

Then,

(1) $y = h(x)$ qualifies as a canonical diffeomorphism;

(2) $g = \left(\frac{\partial h}{\partial x}\right)^{-1}\begin{bmatrix} r_1h_1 & \dots & r_{2n}h_{2n} \end{bmatrix}^T$ is a (not necessarily Hamiltonian) symmetry of f;

(3) g can be linearized by $y = h$, thus obtaining in the new coordinates $\tilde{g} = \begin{bmatrix} r_1y_1 & \dots & r_{2n}y_{2n} \end{bmatrix}^T$;

(4) since $\tilde{f} = \frac{\partial h}{\partial x}f\big|_{x=h^{-1}(y)}$ is analytic at $y = 0$ and homogeneous of degree 0 with respect to the standard dilation $\tilde{g}(y) = \begin{bmatrix} r_1y_1 & \dots & r_{2n}y_{2n} \end{bmatrix}^T$, then $\tilde{f}$ can be rendered linear by a finite-dimensional state immersion;

(5) if $r_i = r_{i+n} = 1$, $i = 1,...,n$, then $\tilde{f}$ is linear.

Example 7: Assume that $H(h)$ is polynomial and homogeneous of degree 4 with respect to $r = \begin{bmatrix} 2 & 1 & 2 & 3 \end{bmatrix}^T$:

$$H = a_1h_1h_2^2 + a_2h_3h_2^2 + a_3h_1h_3 + a_4h_2^4 + a_5h_2h_4,$$

with the functions h_i's satisfying the following conditions:

$$\begin{aligned} \{h_1, h_2\} &= 0, \{h_3, h_4\} = 0, \\ \{h_1, h_3\} &= 1, \{h_2, h_4\} = 1, \\ \{h_1, h_4\} &= 0, \{h_2, h_3\} = 0. \end{aligned}$$

These conditions guarantees that $y_i = h_i(x), i = 1,2,3,4$, qualifies as canonical coordinates, such that, setting $H = a_1y_1y_2^2 + a_2y_3y_2^2 + a_3y_1y_3 + a_4y_2^4 + a_5y_2y_4$, the Hamiltonian system is described by the following vector function:

$$\tilde{f} = \begin{bmatrix} \frac{\partial H}{\partial y_3} \\ \frac{\partial H}{\partial y_4} \\ -\frac{\partial H}{\partial y_1} \\ -\frac{\partial H}{\partial y_2} \end{bmatrix} = \begin{bmatrix} a_2y_2^2 + a_3y_1 \\ a_5y_2 \\ -a_1y_2^2 - a_3y_3 \\ -2a_1y_1y_2 - 2a_2y_2y_3 - 4a_4y_2^3 - a_5y_4 \end{bmatrix}.$$

This system can be linearized by the following state immersion: $y_5 = y_2^2, y_6 = y_2^3, y_7 = y_1y_2, y_8 = y_2y_3$, obtaining an extended linear system $\dot{y}_e = A_e y_e$ with the following dynamic matrix A_e:

$$\begin{bmatrix} a_3 & 0 & 0 & 0 & a_2 & 0 & 0 & 0 \\ 0 & a_5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2a_1y_2 & -a_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -a_5 & 0 & -4a_4 & -2a_1 & -2a_2 \\ 0 & 0 & 0 & 0 & 2a_5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3a_5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & a_2 & a_3+a_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & -a_1 & 0 & a_5-a_3 \end{bmatrix}.$$

VI. CONCLUDING REMARKS

The procedure followed in the previous sections for transforming a Hamiltonian system into a polynomial form, homogeneous of degree 0 with respect to an integer dilation, can be easily extended for transforming a Hamiltonian system into some canonical forms, such as the Poincaré-Dulac normal form [8] if its linear part is semi-simple or the Belitskii normal form [22] if its linear part is nilpotent, as briefly described in the following.

Consider the Hamiltonian function $H(x) = \frac{1}{2}x^T Bx + H_3(x)$, with $B \neq 0$ and H_3 containing third and higher order terms about $x = 0$. The Hamiltonian system associated with H is described by $f(x) = Ax + h(x)$, with $A := SB$ and $h := S\left(\frac{\partial H_3}{\partial x}\right)^T$. Assume that A is semi-simple and let $H_2 := \frac{1}{2}x^T Bx$: clearly, $\dot{x} = Ax$ is the Hamiltonian system associated with H_2, namely $Ax = S\left(\frac{\partial H_2}{\partial x}\right)^T$. Then, $f(x)$ is in the Poincaré-Dulac normal form (see [8] for an introduction to the Poincaré-Dulac normal form) if $H_3 = K(H_2)$, with $K(H_2)$ being an arbitrary function of H_2 (notice that the condition $H_3 = K(H_2)$ is, in the planar case, equivalent to the condition $H_3(e^{At}x) = H_3(x)$ given in [23]). As a matter of fact, in such a case $h = S\left(\frac{\partial H_3}{\partial x}\right)^T = K' Ax$, with $K' = \frac{\partial K}{\partial H_2}$; clearly, $[Ax, K'Ax] = K'[Ax, Ax] + AxL_{Ax}K' = 0$.

Consider the Hamiltonian function $H(h_1, h_2) = \frac{1}{2}h_2^2 + K(h_1)$, with $h_1(x), h_2(x)$ being two arbitrary functions such that $\{h_1, h_2\} = 1$ and K being an arbitrary function of h_1, containing third and higher order terms with respect to $h_1 = 0$. The Hamiltonian system associated with H is described by $f = \begin{bmatrix} h_2\frac{\partial h_2}{\partial x_2} + K'\frac{\partial h_1}{\partial x_2} & -h_2\frac{\partial h_2}{\partial x_1} - K'\frac{\partial h_1}{\partial x_1} \end{bmatrix}^T$, with $K' = \frac{\partial K}{\partial h_1}$. The Belitskii normal form of $f(x)$ is (its linear part is nilpotent) $\tilde{f}(y) = \begin{bmatrix} y_2 & -K'(y_1) \end{bmatrix}^T$ and the diffeomorphism transforming f into $\tilde{f}$ is $y_1 = h_1, y_2 = h_2$.

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