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A fast algorithm to compute the controllability, decentralized fixed-mode, and minimum-phase radius of LTI systems

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Abstract— **In this paper, an efficient algorithm is presented for solving the 2-D optimization problem which is associated with computing the robustness measures of a number of linear time-invariant (LTI) system properties such as the controllability radius, the decentralized fixed-mode radius, and the minimum-phase radius. Unlike methods such as gradient search methods, the proposed algorithm works well independent of the initial trial point, and obtains the minimum with a high level of confidence that it is indeed global. A numerical example is included.**

I. INTRODUCTION

Consider the following LTI multivariable system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned} \quad (1)$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, and $y \in \mathbb{R}^r$ are respectively the state, input, and output vectors, and A, B, C, and D are constant matrices with the appropriate dimensions for $n \geq 1$, $m \geq 1$, $r \geq 1$, and $\max(r, m) \leq n$. It is well known that there exists a LTI controller that can assign the eigenvalues of the closed-loop system to any arbitrary spectrum if and only if the system is controllable and observable. However, when a controllable system is subject to parametric perturbations (i.e. $A \rightarrow A + \Delta_A$ and $B \rightarrow B + \Delta_B$), the system may be very "close" to becoming uncontrollable. Hence, a continuous controllability measure is more informative than the traditional 'yes/no' controllability metric, which simply determines whether a system is controllable or not. The same can be said about other system properties such as observability, stability, minimum-phase, etc.

In the current literature, various continuous measures have been proposed to measure the robustness of various system properties with respect to parametric perturbations. In particular, the *controllability radius* introduced in [1], [2] measures how close a controllable system is to being uncontrollable, the *stability radius* in [3] measures how close a stable system is to an unstable one, and the *decentralized fixed-mode (DFM) radius* ([4], [5]) measures how close a system with no DFMs is to having one. More recently, such robustness measures have been extended to characterize the robustness of a system's transmission zero properties. For instance, the *minimum-phase radius* in [6] measures how close a minimum-phase system is to becoming non-minimum phase.

Computing the stability radius mentioned above ([3]) requires solving only a 1-D optimization problem, and a fast algorithm to solve this is proposed in [7]. On the other hand, computing the other radii mentioned above (i.e. except the stability radius) requires solving a 2-D optimization problem.

As of now, the 2-D problem is solved using nonlinear search methods such as gradient search methods. A disadvantage with using such nonlinear techniques is that the minimum obtained may be a local minimum. Hence selecting the initial trial points are critical, and a large number of initial points are generally tested in order to achieve a certain level of confidence that the obtained minimum is global. In this paper, we propose a more efficient algorithm for solving the 2-D optimization problem. The proposed algorithm has some similar features as the one in [7] (in particular, the use of a so-called "minimizing set"), and has the advantage that it is not dependent on the initial test values, and obtains the global minimum with a high degree of confidence.

The paper is organized as follows. In Section II, the controllability radius is reviewed and the 2-D optimization problem to be solved is described. Then in Section III, the few useful tools are developed, followed by a thorough description of the algorithm in Section IV. Finally, Section V gives a numerical example to demonstrate the effectiveness of the algorithm.

II. NOTATION AND BACKGROUND

In this paper, the field of real and complex numbers are denoted by $\mathbb{R}$ and $\mathbb{C}$ respectively. $\mathbb{C}_+$ and $\mathbb{C}_U$ denotes the closed right half and the closed upper half of the complex plane respectively, and $\mathbb{R}_+$ denotes the nonnegative real line $[0, +\infty)$. The i-th singular value of a matrix $M \in \mathbb{C}^{p \times m}$ is denoted by $\sigma_i(M)$, where $\sigma_1(M) \geq \sigma_2(M) \geq \cdots$. $\|M\|$ denotes the spectral norm of a matrix M and is equal to $\sigma_1(M)$. Also, $\overline{M}$, M^T, M^H, and M^+ denote respectively the complex conjugate, transpose, complex conjugate transpose, and Moore-Penrose pseudoinverse of M. The real and imaginary components of the matrix M are given by $\Re M$ and $\Im M$ respectively. Finally, the set of eigenvalues of a square matrix $A \in \mathbb{C}^{n \times n}$ is denoted by $\lambda(A)$.

A. Controllability radius

The following definition is made.

Definition 2.1: Given a LTI system (1), the *controllability radius*, $r^c_{\mathbb{F}}$, is defined to be:

$$r^c_{\mathbb{F}}(A, B) = \inf\{\|[\Delta_A, \Delta_B]\| \mid \Delta_A \in \mathbb{F}^{n \times n}, \Delta_B \in \mathbb{F}^{n \times m}, (A + \Delta_A, B + \Delta_B) \text{ is uncontrollable}\} \quad (2)$$

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where $\mathbb{F} \in \{\mathbb{C}, \mathbb{R}\}$[1]. Similarly, the *stabilizability radius*, $r_{\mathbb{F}}^{c+}$, is defined as:

$$r_{\mathbb{F}}^{c+}(A,B) \;=\; \inf\{\|[\Delta_A, \Delta_B]\| \mid \Delta_A \in \mathbb{F}^{n\times n}, \Delta_B \in \mathbb{F}^{n\times m}, \; (A+\Delta_A, B+\Delta_B) \text{ is unstabilizable } \} \quad (3)$$

In [1] and [2], the complex and real controllability radii are respectively shown to be given by:

$$r_{\mathbb{C}}^{c}(A,B) = \min_{s\in\mathbb{C}} \sigma_n\big(\left[\begin{array}{cc} A - sI_n & B \end{array}\right]\big) \quad (4)$$

and

$$r_{\mathbb{R}}^{c}(A,B) = \min_{s\in\mathbb{C}} \tau_n\big(\left[\begin{array}{cc} A - sI_n & B \end{array}\right]\big) \quad (5)$$

where for $W \in \mathbb{C}^{p\times q}$,[2]

$$\tau_n(W) := \sup_{\gamma\in(0,1]} \sigma_{2n-1}\left(\left[\begin{array}{cc} \operatorname{Re} W & -\gamma \operatorname{Im} W \\ \gamma^{-1} \operatorname{Im} W & \operatorname{Re} W \end{array}\right]\right) \quad (6)$$

For notational convenience in the paper, denote

$$P(\gamma, W) := \left[\begin{array}{cc} \operatorname{Re} W & -\gamma \operatorname{Im} W \\ \gamma^{-1} \operatorname{Im} W & \operatorname{Re} W \end{array}\right] \quad (7)$$

Similarly, the complex and real stabilizability radii are respectively shown to be:

$$r_{\mathbb{C}}^{c+}(A,B) = \min_{s\in\mathbb{C}_+} \sigma_n\big(\left[\begin{array}{cc} A - sI_n & B \end{array}\right]\big) \quad (8)$$

and

$$r_{\mathbb{R}}^{c+}(A,B) = \min_{s\in\mathbb{C}_+} \tau_n\big(\left[\begin{array}{cc} A - sI_n & B \end{array}\right]\big) \quad (9)$$

B. 2-D optimization problem

It can be seen from (4)–(9) that a 2-D optimization problem in the complex plane is required to be solved when computing the complex and real controllability/stabilizability radii. A similar 2-D problem is to be solved when computing the DFM radius ([4], [5]), and the minimum-phase radius ([6]). In this paper, a fast algorithm is proposed to solve the two standard general problems associated with such 2-D optimization problems; i.e. for given constant real matrices (A, B, C, D) and $i \in \{1, \ldots, n + \min(r, m)\}$:

Problem 1: Find

$$r_{\mathbb{C}}^{i}(C,A,B,D) = \min_{s\in\mathbb{C}} \sigma_i\left(\left[\begin{array}{cc} A - sI & B \\ C & D \end{array}\right]\right) \quad (10)$$

and

Problem 2: Find

$$r_{\mathbb{R}}^{i}(C,A,B,D) = \min_{s\in\mathbb{C}} \tau_i\left(\left[\begin{array}{cc} A - sI & B \\ C & D \end{array}\right]\right) \quad (11)$$

For the sake of simplicity, denote $G(s) := \left[\begin{array}{cc} A - sI & B \\ C & D \end{array}\right]$. To avoid the trivial case where the radius is zero for all $s \in \mathbb{C}$, we will assume that the two problems (10) and (11) satisfy the following assumption:

$$\operatorname{rank}\left(\left[\begin{array}{cc} A & B \\ C & D \end{array}\right]\right) \geq i \quad (12)$$

III. Preliminary Results

Before presenting an efficient algorithm for solving (10) and (11), some preliminary tools are needed; in particular Theorem 3.1 & 3.2 given below.

Definition 3.1 (The $\mathcal{H}$ matrix): Given real $x \geq 0$, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{r\times n}$, and $D \in \mathbb{R}^{r\times m}$, and nonsingular matrices $\Delta \in \mathbb{C}^{n\times n}$, $E \in \mathbb{C}^{(n+r)\times(n+r)}$, and $F \in \mathbb{C}^{(n+m)\times(n+m)}$, define:

$$\mathcal{H}(x, A, B, C, D, \Delta, E, F) := \tilde{\Delta}^{-1}\left(\tilde{A} - \tilde{B}\tilde{D}^{-1}\tilde{C}\right) \quad (13)$$

where $\tilde{A} = \left[\begin{array}{cc} A & 0 \\ 0 & A^T \end{array}\right] + T_{11}$, $\tilde{B} = \left[\begin{array}{cc} B & 0 \\ 0 & C^T \end{array}\right] + T_{12}$, $\tilde{C} = \left[\begin{array}{cc} C & 0 \\ 0 & B^T \end{array}\right] + T_{21}$, $\tilde{D} = \left[\begin{array}{cc} D & 0 \\ 0 & D^T \end{array}\right] + T_{22}$, and $\tilde{\Delta} = \left[\begin{array}{cc} \Delta & 0 \\ 0 & \Delta^H \end{array}\right]$, and where $T := \left[\begin{array}{cc} T_{11} & T_{12} \\ T_{21} & T_{22} \end{array}\right]$ satisfies

$$\left[\begin{array}{cc} T_{11} & T_{12} \\ T_{21} & T_{22} \end{array}\right] = P_l \left[\begin{array}{cc} -x\left(E^H E\right)^{-1} & 0 \\ 0 & -x\left(F F^H\right)^{-1} \end{array}\right] P_r$$

for the permutation matrices $P_l = \left[\begin{array}{cccc} I_n & 0 & 0 & 0 \\ 0 & 0 & I_n & 0 \\ 0 & I_r & 0 & 0 \\ 0 & 0 & 0 & I_m \end{array}\right]$ and $P_r = \left[\begin{array}{cccc} 0 & I_n & 0 & 0 \\ 0 & 0 & 0 & I_r \\ I_n & 0 & 0 & 0 \\ 0 & 0 & I_m & 0 \end{array}\right]$.

The following preliminary result is obtained.

Theorem 3.1: Let $M := E\left[\begin{array}{cc} A - s\Delta & B \\ C & D \end{array}\right]F$, where $s \in \mathbb{R}$, and $E \in \mathbb{C}^{(n+r)\times(n+r)}$, $F \in \mathbb{C}^{(n+m)\times(n+m)}$, and $\Delta \in \mathbb{C}^{n\times n}$ are all nonsingular. Then, for given $x \in \mathbb{R}_+$,

$$x \in \sigma(M) \Leftrightarrow s \in \lambda(\mathcal{H}(x, A, B, C, D, \Delta, E, F)) \quad (14)$$

Proof:

$$\begin{aligned}
x \in \sigma(M) &\Leftrightarrow x \in \lambda\left(\left[\begin{array}{cc} 0 & M \\ M^H & 0 \end{array}\right]\right) \\
&\Leftrightarrow \det\left(\left[\begin{array}{cc} -xI_{n+r} & M \\ M^H & -xI_{n+m} \end{array}\right]\right) = 0 \\
&\Leftrightarrow \det\left(P_l\left[\begin{array}{cc} -xI_{n+r} & M \\ M^H & -xI_{n+m} \end{array}\right]P_r\right) = 0 \\
&\Leftrightarrow \det\left(\left[\begin{array}{cc|cc} A - s\Delta & 0 & B & 0 \\ 0 & A^T - s\Delta^H & 0 & C^T \\ \hline C & 0 & D & 0 \\ 0 & B^T & 0 & D^T \end{array}\right] + \left[\begin{array}{cc} T_{11} & T_{12} \\ T_{21} & T_{22} \end{array}\right]\right) = 0 \\
&\Leftrightarrow \det\left(\left[\begin{array}{c|c} \tilde{A} - s\tilde{\Delta} & \tilde{B} \\ \hline \tilde{C} & \tilde{D} \end{array}\right]\right) = 0 \qquad (15) \\
&\Leftrightarrow \det\left(\tilde{D}\right)\det\left(\tilde{A} - s\tilde{\Delta} - \tilde{B}\tilde{D}^{-1}\tilde{C}\right) = 0 \\
&\Leftrightarrow s \in \lambda\left(\tilde{\Delta}^{-1}\left(\tilde{A} - \tilde{B}\tilde{D}^{-1}\tilde{C}\right)\right)
\end{aligned}$$

■

[1] To distinguish which field is consider, $r_{\mathbb{C}}^{c}$ (i.e. $\mathbb{F} = \mathbb{C}$) is called the *complex* controllability radius and $r_{\mathbb{R}}^{c}$ is called the *real* controllability radius.

[2] $\tau_n(W)$ is referred to as the n-th *real perturbation value* of W (see [8]).

Remark 3.1: If $\tilde{D}$ in (13) is singular, then by (15), (14) can be modified such that s in (14) is a generalized eigenvalue of the matrix pair $\left(\begin{bmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{bmatrix}, \begin{bmatrix} \tilde{\Delta} & 0 \\ 0 & 0 \end{bmatrix}\right)$ instead; i.e. s is in the set of $\lambda \in \mathbb{C}$ such that $\begin{bmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{bmatrix} x = \lambda \begin{bmatrix} \tilde{\Delta} & 0 \\ 0 & 0 \end{bmatrix} x$ for some non-zero eigenvector $x \in \mathbb{C}^{2n+m+r}$.

The following result is also obtained.

Theorem 3.2: Let $\gamma \in (0,1]$, $\theta \in [0,2\pi]$ and real $x \geq 0$ be given. Then, for all $s \in \mathbb{R}_+$,

$$\begin{aligned} x \in \sigma\big(P\big(\gamma, G\big(se^{i\theta}\big)\big)\big) \quad &\Leftrightarrow \\ s \in \lambda\Big(\mathcal{H}\Big(x, \hat{A}, \hat{B}, \hat{C}, \hat{D}, \hat{\Delta}, T_{\gamma,n+r}^{-1} P_{nr}, P_{nm} T_{\gamma,n+m}\Big)\Big) \end{aligned}$$

where $\hat{A} = \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}$, $\hat{B} = \begin{bmatrix} B & 0 \\ 0 & B \end{bmatrix}$, $\hat{C} = \begin{bmatrix} C & 0 \\ 0 & C \end{bmatrix}$, $\hat{D} = \begin{bmatrix} D & 0 \\ 0 & D \end{bmatrix}$, $\hat{\Delta} = \begin{bmatrix} e^{i\theta} I_n & 0 \\ 0 & e^{-i\theta} I_n \end{bmatrix}$, $P_{nr} = \begin{bmatrix} I_n & 0 & 0 & 0 \\ 0 & 0 & I_r & 0 \\ 0 & I_n & 0 & 0 \\ 0 & 0 & 0 & I_r \end{bmatrix}$, $P_{mr} = \begin{bmatrix} I_n & 0 & 0 & 0 \\ 0 & 0 & I_n & 0 \\ 0 & I_m & 0 & 0 \\ 0 & 0 & 0 & I_m \end{bmatrix}$, and $T_{\gamma,n} := \frac{1}{2\gamma} \begin{bmatrix} \sqrt{1+\gamma^2} I_n & i\gamma\sqrt{1+\gamma^2} I_n \\ \sqrt{1+\gamma^2} I_n & -i\gamma\sqrt{1+\gamma^2} I_n \end{bmatrix}$.

Proof: It can easily be verified (e.g. see [8]) that

$$\begin{aligned} T_{\gamma,n+r} P\big(\gamma, G\big(se^{i\theta}\big)\big)\, T_{\gamma,n+m}^{-1} &= \begin{bmatrix} G\big(se^{i\theta}\big) & 0 \\ 0 & \overline{G(se^{i\theta})} \end{bmatrix} \\ &= P_{nr} \left[\begin{array}{cc|cc} A - se^{i\theta} I_n & 0 & B & 0 \\ 0 & A - se^{-i\theta} I_n & 0 & B \\ \hline C & 0 & D & 0 \\ 0 & C & 0 & D \end{array}\right] P_{nm} \end{aligned}$$

Therefore,

$$P\big(\gamma, G\big(se^{i\theta}\big)\big) = T_{\gamma,n+r}^{-1} P_{nr} \begin{bmatrix} \hat{A} - s\hat{\Delta} & \hat{B} \\ \hat{C} & \hat{D} \end{bmatrix} P_{nm} T_{\gamma,n+m}$$

and the proof follows immediately from Theorem 3.1. ■

IV. Algorithm

An efficient iterative algorithm for solving (10) and (11) for given constant real matrices (A, B, C, D) and $i \in \{1, \ldots, n + \min(m, r)\}$ is now presented in this section. The following development focuses mainly on solving (11), but it will be shown later that with a slight modification, the algorithm also applies to solving (10).

Firstly, let the global minimum of (11) be denoted by:

$$r^* = \min_{s \in \mathbb{C}} \tau_i(G(s)) = \min_{s \in \mathbb{C}} \sup_{\gamma \in (0,1]} \sigma_{2i-1}(P(\gamma, G(s)))$$

and is achieved at $s^* \in \mathbb{C}$ and $\gamma^* \in (0,1]$, i.e. $r^* = \sigma_{2i-1}(P(\gamma^*, G(s^*)))$. Now suppose that at the k-th iteration, for $k = 1, 2, \ldots$, we are given r_{k-1}, γ_{k-1}, and s_{k-1} (i.e. obtained from the previous iteration), which are approximations of r^*, γ^*, and s^* respectively, where $r_{k-1} = \sigma_{2i-1}(P(\gamma_{k-1}, G(s_{k-1})))$. Furthermore, suppose we are also given the so-called "minimizing set", S_{k-1}, which is a (closed) set in the complex plane that contains the global minimizer s^*. The basic idea of the algorithm is then to use the information of r_{k-1}, γ_{k-1}, and s_{k-1}, together with Theorem 3.2, to reduce the size of S_{k-1}; i.e. to reduce the region containing the global minimizer s^*. Denoting the reduced set by S_k ($\subseteq S_{k-1}$), we then search within S_k for a new point $s_k \in S_k$ such that

$$\tau_i(G(s_k)) < r_{k-1}$$

and assign $r_k = \tau_i(G(s_k))$. Also, assign γ_k to be the value of γ that achieves r_k; i.e. $r_k = \sigma_{2i-1}(P(\gamma_k, G(s_k)))$. Note that since $r_k < r_{k-1}$, then r_k is a new and better approximation of the global minimum r^*. The procedure is then repeated until either the size of S_k is smaller than an user-specified tolerance, or until r_k becomes "very close" to zero (more on this later).

A. The minimizing set, S_k

In the algorithm, the "minimizing set" is defined to be the union of particular sectors of interest in the closed upper half of the complex plane, and can be described by $S_k \subseteq [0, \pi]$. The reason why the "minimizing set" is constrained to the closed upper half of the complex plane is due to the following (e.g. see [6]):

$$\sigma_i\left(\begin{bmatrix} A - sI & B \\ C & D \end{bmatrix}\right) = \sigma_i\left(\begin{bmatrix} A - \bar{s}I & B \\ C & D \end{bmatrix}\right)$$

and

$$\tau_i\left(\begin{bmatrix} A - sI & B \\ C & D \end{bmatrix}\right) = \tau_i\left(\begin{bmatrix} A - \bar{s}I & B \\ C & D \end{bmatrix}\right)$$

In other words, the radii in (10) and (11) are both symmetrical with respect to the real axis; hence one only needs to search within either the closed upper or lower half of the complex plane for the global minimum.

It should be pointed out that S_k is a subset of $[0, \pi]$, and that the underlying "minimizing set" actually consists of the sectors in the complex plane described by S_k; i.e. the "minimizing set" is $\{s \in \mathbb{C} | \angle s \in S_k\}$, where $\angle s$ denotes the angle of $s \in \mathbb{C}$. For convenience, however, we will sometimes refer to the "minimizing set" as S_k.

To reduce the size of S_k based on given r_{k-1}, γ_{k-1}, and s_{k-1}, where $r_{k-1} = \sigma_{2i-1}(P(\gamma_{k-1}, G(s_{k-1})))$, we first construct the set R_k:

$$R_k = \{s \in \mathbb{C}_U | \sigma_{2i-1}(P(\gamma_{k-1}, G(s))) < r_{k-1}\}$$ [3]

The significance of R_k is that for all points not in R_k, the radius can never be smaller than r_{k-1}. This is because for all $s \notin R_k$,

$$\begin{aligned} \tau_i(G(s)) &= \sup_{\gamma \in (0,1]} \sigma_{2i-1}(P(\gamma, G(s))) \\ &\geq \sigma_{2i-1}(P(\gamma_{k-1}, G(s))) \geq r_{k-1} \end{aligned}$$

Hence, one only needs to search within R_k to find points in the complex plane that achieve a radius smaller than r_{k-1}.

[3] Note that R_k can also be a (tighter) "minimizing set". For ease of implementation, however, we chose the "minimizing set" to be the sectors containing R_k (i.e. as described by S_k) instead.

To obtain R_k, we first fix a particular $\theta \in [0,\pi]$, and then find the set

$$R_k^\theta = \left\{w \in \mathbb{R}_+ | \sigma_{2i-1}\left(P\left(\gamma_{k-1}, G\left(we^{i\theta}\right)\right)\right) < r_{k-1}\right\} \quad (16)$$

Therefore,

$$R_k = \bigcup_{\theta\in[0,\pi]} R_k^\theta \quad (17)$$

Note that R_k^θ for a particular $\theta \in [0,\pi]$ can be obtained by applying Theorem 3.2. In particular, by Theorem 3.2, the values along the ray, $\mathcal{R}_\theta := \left\{se^{i\theta} \in \mathbb{C} | s \in \mathbb{R}_+\right\}$, that achieve a radius equal to r_{k-1} are among the real nonnegative eigenvalues of $\mathcal{H}\left(r_{k-1}, \hat{A}, \hat{B}, \hat{C}, \hat{D}, \hat{\Delta}, T^{-1}_{\gamma_{k-1},n+r}P_{nr}, P_{nm}T_{\gamma_{k-1},n+m}\right)$. Hence, by solving for $\Lambda = \lambda\left(\mathcal{H}\left(r_{k-1}, \hat{A}, \hat{B}, \hat{C}, \hat{D}, \hat{\Delta}, T^{-1}_{\gamma_{k-1},n+r}P_{nr}, P_{nm}T_{\gamma_{k-1},n+m}\right)\right)$ and then finding all real nonnegative $s \in \Lambda$ such that

$$\sigma_{2i-1}\left(P\left(\gamma_{k-1}, G\left(se^{i\theta}\right)\right)\right) = r_{k-1} \quad (18)$$

we obtain the endpoints of the intervals along the ray $\mathcal{R}_\theta$ where $\sigma_{2i-1}\left(P\left(\gamma_{k-1}, G\left(se^{i\theta}\right)\right)\right) < r_{k-1}$ and where $\sigma_{2i-1}\left(P\left(\gamma_{k-1}, G\left(se^{i\theta}\right)\right)\right) > r_{k-1}$. To determine which interval is the former (i.e. the one of interest), one can simply pick a trial point, w, within a particular interval and evaluate $\sigma_{2i-1}\left(P\left(\gamma_{k-1}, G\left(we^{i\theta}\right)\right)\right)$. If there are no real nonnegative eigenvalues, $s \in \Lambda$, such that $\sigma_{2i-1}\left(P\left(\gamma_{k-1}, G\left(se^{i\theta}\right)\right)\right) = r_{k-1}$, then either $R_k^\theta = \mathbb{R}_+$, or $R_k^\theta = \emptyset$. Again, a straightforward test using a trial point can confirm which is true.

If $R_k^\theta = \emptyset$ is true for a particular $\theta \in [0,\pi]$, then this implies that all the values along the ray $\mathcal{R}_\theta$ achieve a radius larger than r_{k-1}, and hence θ can be eliminated from the "minimizing set", S_k. Therefore the "minimizing set" for the next iteration is updated as follows. Let $\Theta_k \subseteq [0,\pi]$ be the set of intervals that describes the smallest union of sectors that contains R_k; i.e.

$$\Theta_k = \left\{\theta \in [0,\pi] \,| R_k^\theta \neq \emptyset\right\} \quad (19)$$

Then update $S_k = S_{k-1} \cap \Theta_k$.

B. Updating the current minimum, r_k

To obtain a radius smaller than r_{k-1} (i.e. a better approximation of the global minimum r^*), one can perform a search (e.g. a grid or random search) within the set $\left\{R_k^\theta | \theta \in S_k|\right\}$. From experience though, it is found that one can often obtain r_k by letting $s_k = w_m e^{i\theta_m}$ (i.e. $r_k = \tau_i(G(s_k))$), where θ_m is the midpoint of the largest interval of S_k, and w_m is the midpoint of the largest interval of $R_k^{\theta_m}$.

C. Stopping criteria

The algorithm has two main stopping criteria. Firstly, the algorithm stops when the "size" of the "minimizing set", S_k, and the size of R_k in (17) are both respectively smaller than user-specified tolerances, TOL_{S_k} and TOL_{R_k}, where the size of S_k is chosen to be the length of the largest interval in S_k, and the size of R_k is chosen to be:

$$size(R_k) = \max_\theta \left\{\text{length of largest interval in } R_k^\theta\right\}$$

The second stopping criteria handles the special case when the radius is zero in spite of the fact that assumption (12) is satisfied. For example, in [6], it is shown that the complex and real minimum-phase radius are both 0 when $D = 0$; i.e.

$$r_{\mathbb{F}}^{n+\min(r,m)}(A,B,C,0) = 0$$

where $\mathbb{F} \in \{\mathbb{C}, \mathbb{R}\}$, and this is achieved as (real) $s \to \infty$. Therefore, to prevent the algorithm from searching off into infinity, the second stopping criteria is added to quit when r_k is smaller than an user-specified tolerance, TOL_{r_k}.

D. Algorithm outline

The algorithm can be summarized as follows:

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Algorithm 4.1:
Input: $(A,B,C,D)$ and $i$, where (12) is satisfied
Input tolerances: $TOL_{S_k}$, $TOL_{R_k}$ and $TOL_{r_k}$
Output: $r^*$, $s^*$, and $\gamma^*$, where $r^* = \sigma_{2i-1}(P(\gamma^*, G(s^*)))$
1) Initialization:
    • Set $S_0 = [0,\pi]$.
    • Choose an arbitrary $s_0 \in \mathbb{C}_U$, and compute $r_0$ and
      $\gamma_0$, where $r_0 = \sigma_{2i-1}(P(\gamma_0, G(s_0)))$.
2) Iteration $k(= 1, 2, \ldots)$
    a) Given $r_{k-1}$, $\gamma_{k-1}$, $s_{k-1}$, and $S_{k-1}$.
    b) Reset $\Theta_k = \emptyset$.
    c) For $\theta \in S_{k-1}$ ($\theta$ can be discretized steps of $S_{k-1}$)
        • Compute $R_k^\theta$ by finding all real nonnegative
              $s \in \lambda\left(\mathcal{H}\left(r_{k-1}, \hat{A}, \hat{B}, \hat{C}, \hat{D}, \hat{\Delta}, T^{-1}_{\gamma_{k-1},n+r}P_{nr}, P_{nm}T_{\gamma_{k-1},n+m}\right)\right)$
          such that $\sigma_{2i-1}\left(P\left(\gamma_{k-1}, G\left(se^{i\theta}\right)\right)\right) = r_{k-1}$.
        • If $R_k^\theta \neq \emptyset$, then update $\Theta_k \leftarrow \Theta_k \cup \theta$.
    d) Update $S_k = S_{k-1} \cap \Theta_k$.
    e) Quit if the length of the largest interval of $S_k <$
       $TOL_{S_k}$ and $size(R_k) < TOL_{R_k}$.
    f) Find one point $s_k \in R_k^\theta$ such that $r_k < r_{k-1}$,
       where $\theta \in S_k$, and $r_k = \tau_i(G(s_k))$.
3) Quit if $r_k < TOL_{r_k}$.
4) Update $k \leftarrow k + 1$ and goto step 2.
```

E. Algorithm for solving problem (10)

To solve problem (10), only two slight modifications of Algorithm 4.1 are needed, which actually results in a simpler algorithm. Firstly, instead of computing the eigenvalues of $\mathcal{H}\left(r_{k-1}, \hat{A}, \hat{B}, \hat{C}, \hat{D}, \hat{\Delta}, T^{-1}_{\gamma_{k-1},n+r}P_{nr}, P_{nm}T_{\gamma_{k-1},n+m}\right)$ in step (2-c) to obtain R_k^θ, one is only required to compute the eigenvalues of $\mathcal{H}\left(r_{k-1}, A, B, C, D, e^{i\theta}I_n, I_{n+r}, I_{n+m}\right)$ (the proof is trivial and follows directly from Theorem 3.1). Furthermore, when solving problem (10), all the values within R_k actually achieve a smaller radius than r_{k-1} (and all the values outside R_k do not). This is not true when solving problem (11), where R_k only achieves a bound. Hence, step (2-f) for solving problem (10) can very easily be accomplished by selecting any point in R_k.

F. Algorithm for computing the real stabilizability radius

To compute the real stabilizability radius, the real unstable DFM radius, and the real minimum-phase radius, where the search is for a minimum in the closed right half of the complex plane – i.e. by solving the following problem

$$\tilde{r}^i_{\mathbb{R}}(A,B,C,D) = \min_{s\in\mathbb{C}_+} \tau_i\left(\begin{bmatrix} A-sI & B \\ C & D \end{bmatrix}\right) \qquad (20)$$

for given (A,B,C,D) and $i \in \{1,\ldots,n+\min(r,m)\}$ – only one very simply modification of Algorithm 4.1 is needed. In particular, one only needs to change the initialization step (1) of Algorithm 4.1 from $S_0 = [0,\pi]$ to $S_0 = \left[0,\frac{\pi}{2}\right]$ (the proof is trivial).

G. Operations count

To provide an idea of the computational requirements of using Algorithm 4.1, the number of operations required in terms of the number of singular value (σ), real perturbation value (τ), and eigenvalue (λ) problems are noted in Table I.

It is to be noted that every time the radius is evaluated at a particular $s \in \mathbb{C}$, a real perturbation value problem is solved. This occurs in step (1) and (2-f). In step (1), the real perturbation value problem is evaluated only once, but step (2-f) may require multiple evaluations, depending on the number of trials needed before $r_k < r_{k-1}$ is obtained. However, it is found from experiment that the number of trials is typically small (i.e. close to 1 when using the method outlined in Section IV-B).

It is also to be noted that the main portion of the total operations count is from computing R^θ_k for all $\theta \in S_k$ in step (2-c). In terms of implementation, S_k is discretized into a number of points. From experiment, it is found that it is generally sufficient to discretize S_k into 15 to 20 points, or a minimum of 0.5° (i.e. a maximum of 360 ($= \frac{\pi}{0.5^\circ}$) points), whichever results in a smaller angular division. For a given $\theta \in S_k$, computing R^θ_k requires: i) solving for the eigenvalues, Λ, of the $2n \times 2n$ matrix $\mathcal{H}\Big(r_{k-1}, \hat{A}, \hat{B}, \hat{C}, \hat{D}, \hat{\Delta}, T^{-1}_{\gamma_{k-1},n+r} P_{nr}, P_{nm} T_{\gamma_{k-1},n+m}\Big)$; and ii) evaluating a number of $2(n+r) \times 2(n+m)$-sized singular value problems to verify which real nonnegative eigenvalues, $s \in \Lambda$, (there are a maximum of $2n$ such values) have the property that they satisfy (18). Furthermore, since there are at most $2n$ real nonnegative eigenvalues $s \in \Lambda$ that may satisfy (18), R^θ_k contains at most $2n+1$ intervals. Hence, to determine which intervals in R^θ_k achieve a radius less than r_{k-1}, at most $2n+1$ points are tested, resulting in $2n+1$ additional $2(n+r) \times 2(n+m)$-sized singular value problems.

Remark 4.1: It should be noted that Algorithm 4.1 solves only a limited number of real perturbation value problems, which by itself is a 1-D optimization problem involving multiple $2(n+r) \times 2(n+m)$-sized singular value problems. Hence, this proves to be an advantage of Algorithm 4.1, as compared to say, a gradient search method, which requires solving a real perturbation value problem at each trial point, and at each step involved with computing the steepest gradient.

TABLE I

SUMMARY OF OPERATIONS COUNT OF ALGORITHM 4.1

Step	# of (σ,τ,λ)	Size of matrix
1	$1 \times \tau$	$(n+r) \times (n+m)$
2-c (for each $\theta \in S_{k-1}$)	$1 \times \lambda$ at most $(4n+1) \times \sigma$	$2n \times 2n$ $2(n+r) \times 2(n+m)$
2-f	at least $1 \times \tau$	$(n+r) \times (n+m)$

Here, we denote σ, τ, and λ as singular value, real perturbation value, and eigenvalue problems.

V. Numerical Example

The following example can be found in [2], where the real controllability radius of

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0.1 & 3 & 5 \\ 0 & -1 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 0.1 \\ 0 \end{bmatrix}$$

is found to be 0.0492, and is achieved at $s = 0.972 + j0.982$. Using Algorithm 4.1 with the (arbitrary) starting point $s_0 = 1j$, the same radius and global minimizer are found to within 3 significant figures in 5 iterations and to 6 significant figures in 7 iterations. Table II lists r_k, s_k, and the "minimizing set", S_k for each iteration $k = 0,\ldots,9$. Figure 1 provides a grid plot of the real controllability radius with respect to a given $s \in \mathbb{C}$. Figures 2 to 5 plot R_k (outlined by the dots) obtained at Iterations 1, 3, 4, and 8 respectively, superimposed on a contour plot of Figure 1. The straight lines originating from the origin depict S_k.

It is interesting to note that on a computer with a Pentium IV 2.0GHz processor, 512MB of RAM, and MATLAB 7.0, Algorithm 4.1 took a total of about 6 sec. to complete, which is approximately the same amount of time a gradient search method requires to obtain a (possibly local) minimum from a single starting point. Hence, if the gradient search method has to test 20 initial points in order to achieve a certain level of confidence that the obtained minimum is global, then the gradient search method will take approximately 20 times longer to run than Algorithm 4.1.

VI. Conclusions

In this paper, an efficient algorithm is presented for solving the general 2-D optimization problems (10) and (11), which are essential for computing the complex and

TABLE II

ESTIMATES OF THE GLOBAL MINIMUM RADIUS (r_k), THE MINIMIZER (s_k), AND THE MINIMIZING SET (S_k) AT EACH ITERATION k

Iter. (k)	r_k	s_k	S_k
0	0.745637	$j1$	$[0.00^\circ, 180.00^\circ]$
1	0.740724	$3.0616_{10^{-17}} + j0.5$	$[0.00^\circ, 180.00^\circ]$
2	0.218632	0.46766	$[0.00^\circ, 180.00^\circ]$
3	0.117352	$0.98098 + j0.58561$	$[0.00^\circ, 61.67^\circ]$
4	$5.33004_{10^{-2}}$	$0.97584 + j0.91703$	$[30.33^\circ, 56.11^\circ]$
5	$4.92304_{10^{-2}}$	$0.97060 + j0.98023$	$[42.70^\circ, 47.86^\circ]$
6	$4.92191_{10^{-2}}$	$0.97214 + j0.98179$	$[45.02^\circ, 45.55^\circ]$
7	$4.92186_{10^{-2}}$	$0.97176 + j0.98203$	$[45.27^\circ, 45.33^\circ]$
8	$4.92186_{10^{-2}}$	$0.97186 + j0.98194$	$[45.29^\circ, 45.30^\circ]$
9	$4.92186_{10^{-2}}$	$0.97184 + j0.98197$	$[45.30^\circ, 45.30^\circ]$

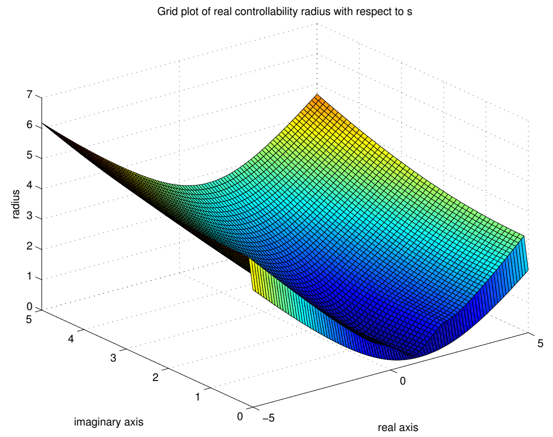

Fig. 1. Grid plot of the controllability radius with respect to a given $s \in \mathbb{C}$.

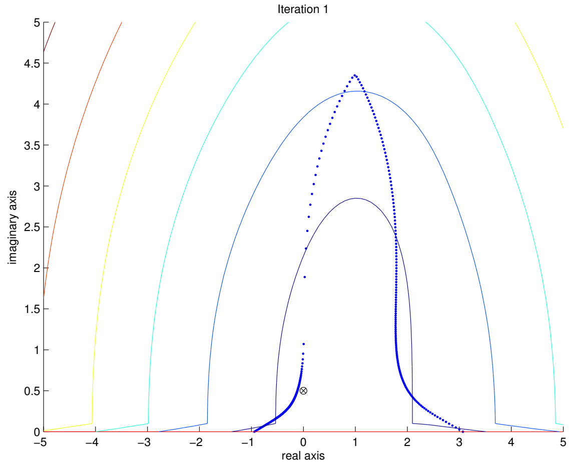

Fig. 2. Iteration 1

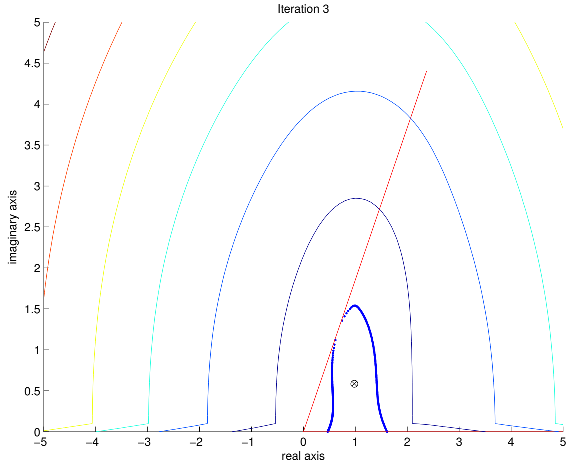

Fig. 3. Iteration 3

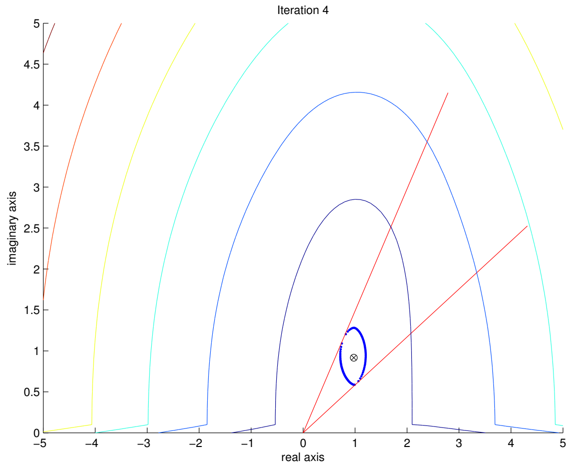

Fig. 4. Iteration 4

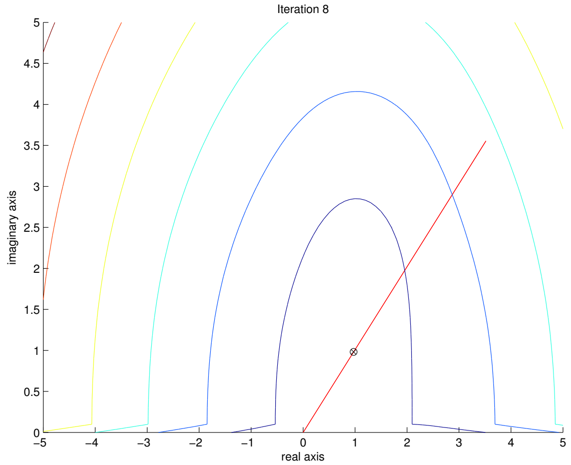

Fig. 5. Iteration 8

real controllability radius, the minimum-phase radius, the DFM radius, etc. The algorithm works by iteratively reducing the size of the so-called "minimizing set" which contains the global minimizer. Unlike methods such as the gradient search method, the choice of the initial point of the proposed algorithm is not crucial, and the minimum is obtained with a high degree of confidence that it is indeed global.

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