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A Graph Theory Based Characterization of Controllability for Multi-Agent Systems with Fixed Topology

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Abstract— **The paper studies a class of formation control problem, i.e. the controllability for multi-agent systems under leader-follower framework. A new concept *leader-follower connectedness* is proposed to deal with what is the desired extent of connectivity between the leader and follower subgraphs for a multi-agent system to be controllable. Analysis based on this concept yields graph theory based characterizations for the controllability of interconnection dynamic networks with fixed topologies.**

I. INTRODUCTION

Distributed coordination of networks of dynamic agents has attracted a great deal of attention in recent years [1], [11], [14], [8], [9], [10], [2], [3], [13], [7]. This is partly due to broad applications of multi-agent systems in, e.g. the cooperative control of unmanned aerial vehicles, and technology improvements allowing smaller, more versatile robots and other types of agents.

The controllability problem was put forward for the first time for multi-agent systems by Tanner in [13], and then developed in [3], [2], [10], [9], [8]. The problem is on how the interconnected systems can be steered to specific positions by regulating the motion of a single system that plays the role of the group leader [13]. This is what the so-called the group can be controlled. This requires the characterization of conditions under which the leaders can move the followers into any desired position or configuration [3]. That is, to derive conditions for a group of systems interconnected via neighbor rules, to be controllable by one of them acting as a leader [13].

It is essentially a kind of formation control problem. The problem is transformed to a classical notion of controllability in [13] with respect to a fixed interconnection topology and a switched controllability problem in [9], [8], [4] with respect to a switching topology where the results established in [6], [5], [12] are employed. One of the features for the controllability problem studied in [13], [9], [8] is that the leader is assumed unidirectional, i.e. the leader's neighbors still obey the interconnection neighbor rules, but the leader is indifferent, and is free to pick any agent [13]. Accordingly the leader does not participate in the typical configuration updates, and merely acts as an external control signals. The leader is not affected by the members whereas each member is influenced by the leader and the other members.

Central to the investigation of formation control is the nature of interconnection topologies. Some preliminary results on formation control were derived with respect to the fixed topology, which is a necessary step toward the more realistic dynamic setting. For example, in addition to [14], [2], the feasibility problem of achieving a specified geometric formation of a group of unicycles was investigated in [7], where necessary and sufficient graphical conditions for the existence of local information controller to assure the asymptotic convergence of the closed system were derived. Our goal is to consider the formation control, which is reformulated as a controllability problem in this paper, where the dynamics are influenced by multiple leaders. The results are graph theory based and accordingly provide new insights into some related algebraic conditions from the viewpoint of graph theory.

II. GRAPH THEORY PRELIMINARIES

An undirected graph $\mathcal{G}$ consists of a node set $\mathcal{V} = \{v_1, \cdots, v_N\}$ and an edge set $\mathcal{E} = \{(v_i, v_j)|\ v_i, v_j \in \mathcal{V}\}$, where an edge is an unordered pair of distinct nodes of $\mathcal{V}$. A graph with node set $\mathcal{V}$ is said to be a graph on $\mathcal{V}$, and it can be visually depicted by drawing a dot for each node and a line for each edge. The number of nodes of a graph $\mathcal{G}$ is its order, and its total number of edges is its degree. If we use $|\cdot|$ to denote cardinality, we have that the order of $\mathcal{G}$ is $|\mathcal{V}(\mathcal{G})|$, or simply $|\mathcal{V}|$, and its degree of $\mathcal{G}$ is $|\mathcal{E}(\mathcal{G})|$, or $|\mathcal{E}|$. Two nodes v_i and v_j are neighbors if $(v_i, v_j) \in \mathcal{E}$, and the neighboring relation is indicated with $v_j \sim v_i$. In this case we say that v_j is a neighbor of v_i. The number of neighbors of each node is its valency or degree. If all the nodes of $\mathcal{G}$ are pairwise adjacent, then $\mathcal{G}$ is complete. A path $v_{i_0} v_{i_1} \cdots v_{i_s}$ is a finite sequence of nodes such that $v_{i_{k-1}} \sim v_{i_k}$, $k = 1, \cdots, s$, and a graph $\mathcal{G}$ is connected if there is a path between any pair of distinct nodes. Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ and $\mathcal{G}' = (\mathcal{V}', \mathcal{E}')$ be two graphs. We call $\mathcal{G}'$ a subgraph of $\mathcal{G}$ (and $\mathcal{G}$ a supergraph of $\mathcal{G}'$) if $\mathcal{V}' \subseteq \mathcal{V}$ and $\mathcal{E}' \subseteq \mathcal{E}$, and we denote this by $\mathcal{G}' \subseteq \mathcal{G}$. A subgraph $\mathcal{G}'$ is said to be induced from the original graph $\mathcal{G}$ if $\mathcal{E}' = \mathcal{E} \cap \mathcal{V}' \times \mathcal{V}'$. In other words, it is obtained by deleting a subset of nodes and all the edges connecting to those nodes. $\mathcal{G}' \subseteq \mathcal{G}$ is a spanning subgraph of $\mathcal{G}$ if $\mathcal{V}' = \mathcal{V}$.

The adjacency matrix $\mathcal{A}(\mathcal{G})$ of $\mathcal{G}$ is an $|\mathcal{V}| \times |\mathcal{V}|$ matrix of whose (i, j)-entry is 1 if (v_i, v_j) is one of $\mathcal{G}$'s edges and 0 if it is not. Any undirected graph can be represented by its adjacency matrix, $\mathcal{A}(\mathcal{G})$, which is a symmetric matrix with 0-1 elements. The valency matrix $\triangle(\mathcal{G})$ of a graph

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$\mathcal{G}$ is a diagonal matrix with rows and columns indexed by $\mathcal{V}$, in which the (i,j)-entry is the valency of node v_i. The incidence matrix $\mathrm{In}(\mathcal{G})$ of $\mathcal{G}$ is an $|\mathcal{V}| \times |\mathcal{E}|$ matrix, with one row for each node and one column for each edge. Suppose edge $e = (v_i, v_j)$. Then column e of $\mathrm{In}(\mathcal{G})$ is zero except for the ith and jth entries, which are $+1$ and -1, respectively. The Laplacian matrix $L(\mathcal{G})$ (simply, L) of a graph $\mathcal{G}$, where $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is an undirected, unweighted graph without graph loops (i,i) or multiple edges from one node to another, is an $|\mathcal{V}| \times |\mathcal{V}|$ symmetric matrix with one row and column for each node defined by

$$L(\mathcal{G})_{i,j} = \begin{cases} d_i, & \text{if } i = j \text{ (number of incident edges)} \\ -1, & \text{if } i \neq j \text{ and } \exists \text{ edge } (v_i, v_j) \\ 0, & \text{otherwise.} \end{cases}$$

Given a graph $\mathcal{G}$, its associated matrices $\mathrm{In}(\mathcal{G})$ and $L(\mathcal{G})$ have the following properties: (a) $L(\mathcal{G})$ is always symmetric and positive semidefinite; (b) zero is always a eigenvalue of $L(\mathcal{G})$ with $\mathbf{1}_n$, the vector of ones, being the associated eigenvector, and the algebraic multiplicity of the zero eigenvalue is equal to the number of connected components in the graph; (c) $\mathrm{In}(\mathcal{G})(\mathrm{In}(\mathcal{G}))^T = L(\mathcal{G})$, and $L(\mathcal{G}) = \triangle(\mathcal{G}) - \mathcal{A}(\mathcal{G})$.

III. Problem formulation

Consider a multi-agent system consisting of $N+n_l$ agents with simple, first order dynamics:

$$\mathfrak{M}: \begin{cases} \dot{x}_i = u_i, & i = 1, \ldots, N \\ \dot{x}_{N+j} = u_{N+j}, & j = 1, \ldots, n_l \end{cases} \tag{1}$$

where x_i is the state of the ith agent, $i = 1, \cdots, N+n_l$. The dimension of x_i could be arbitrary, as long as it is the same for all agents. In order to facilitate presentation, we will analyze only the one-dimensional case. The analysis is valid for any dimension n, with the difference being that expressions should be rewritten in terms of Kronecker products. Once the linkages between agents are known, an interconnection graph can be defined to describe the interconnection network.

Definition 1: [13] The interconnection graph, $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$, is being defined as an undirected graph consisting of :

- a set of nodes, $\mathcal{V} = \{v_1, \ldots, v_N, v_{N+1}, \ldots, v_{N+n_l}\}$, indexed by the agents in the group, and
- a set of edges, $\mathcal{E} = \{(v_i, v_j) \in \mathcal{V} \times \mathcal{V} | v_i \sim v_j\}$, containing unordered pairs of nodes that correspond to interconnected agents.

Interconnections come true through the input u_i

$$u_i = -\sum_{j \in \mathcal{N}_i} (x_i - x_j), \quad i = 1, \cdots, N+n_l, \tag{2}$$

where $\mathcal{N}_i = \{j \mid v_i \sim v_j; j \neq i\}$ is the set of indices of the agents that are interconnected to v_i, i.e., the neighboring set of v_i. Under the protocol (2) and with $x = (x_1, \cdots, x_{N+n_l})^T$ being the stack vector of all the agent states, we will have

$$\dot{x} = -Lx, \tag{3}$$

where L is the Laplacian matrix of the interconnection graph.

Let us now select $x_{N+1}, \cdots, x_{N+n_l}$ to take the leaders' role. Interconnections with the leaders are now assumed unidirectional: the leaders' neighbors still obey (2), but the leaders are indifferent, and are free to pick $u_{N+j}, j = 1, \ldots, n_l$ arbitrarily. Rename the agents and then the multi-agent system reads

$$\mathfrak{M}: \begin{cases} y_i \triangleq x_i, & i = 1, \ldots, N \\ z_j \triangleq x_{N+j}, & j = 1, \ldots, n_l \end{cases}$$

with y being the stack vector of all y_i, z the stack vector of all z_j, and u the stack vector of all u_{N+j}, $j = 1, \ldots, n_l$, one can write the system in the form:

$$\begin{bmatrix} \dot{y} \\ \dot{z} \end{bmatrix} = -\begin{bmatrix} F & R \\ 0 & 0 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} + \begin{bmatrix} 0 \\ u \end{bmatrix}$$

where F is the matrix obtained from L after deleting the last n_l rows and n_l columns, and R is the $N \times n_l$ submatrix consisting of the first N elements of the deleted columns. Then the dynamics of the followers that correspond to the y component of the equation can be extracted as

$$\dot{y} = -Fy - Rz. \tag{4}$$

It should be noted that the selection of leaders $x_{N+j}, j = 1, \ldots, n_l$, is indifferent, and it is free to pick any agents. The subsequent analysis is effective for any selected leaders.

Definition 2: A follower subgraph $\mathcal{G}_f$ of the interconnection graph is the subgraph induced by the follower set $\mathcal{V}_f$. Similarly, A leader subgraph $\mathcal{G}_f$ is the subgraph induced by the leader set $\mathcal{V}_l$.

To formulate the problem clearly, we give the following definition for controllability under fixed topology, where each agent is interconnected to a fixed number of other agents.

Definition 3: The multi-agent system (1) is said to be controllable under leaders $x_{N+j}, j = 1, \ldots, n_l$, and fixed topology if system (4) is controllable.

IV. Supporting lemmas

Denote by $\mathcal{G}_{c_1}, \ldots, \mathcal{G}_{c_\gamma}$, the γ connected components in the follower subgraph $\mathcal{G}_f$, we give the following definition.

Definition 4: (leader-follower connected topology) A graph $\mathcal{G}$ is said to be leader-follower connected if for each connected component $\mathcal{G}_{c_i}$ in the follower subgraph $\mathcal{G}_f$, there exists a leader in the leader subgraph $\mathcal{G}_l$, so that there is an edge between this leader and a node in $\mathcal{G}_{c_i}$, $i = 1, \cdots, \gamma$.

If there is *no* leader in $\mathcal{G}_l$, so that there is an edge between the leader and a node in $\mathcal{G}_{c_i}$. We say that $\mathcal{G}_{c_i}$ is not connected to $\mathcal{G}_l$. Throughout the paper, we make the following assumption.

Assumption 1: The interconnection graph $\mathcal{G}$ is leader-follower connected.

It is worth noting that the assumption does not require the interconnection graph $\mathcal{G}$ be connected.Accordingly it is a less conservative condition than connectedness. Note that the interconnection graph being connected is a prerequisite for derivations of all the existing results on controllability, see e.g. [13], [3], [2], [10], [9], [8].

Let $L=(a_{ij})$ be the $(N+n_l)\times(N+n_l)$ Laplacian matrix of $\mathcal{G}$ associated with multi-agent systems (1). Assume that $L_{i_1,\ldots,i_\eta}$ is such a submatrix obtained by *deleting* the i_1th,..., i_ηth rows and i_1th,..., i_ηth columns of L, $i_1,\ldots,i_\eta\in\{1,\ldots,N+n_l\}$. The following is required for investigation of the controllability.

Lemma 1: Under Assumption 1, $L_{N+1,\ldots,N+n_l}$ is a positive definite $N\times N$ matrix.

Proof: Assume, without loss of generality, that $\mathcal{G}_f$ consists of γ connected components, $\mathcal{G}_{c_1},\ldots,\mathcal{G}_{c_\gamma}$, with $\{v_1,\ldots,v_{n_1}\}$, $\{v_{n_1+1},\ldots,v_{n_2}\},\ldots$, and $\{v_{n_{\gamma-1}+1},\ldots,v_N\}$ being their node sets, respectively. For the convenience of statement and the notational simplicity, the result will be proved only for the situation that the leader set contains two agents x_{N+1}, and x_{N+2}. That is to say, it will be shown that $L_{N+1,N+2}$ is positive definite. The general case can be proved in the same way.

Since the leader subgraph $\mathcal{G}_l$ is connected to $\mathcal{G}_{c_1}$, it can be assumed that there are τ_1 nodes $v_{h_{1,1}},\ldots,v_{h_{1,\tau_1}}$ in the node set of $\mathcal{G}_{c_1}$, with both $(v_{h_{1,1}},v_{N+1}),\ldots,(v_{h_{1,\tau_1}},v_{N+1})$ and $(v_{h_{1,1}},\ v_{N+2}),\ \ldots,\ (v_{h_{1,\tau_1}},v_{N+2})$ are edges in the interconnection graph $\mathcal{G}$. In other words, $v_{h_{1,1}},\ldots,v_{h_{1,\tau_1}}$ are the nodes which constitute edges with both x_{N+1} and x_{N+2}. At the same time one can assume that there are nodes $v_{i_{1,1}},\ldots,v_{i_{1,\xi_1}}$ and $v_{j_{1,1}},\ldots,v_{j_{1,\eta_1}}$ in the node set of $\mathcal{G}_{c_1}$, with $(v_{i_{1,1}},\ v_{N+1}),\ldots,(v_{i_{1,\xi_1}},v_{N+1})$ and $(v_{j_{1,1}},v_{N+2}),\ldots,(v_{j_{1,\eta_1}},v_{N+2})$ being edges, where $v_{i_{1,k}}\neq v_{j_{1,t}},\forall k=1,\ldots,\xi_1;t=1,\ldots,\eta_1$, i.e. any two nodes in the set $\{v_{i_{1,1}},\ldots,v_{i_{1,\xi_1}},v_{j_{1,1}},\ldots,v_{j_{1,\eta_1}}\}$ are not identical.

Similarly, one can assume, with respect to $\mathcal{G}_{c_s}$, $s=2,\ldots,\gamma$, that there are τ_s nodes $v_{h_{s,1}},\ \ldots,v_{h_{s,\tau_s}}$ in the node set of $\mathcal{G}_{c_s}$, with both $(v_{h_{s,1}},v_{N+1}),\ldots,(v_{h_{s,\tau_s}},v_{N+1})$ and $(v_{h_{s,1}},v_{N+2}),\ldots,(v_{h_{s,\tau_s}},v_{N+2})$ are edges. Meanwhile, it can be assumed that there are nodes $v_{i_{s,1}},\ldots,v_{i_{s,\xi_s}}$ and $v_{j_{s,1}},\ldots,v_{j_{s,\eta_s}}$ in the node set of $\mathcal{G}_{c_s}$, with $(v_{i_{s,1}},\ v_{N+1}),\ldots,(v_{i_{s,\xi_s}},v_{N+1})$ and $(v_{j_{s,1}},\ v_{N+2}),\ldots,(v_{j_{s,\eta_s}},v_{N+2})$ being edges, where $v_{i_{s,k}}\neq v_{j_{s,t}},\forall k=1,\ldots,\xi_s;t=1,\ldots,\eta_s$.

Since $L_{N+1,N+2}$ is a submatrix which is obtained by deleting the $(N+1)$-th, $(N+2)$-th rows and the $(N+1)$-th, $(N+2)$-th columns of $L(\mathcal{G})$, it follows from the definition of $L(\mathcal{G})$ that $L_{N+1,N+2}$ has the following property for $l,p=1,\ldots,N$,

$$a_{ll}+\sum\nolimits_{p\neq l}a_{lp}=\begin{cases}0, & l\neq h_{s,q},l\neq i_{s,r},l\neq j_{s,w};\\ & s=1,\ldots,\gamma;q=1,\ldots,\tau_s;\\ & r=1,\ldots,\xi_s;w=1,\ldots,\eta_s\\ 1, & l=i_{s,r}\text{ or }l=j_{s,w};\ s=1,\ldots,\gamma;\\ & r=1,\ldots,\xi_s;\ w=1,\ldots,\eta_s\\ 2, & l=h_{s,q};s=1,\ldots,\gamma;q=1,\ldots,\tau_s\end{cases}\tag{5}$$

Because $\mathcal{G}_{c_1},\ldots,\ \mathcal{G}_{c_\gamma}$ are connected components of $\mathcal{G}_f$, it can be concluded from (5) and the definition of $L_{N+1,N+2}$ that

$$\begin{aligned}x^TL_{N+1,N+2}\,x\\ =&\sum_{i,j=1,\ldots,n_1;i\neq j}(x_i-x_j)^2+\sum_{i,j=n_1+1,\ldots,n_2;i\neq j}(x_i-x_j)^2\\ &+\cdots+\sum_{i,j=n_{\gamma-1}+1,\ldots,N;i\neq j}(x_i-x_j)^2+\sum_{k=i_{s,r};s=1,\ldots,\gamma}^{r=1,\ldots,\xi_s}x_k^2\\ &+\sum_{k=j_{s,w};s=1,\ldots,\gamma}^{w=1,\ldots,\eta_s}x_k^2+\sum_{k=h_{s,q};s=1,\ldots,\gamma}^{q=1,\ldots,\tau_s}2x_k^2,\end{aligned}\tag{6}$$

where $x=[x_1,\ldots,x_N]^T$. Denote

$$\mathrm{X}\triangleq\sum_{k=i_{s,r};s=1,\ldots,\gamma}^{r=1,\ldots,\xi_s}x_k^2+\sum_{k=j_{s,w};s=1,\ldots,\gamma}^{w=1,\ldots,\eta_s}x_k^2+\sum_{\substack{k=h_{s,q}\\ s=1,\ldots,\gamma}}^{q=1,\ldots,\tau_s}2x_k^2$$

X can be written as

$$\mathrm{X}=\mathrm{X}_1+\cdots+\mathrm{X}_\gamma,$$

where

$$\mathrm{X}_s\triangleq\sum_{k=i_{s,r}}^{r=1,\ldots,\xi_s}x_k^2+\sum_{k=j_{s,w}}^{w=1,\ldots,\eta_s}x_k^2+\sum_{k=h_{s,q}}^{q=1,\ldots,\tau_s}2x_k^2,s=1,\ldots,\gamma.$$

Obviously, the sum X_s corresponds to the connected component $\mathcal{G}_{c_s}$, $s=1,\ldots,\gamma$. This is because each adding term x_k in X_s corresponds to a node of $\mathcal{G}_{c_s}$.

The sum X of the last three adding terms in the equation (6) originates from the fact that if x_k corresponds to the node v_k, and there are ϱ leaders with each one of them constituting an edge together with v_k, then the number of $x_k's$ appeared in X is also ϱ. In other words, the number of $x_k's$ in the sum X equals the number of leaders which constitute edges together with the node corresponding to x_k. For example, for $k=h_{s,q}$, the node corresponding to x_k is $v_{h_{s,q}}$. Because the number of leaders constituting edges together with $v_{h_{s,q}}$ is two, then there are two $x_k's$, i.e. $x'_{h_{s,q}}s$ included in X. On the other hand, Assumption 1 implies that for each connected component $\mathcal{G}_{c_s}$ of $\mathcal{G}_l$, there exists at least one leader which constitutes an edge together with a follower in the very connected component $\mathcal{G}_{c_s}$. This, together with the definition of X_s, shows that there is at least one adding term x_k in the sum X_s for each $s=1,\ldots,\gamma$. Hence it can be said in this sense that each X_s is not 'empty'. At the same time, it can be observed that the sth adding term in the first γ adding terms in (6) corresponds to the sth connected component $\mathcal{G}_{c_s}$. Combining these analyses with equation (6) give rise to $x^TL_{N+1,N+2}\,x\geq 0$, and $x^TL_{N+1,N+2}\,x=0\Longleftrightarrow x=0$. So, $L_{N+1,N+2}$ is positive definite. The assertion that $L_{N+1,\ldots,N+n_l}$ is positive definite can be proved in the same way. ∎

Suppose the connected component $\mathcal{G}_{c_i}$ of follower subgraph $\mathcal{G}_f$ is on the node set $\{v_{n_{i-1}+1},\ \ldots,v_{n_i}\}$, $i=1,\ldots,\gamma$, with $n_0=1,n_\gamma=N$. Denote by $S_{n_{i-1}+1,\ldots,n_i}$ the submatrix which is obtained by *selecting* the $(n_{i-1}+1)$-th,..., n_i-th rows and $(n_{i-1}+1)$-th,..., n_i-th columns of L. The following result can be derived.

Lemma 2: $L_{N+1,\ldots,N+n_l}=diag\{S_{1,\ldots,n_1},\ S_{n_1+1,\ldots,n_2},\ \ldots,S_{n_{\gamma-1}+1,\ldots,N}\}$, and under Assumption 1, each $S_{n_{i-1}+1,\ldots,n_{i+1}}$ is positive definite, $i=1,\ldots,\gamma$.

Proof: The equality $L_{N+1,\ldots,N+n_l} = diag\{S_{1,\ldots,n_1}, S_{n_1+1,\ldots,n_2}, \ldots, S_{n_{\gamma-1}+1,\ldots,N}\}$ follows from the following observations (i)-(iii):

(i) $L_{N+1,\ldots,N+n_l}$ is obtained by selecting the first N rows and N columns of L, and these rows and columns correspond to the node set $\{v_1,\ldots,v_N\}$ of the follower subgraph $\mathcal{G}_f$.

(ii) $S_{n_{i-1}+1,\ldots,n_i}$ corresponds to the connected component $\mathcal{G}_{c_i}, i = 1,\ldots,\gamma$, in the sense that $S_{n_{i-1}+1,\ldots,n_i}$ is obtained by selecting the $(n_{i-1}+1)$-th,..., n_i-th rows and $(n_{i-1}+1)$-th,..., n_i-th columns of L, and these rows and columns correspond to the node set $\{v_{n_{i-1}+1}, \ldots, v_{n_i}\}$ of $\mathcal{G}_{c_i}$.

(iii) The follower subgraph $\mathcal{G}_f$ constitutes of the connected components $\mathcal{G}_{c_1},\ldots,\mathcal{G}_{c_\gamma}$, and there are no common nodes between any two connected components $\mathcal{G}_{c_i}$ and $\mathcal{G}_{c_j}, i \neq j$.

Next, we show the second part of the result. Since each adding term x_k in X_i corresponds to a node of $\mathcal{G}_{c_i}$, it follows from the definitions of $S_{n_{i-1}+1,\ldots,n_{i+1}}$ and X_i that

$$x^T S_{n_{i-1}+1,\ldots,n_i} x = \sum_{i,j=n_{i-1}+1,\ldots,n_i; i\neq j} (x_i - x_j)^2 + \mathrm{X}_i, \tag{7}$$

where $x = [x_{n_{i-1}+1},\ldots,x_{n_i}]$. Noticing that Assumption 1 implies that for each connected component $\mathcal{G}_{c_j}$ of $\mathcal{G}_l$, there exists at least one leader which constitutes an edge with a follower in $\mathcal{G}_{c_j}$, it can be claimed that there is at least one adding term x_k in the sum X_j for each $j = 1,\ldots,\gamma$. This, together with (7), yields that $S_{n_{i-1}+1,\ldots,n_i}$ is positive definite. ■

V. MAIN RESULTS

Lemma 3: If multi-agent system (1) with fixed topology is controllable, then the interconnection graph $\mathcal{G}$ is leader-follower connected.

Proof: The proof is conducted by contradiction. Let $\mathcal{G}_{c_1},\ldots,\mathcal{G}_{c_\varrho},\ldots,\mathcal{G}_{c_\varphi},\ldots,\mathcal{G}_{c_\gamma}$ represent the γ connected components of $\mathcal{G}_f$, where $1 \le \varrho < \varphi \le \gamma$. Denote by $\{v_{n_{\varrho-1}+1},\ldots,v_{n_\varrho}\}$ and $\{v_{n_{\varphi-1}+1},\ldots,v_{n_\varphi}\}$ the node sets of $\mathcal{G}_{c_\varrho}$ and $\mathcal{G}_{c_\varphi}$, respectively, where $n_0 = 1, n_\gamma = N$. For notational simplicity, we assume that there are only two connected components $\mathcal{G}_{c_\varrho}$ and $\mathcal{G}_{c_\varphi}$ not connected to the leader subgraph $\mathcal{G}_l$. The general case can be shown in the same manner.

Since R is the $N \times n_l$ submatrix consisting of the first N elements of the deleted last n_l columns of the Laplacian matrix L, and $\mathcal{G}_{c_\varrho}$, $\mathcal{G}_{c_\varphi}$ are not connected to $\mathcal{G}_l$, R can be expressed as

$$R = \left[\widetilde{R}_1^T,\cdots,\widetilde{R}_{\varrho-1}^T,\widetilde{0}_\varrho^T,\widetilde{R}_{\varrho+1}^T,\cdots,\widetilde{0}_\varphi^T,\cdots,\widetilde{R}_\gamma^T\right]^T \tag{8}$$

with $\widetilde{R}_i : (n_i - n_{i-1}) \times n_l$, $\forall i \in \{1,\ldots,\gamma; i \neq \varrho,\varphi\}$; $\widetilde{0}_\varrho : (n_\varrho - n_{\varrho-1}) \times n_l$; and $\widetilde{0}_\varphi : (n_\varphi - n_{\varphi-1}) \times n_l$. Accordingly, it follows from Lemma 2 that matrix F can be partitioned as

$$F = diag\left\{\widetilde{F}_1,\cdots,\widetilde{F}_\varrho,\cdots,\widetilde{F}_\varphi,\cdots,\widetilde{F}_\gamma\right\} \tag{9}$$

where $\widetilde{F}_i : (n_i - n_{i-1}) \times (n_i - n_{i-1})$ plays the same role as $S_{n_{i-1}+1,\ldots,n_i}$ in Lemma 2. Denote by $\mathbb{C}$ the controllable matrix of system (4). From (8) and (9), it can be readily seen that

$$\mathbb{C} = \begin{bmatrix} -\widetilde{R}_1 & \widetilde{F}_1\widetilde{R}_1 & -\widetilde{F}_1^2\widetilde{R}_1 & \cdots & (-1)^N\widetilde{F}_1^{N-1}\widetilde{R}_1 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \widetilde{0}_\varrho & \widetilde{0}_\varrho & \widetilde{0}_\varrho & \cdots & \widetilde{0}_\varrho \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \widetilde{0}_\varphi & \widetilde{0}_\varphi & \widetilde{0}_\varphi & \cdots & \widetilde{0}_\varphi \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ -\widetilde{R}_\gamma & \widetilde{F}_\gamma\widetilde{R}_\gamma & -\widetilde{F}_\gamma^2\widetilde{R}_\gamma & \cdots & (-1)^N\widetilde{F}_\gamma^{N-1}\widetilde{R}_\gamma \end{bmatrix}.$$

As a consequence,

$$rank\,\mathbb{C} \le N - (n_\varrho - n_{\varrho-1}) - (n_\varphi - n_{\varphi-1}), \tag{10}$$

where $n_\varrho - n_{\varrho-1} \ge 1$, $n_\varphi - n_{\varphi-1} \ge 1$, as the node sets of $\mathcal{G}_{c_\varrho}$ and $\mathcal{G}_{c_\varphi}$ are both nonempty. Hence, the multi-agent system (1) is not controllable. This completes the proof. ■

The following observation is a direct consequence of the result.

Corollary 1: In case of a single leader, a necessary condition for the controllability under fixed topology is that the interconnection graph $\mathcal{G}$ is connected.

Proof: The result comes from the fact that under the circumstance of a single leader, Assumption 1 is equivalent to the connectedness of $\mathcal{G}$. ■

Corollary 2: If Assumption 1 is not satisfied, with δ connected components $\mathcal{G}_{c_{i_1}},\ldots,\mathcal{G}_{c_{i_\delta}}$ of $\mathcal{G}_f$ not connected to $\mathcal{G}_l$, where $i_1,\ldots,i_\delta \in \{1,\ldots,\gamma\}$, then the dimension of the controllable subspace is not more than $N - \sum_{j=1}^{\delta}(n_{i_j} - n_{i_{j-1}})$, where it is assumed that $\mathcal{G}_{c_{i_j}}$ is on the node set $\left\{v_{n_{i_{j-1}}+1},\ldots,v_{n_{i_j}}\right\}$.

Proof: The result can be proved in the same manner as that for (10), which is a special case of $\delta = 2$. ■

Lemma 4: If multi-agent system (1) with fixed topology is controllable, so is the matrix pair $(-\widetilde{F}_i, -\widetilde{R}_i)$, $\forall i \in \{1,\ldots,\gamma\}$.

Proof: Set $\widetilde{n}_i \triangleq n_i - n_{i-1}$, $i = 1,\ldots,\gamma$, and denote

$$\widetilde{C}_i \triangleq \left[-\widetilde{R}_i, \widetilde{F}_i\widetilde{R}_i, -\widetilde{F}_i^2\widetilde{R}_i, \cdots, (-1)^N\widetilde{F}_i^{N-1}\widetilde{R}_i,\right],$$

$$\mathbb{C}_i \triangleq \left[-\widetilde{R}_i, \widetilde{F}_i\widetilde{R}_i, -\widetilde{F}_i^2\widetilde{R}_i, \cdots, (-1)^{\widetilde{n}_i}\widetilde{F}_i^{\widetilde{n}_i-1}\widetilde{R}_i,\right].$$

It can be seen that $\mathbb{C}_i$ is the controllable matrix of systems $(-\widetilde{F}_i, -\widetilde{R}_i)$. Noticing that $\widetilde{R}_i$ is a $\widetilde{n}_i \times \widetilde{n}_i$ matrix and $\widetilde{n}_i \le N$, one can get from the Cayley-Hamilton theorem that

$$rank\,\mathbb{C}_i = rank\,\widetilde{C}_i.$$

On the other hand, it follows from the proof of Lemma 3 that the controllable matrix $\mathbb{C}$ of multi-agent systems (1) is given by

$$\mathbb{C} = \left[\widetilde{C}_1^T,\ldots,\widetilde{C}_i^T,\ldots,\widetilde{C}_\gamma^T\right]^T. \tag{11}$$

So, if the system $(-\widetilde{F}_i, -\widetilde{R}_i)$ is uncontrollable, $rank\,\mathbb{C}_i < \widetilde{n}_i$. This, together with (11), give rise to $rank\,\mathbb{C} < N$. The conclusion is then proved by contradiction. ■

Definition 5: (controllable interconnection graph) An interconnection graph $\mathcal{G}$ is said to be controllable if its corresponding multi-agent system is controllable. The eigenvalues(eigenvectors) of the matrix F introduced in (4) are said to be the eigenvalues(eigenvectors) of the interconnection graph $\mathcal{G}$.

As mentioned above, let $\mathcal{G}_{c_1}, \ldots, \mathcal{G}_{c_\gamma}$ be the γ connected components of $\mathcal{G}_f$, with $\mathcal{G}_{c_i}$ on the node set $\{v_{n_{i-1}+1}, \ldots, v_{n_i}\}$, $i = 1, \ldots, \gamma$, $n_0 = 1, n_\gamma = N$; and $\mathcal{G}_l$ on the node set $\mathcal{V}_l = \{v_{N+1}, \ldots, v_{N+n_l}\}$. Denote by $\mathcal{G}(i)$ an induced subgraph of $\mathcal{G}$, which is on the node set $\{v_{n_{i-1}+1}, \ldots, v_{n_i}, v_{N+1}, \ldots, v_{N+n_l}\}$. That is, the node set of $\mathcal{G}(i)$ is the union of the node sets of $\mathcal{G}_{c_i}$ and $\mathcal{G}_l$. It can be seen that $\mathcal{G}(i)$ is the interconnection subgraph associated with a 'smaller' multi-agent system with its follower set being $\mathcal{G}_{c_i}$ and leader set still being $\mathcal{G}_l$. Specifically, the multi-agent system is described as follows:

$$\mathfrak{M}(i): \begin{cases} y_1 \triangleq x_{n_{i-1}+1}, \ \ldots, \ y_{\tilde{n}_i} \triangleq x_{n_i}, \\ z_j \triangleq x_{N+j}, \qquad j = 1, \ldots, n_l, \end{cases} \tag{12}$$

where $\tilde{n}_i = n_i - n_{i-1}$, and the linkages between agents in the system $\mathfrak{M}(i)$ are described by the subgraph $\mathcal{G}(i)$. Accordingly, $(\widetilde{F}_i, \widetilde{R}_i)$ is the matrix pair of the induced submulti-agent system $\mathfrak{M}(i)$.

Definition 6: The multi-agent system $\mathfrak{M}(i)$ defined in (12) with its interconnection graph being $\mathcal{G}(i)$ is said to be the induced submulti-agent system of the original multi-agent system $\mathfrak{M}$.

Theorem 1: If multi-agent system (1) with fixed topology is controllable, then the interconnection graph is leader-follower connected, and each subgraph $\mathcal{G}(i)$ is controllable, $i \in \{1, \ldots, \gamma\}$; γ is the number of connected components in $\mathcal{G}_f$.

Proof: The conclusion that $\mathcal{G}_l$ is linked to $\mathcal{G}_f$ follows from Lemma 3. Since $(\widetilde{F}_i, \widetilde{R}_i)$ is the matrix pair of the induced submulti-agent system $\mathfrak{M}(i)$, by Lemma 4, $(-\widetilde{F}_i, -\widetilde{R}_i)$ is controllable, $\forall i \in \{1, \ldots, \gamma\}$. So $\mathfrak{M}(i)$, and accordingly the corresponding subgraph $\mathcal{G}(i)$ is controllable. ■

As for sufficient conditions, we have the following.

Theorem 2: The multi-agent system (1) with fixed topology is controllable under Assumption 1, if the following conditions are fulfilled:

1) The eigenvalues of each induced subgraph $\mathcal{G}(i)$ and those between any two different subgraphs $\mathcal{G}(i)$ and $\mathcal{G}(j)$, are distinct from each other, $\forall i, j \in \{1, \ldots, \gamma\}$;
2) There exists an index l such that for each subgraph $\mathcal{G}(i)$, the eigenvectors of $\mathcal{G}(i)$ are not orthogonal to the l-th linking vector of $\mathcal{G}(i)$, $i = 1, \ldots, \gamma; l \in \{1, \cdots, n_l\}$.

Proof: Set $\widetilde{B} \triangleq -R, \widetilde{H} \triangleq -F$. In the sequel, we shall calculate the controllable matrix $\mathbb{C}$ first. Since F is symmetric, it can be assumed that $\widetilde{H} \triangleq \widetilde{U}\widetilde{D}\widetilde{U}^T$ with $\widetilde{U}$ being an orthogonal matrix. The controllable matrix is thus given by

$$\mathbb{C} = \widetilde{U}\left[\overline{B}, \widetilde{D}\overline{B}, \ldots, \widetilde{D}^{N-1}\overline{B}\right], \tag{13}$$

where $\overline{B} \triangleq \widetilde{U}^T\widetilde{B}$. For simplicity of presentation, we prove the result only for $n_l = 2$, $\gamma = 2$. The general case can be verified in the same way. In this case, $\overline{B}$ can be assumed to be

$$\overline{B} = \left[\overline{B}_1^T, \overline{B}_2^T\right]^T \tag{14}$$

with $\overline{B}_1 = \left[\bar{b}_{11}, \bar{b}_{12}\right] : \tilde{n}_1 \times n_l$, $\overline{B}_2 = \left[\bar{b}_{21}, \bar{b}_{22}\right] : \tilde{n}_2 \times n_l$, where

$$\bar{b}_{i1} = \begin{bmatrix} \bar{b}_{i1}^{(1)} \\ \vdots \\ \bar{b}_{i1}^{(\tilde{n}_i)} \end{bmatrix}, \bar{b}_{i2} = \begin{bmatrix} \bar{b}_{i2}^{(1)} \\ \vdots \\ \bar{b}_{i2}^{(\tilde{n}_i)} \end{bmatrix},$$

$\tilde{n}_1 \triangleq n_1 - n_0, \tilde{n}_2 \triangleq n_2 - n_1; n_0 = 0, n_2 = N$. Denote by $\widetilde{D} \triangleq \begin{bmatrix} \widetilde{D}_1 & \\ & \widetilde{D}_2 \end{bmatrix}$, with

$$\widetilde{D}_1 \triangleq \begin{bmatrix} \bar{d}_{11} & & \\ & \ddots & \\ & & \bar{d}_{1\tilde{n}_1} \end{bmatrix}, \widetilde{D}_2 \triangleq \begin{bmatrix} \bar{d}_{21} & & \\ & \ddots & \\ & & \bar{d}_{2\tilde{n}_2} \end{bmatrix}.$$

It follows from (13) and (14) that

$$\operatorname{Im}\mathbb{C} = \operatorname{Im}\widetilde{U}\left[\overline{\Lambda}_1\Xi, \overline{\Lambda}_2\Xi\right], \tag{15}$$

where $\operatorname{Im}(\cdot)$ denotes the image space of a matrix,

$$\overline{\Lambda}_1 \triangleq diag\left\{\bar{b}_{11}^{(1)}, \cdots, \bar{b}_{11}^{(\tilde{n}_1)}, \bar{b}_{21}^{(1)}, \cdots, \bar{b}_{21}^{(\tilde{n}_2)}\right\},$$

$$\overline{\Lambda}_2 \triangleq diag\left\{\bar{b}_{12}^{(1)}, \cdots, \bar{b}_{12}^{(\tilde{n}_1)}, \bar{b}_{22}^{(1)}, \cdots, \bar{b}_{22}^{(\tilde{n}_2)}\right\},$$

$$\Xi \triangleq \begin{bmatrix} 1 & \bar{d}_{11} & \bar{d}_{11}^2 & \cdots & \bar{d}_{11}^{N-1} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & \bar{d}_{1\tilde{n}_1} & \bar{d}_{1\tilde{n}_1}^2 & \cdots & \bar{d}_{1\tilde{n}_1}^{N-1} \\ 1 & \bar{d}_{21} & \bar{d}_{21}^2 & \cdots & \bar{d}_{21}^{N-1} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & \bar{d}_{2\tilde{n}_2} & \bar{d}_{2\tilde{n}_2}^2 & \cdots & \bar{d}_{2\tilde{n}_2}^{N-1} \end{bmatrix}.$$

On the other hand, since $\gamma = 2$, matrix F and R can be partitioned, respectively, as $F = diag\{\widetilde{F}_1, \widetilde{F}_2\}$, $\widetilde{F}_1 : \tilde{n}_1 \times \tilde{n}_1; \widetilde{F}_2 : \tilde{n}_2 \times \tilde{n}_2$, and $-R = [-\widetilde{R}_1^T, -\widetilde{R}_2^T]^T = [\widetilde{B}_1^T, \widetilde{B}_2^T]^T$; $\widetilde{R}_i : \tilde{n}_i \times n_l; i = 1, 2$. Accordingly, $\widetilde{H}$ and $\widetilde{U}$ can be partitioned, respectively, as $\widetilde{H} = diag\{\widetilde{H}_1, \widetilde{H}_2\}$, with $\widetilde{H}_i = -\widetilde{F}_i, i = 1, 2$; and $\widetilde{U} = diag\{\widetilde{U}_1, \widetilde{U}_2\}$. Because $-\widetilde{F}_i = \widetilde{H}_i = \widetilde{U}_i\widetilde{D}_i\widetilde{U}_i^T$, and the matrix pair $(-\widetilde{F}_i, -\widetilde{R}_i)$ corresponds to $\mathcal{G}(i)$, it follows from Lemma 1 that all the eigenvalues of $\widetilde{F}_i$, i.e., $-\bar{d}_{i1}, \ldots, -\bar{d}_{i\tilde{n}_i}$, are nonzero.

At the same time, since $(\widetilde{F}_i, \widetilde{R}_i)$ is the matrix pair of $\mathcal{G}(i), i = 1, 2$, one can conclude from Conditions 1 that the numbers in the set $\{\bar{d}_{11}, \ldots, \bar{d}_{1\tilde{n}_1}, \bar{d}_{21}, \ldots, \bar{d}_{2\tilde{n}_1}\}$ are different from each other. As a consequence, the Vandermonde matrix Ξ is nonsingular. Noticing that $\widetilde{U}_i^T\widetilde{B}_i \triangleq \overline{B}_i = \left[\bar{b}_{i1}, \bar{b}_{i2}\right]$, $i = 1, 2$, it follows from Condition 2 that for the index $l \in \{1, 2\}$, each number in the set $\{\bar{b}_{1l}^{(1)}, \cdots, \bar{b}_{1l}^{(\tilde{n}_1)}, \bar{b}_{2l}^{(1)}, \cdots, \bar{b}_{2l}^{(\tilde{n}_2)}\}$ is nonzero. Accordingly, the diagonal matrix $\overline{\Lambda}_l$ is nonsingular. Therefore, if Conditions 1,2 are satisfied, it can be seen from (15) that $rank\,\mathbb{C} = N, k$ i.e system (1) with fixed topology is controllable. ■

Theorem 3: The multi-agent system (1) with single leader and fixed topology is controllable if and only if each induced subgraph $\mathcal{G}(i)$ is controllable, and there are no common eigenvalues between any two different subgraphs $\mathcal{G}(i)$ and $\mathcal{G}(j)$, where $i, j \in \{1, \ldots, \gamma\}$, γ is the number of connected components in $\mathcal{G}_f$.

Proof: (Sufficiency) Since the leader is single, the matrix pair of $\mathcal{G}(i)$ is denoted by $(\widetilde{F}_i, \widetilde{r}_i)$ with $\widetilde{F}_i : \widetilde{n}_i \times \widetilde{n}_i$, $\widetilde{r}_i : \widetilde{n}_i \times 1$, where $\widetilde{n}_i = n_i - n_{i-1}$ is the number of elements in the node set of $\mathcal{G}_{c_i}$, which is, by definition, the follower subgraph of $\mathcal{G}(i)$. The controllability of $\mathcal{G}(i)$ implies

$$rank\,\mathbb{C}_i = rank\left[\widetilde{b}_i, \widetilde{H}_i\widetilde{b}_i, \ldots, \widetilde{H}_i^{\widetilde{n}_i-1}\widetilde{b}_i\right] = \widetilde{n}_i, \qquad (16)$$

where $\widetilde{H}_i \triangleq -\widetilde{F}_i$; $\widetilde{b}_i \triangleq -\widetilde{r}_i$; Since $\widetilde{H}_i = \widetilde{U}_i\widetilde{D}_i\widetilde{U}_i^T$, with $\widetilde{U}_i$ being an orthogonal matrix, and

$$\left[\widetilde{b}_i, \widetilde{H}_i\widetilde{b}_i, \ldots, \widetilde{H}_i^{\widetilde{n}_i-1}\widetilde{b}_i\right] = \widetilde{U}_i\left[\overline{b}_i, \widetilde{D}_i\overline{b}_i, \ldots, \widetilde{D}_i^{\widetilde{n}_i-1}\overline{b}_i\right],$$

$\overline{b}_i \triangleq \widetilde{U}_i^T\widetilde{b}_i$, it follows from some computations that

$$\mathbb{C}_i = \widetilde{U}_i\Lambda_i\Xi_i, \qquad (17)$$

where $\overline{b}_i = [b_{i1}, \cdots, b_{i\widetilde{n}_i}]^T$, $\widetilde{D}_i = diag\{d_{i1}, \cdots, d_{i\widetilde{n}_i}\}$, and

$$\Lambda_i = diag\{b_{i1}, \cdots, b_{i\widetilde{n}_i}\}, \Xi_i = \begin{bmatrix} 1 & d_{i1} & \cdots & d_{i1}^{\widetilde{n}_i-1} \\ \vdots & \vdots & \cdots & \vdots \\ 1 & d_{i\widetilde{n}_i} & \cdots & d_{i\widetilde{n}_i}^{\widetilde{n}_i-1} \end{bmatrix}$$

Combining (16) with (17) yields

(i) $b_{ij} \neq 0, \ i = 1, \ldots, \gamma; \ j = 1, \ldots, \widetilde{n}_i$; and
(ii) $d_{ij} \neq d_{ik}$, for an arbitrary given $i \in \{1, \cdots, \gamma\}$, where $\forall j \neq k; \ j, k \in \{1, \cdots, \widetilde{n}_i\}$.

On the other hand, some computations show that the controllable matrix $\mathbb{C}$ can be expressed as

$$\mathbb{C} = \widetilde{U}\Lambda\Xi, \qquad (18)$$

where $\widetilde{U} = diag\{\widetilde{U}_1, \cdots, \widetilde{U}_\gamma\}$, $\Lambda = diag\{b_{11}, \cdots, b_{1\widetilde{n}_1}, \cdots, b_{\gamma 1}, \cdots, b_{\gamma\widetilde{n}_\gamma}\}$,

$$\Xi = \begin{bmatrix} 1 & d_{11} & \cdots & d_{11}^{N-1} \\ \cdots & \cdots & \cdots & \cdots \\ 1 & d_{1\widetilde{n}_1} & \cdots & d_{1\widetilde{n}_1}^{N-1} \\ \cdots & \cdots & \cdots & \cdots \\ 1 & d_{\gamma 1} & \cdots & d_{\gamma 1}^{N-1} \\ \cdots & \cdots & \cdots & \cdots \\ 1 & d_{\gamma\widetilde{n}_\gamma} & \cdots & d_{\gamma\widetilde{n}_\gamma}^{N-1} \end{bmatrix}.$$

By Lemma 1, the controllability of $\mathcal{G}(i)$ means $\widetilde{F}_i$ is positive definite, and consequently d_{ij} is nonzero, for $\forall\, i = 1, \cdots, \gamma; j = 1, \cdots, \widetilde{n}_i$. Since there are no common eigenvalues between any two different subgraphs $\mathcal{G}(i)$ and $\mathcal{G}(j)$, $i, j \in \{1, \ldots, \gamma\}$, it follows from (ii) that the Vandermonde matrix Ξ is nonsingular. Combing this with (i) and (18) gives rise to $rank\,\mathbb{C} = N$. That is, the system is controllable.

(necessity) The controllability of each induced subgraph comes from Theorem 1. Because a real number is an eigenvalue of $\mathcal{G}$ if and only if it is an eigenvalue of some induced subgraph $\mathcal{G}(i)$, it can be concluded that if there is a common eigenvalue between two subgraphs $\mathcal{G}(i)$ and $\mathcal{G}(j)$, the Vandermonde matrix Ξ is singular, which, due to (18), contradicts the controllability of the multi-agent system. ■

VI. CONCLUSIONS

The paper reveals a class of controllable interconnection topologies for a group of systems to be controllable by some of them acting as a leader, which is characterized by the concept of linking between the leader and follower subgraphs and some induced subgraphs $\mathcal{G}(i), i = 1, \ldots, \gamma$. The results indicate to a certain degree how the controllability of the overall interconnected systems can be affected by the structure of the interconnection topology.

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