

From Continuous-Time Design to Sampled-Data Design of Nonlinear Observers

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Abstract—In this work, a sampled-data nonlinear observer is designed using a continuous-time design coupled with an inter-sample output predictor. The proposed sampled-data observer is a hybrid system. It is shown that under certain conditions, the robustness properties of the continuous-time design are inherited by the sampled-data design, as long as the sampling period is not too large. The approach is applied to triangular globally Lipschitz systems.

I. INTRODUCTION

THE problem of designing sampled-data nonlinear observers is a very challenging problem that has attracted a lot of attention in the literature. Continuous-time nonlinear observer designs [12,18,20,25,26,27] are meant to be used only for very small sampling periods, whereas their potential “redesign” for the purpose of digital implementation, even though straightforward and popular for linear systems [5], poses significant challenges in the nonlinear case. For this reason, the main line of attack has been through the use of an exact or approximate discrete-time description of the dynamics as the starting point for observer design [3,4,7,8,9,10,11,14,19,21,22,26,28]. This is a reasonable point of view, but faces two important difficulties:

- (i) from the moment that the continuous-time system description is abandoned and is substituted by a discrete-time description, the inter-sample dynamic behavior is lost
- (ii) any errors in the sampling schedule, get transferred into errors in the discrete-time description

As a consequence, available design methods (i) do not provide an explicit estimate of the error in between two consecutive sampling times and (ii) do not account for perturbations of the sampling schedule. Moreover, due to observability issues, the magnitude of the sampling period cannot be arbitrary (see [1,26]).

Finally, optimization-based approaches for nonlinear observer design are provided in [2,13,23,24,29]. A hybrid observer design approach was recently proposed in [6], which bears similarities to the above-mentioned optimization-based approach, but the hybrid nature of their observer offers certain advantages. In the present work, our proposed sampled-data observer will also be a hybrid

system; however, it will directly emerge from a continuous-time design of a nonlinear observer.

Consider a single-output continuous-time system:

$$\begin{aligned}\dot{x} &= f(x), x \in \mathfrak{R}^n \\ y &= h(x), y \in \mathfrak{R}\end{aligned}\quad (1)$$

where $f \in C^1(\mathfrak{R}^n; \mathfrak{R}^n)$, $h \in C^2(\mathfrak{R}^n; \mathfrak{R})$ with $f(0) = 0$, $h(0) = 0$. For this system, suppose that a continuous-time observer design is available

$$\begin{aligned}\dot{z} &= F(z, y), z \in \mathfrak{R}^k \\ \hat{x} &= \Psi(z), \hat{x} \in \mathfrak{R}^n\end{aligned}\quad (2)$$

where $F \in C^1(\mathfrak{R}^k \times \mathfrak{R}; \mathfrak{R}^k)$, $\Psi \in C^1(\mathfrak{R}^k; \mathfrak{R}^n)$ with $F(0,0) = 0$, $\Psi(0) = 0$.

The question is whether this design would still be useful in the presence of sampled measurements $y(ih)$, $i=0,1,\dots$, where h is the sampling period, or more generally, at some countable set of time instants $\pi = \{\tau_i\}_{i=0}^\infty$, not necessarily uniformly spaced, but satisfying $0 < \tau_{i+1} - \tau_i \leq r$ for all $i = 0,1,\dots$ for some $r > 0$.

The present work has been motivated by the intuitive expectation that a continuous-time nonlinear observer design would still be useful in the presence of “medium-size” sampling periods, as long as special care is taken in the time-interval between measurements. Holding the most recent measurement (zero-order hold) is not the most intuitively meaningful strategy; instead, the model (1) could be used to predict the evolution of the output, up until the new measurement is received. In particular, in the present paper, we propose a sampled-data observer consisting of the continuous-time observer, coupled with an output predictor for the time interval between two consecutive measurements:

$$\begin{aligned}\dot{z}(t) &= F(z(t), w(t)), t \in [\tau_i, \tau_{i+1}) \\ \dot{w}(t) &= L_f h(\Psi(z(t))), t \in [\tau_i, \tau_{i+1}) \\ w(\tau_{i+1}) &= y(\tau_{i+1}) \\ (z(t), w(t)) &\in \mathfrak{R}^k \times \mathfrak{R} \\ \hat{x}(t) &= \Psi(z(t)), \hat{x} \in \mathfrak{R}^n\end{aligned}\quad (3)$$

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Figure 1 depicts the structure of the sampled-data observer (3) compared to the continuous-time observer (2). The sampled-data observer uses the continuous-time observer as a key ingredient, coupled with an inter-sample output predictor. The latter is initialized by the most recent measurement and integrates the rate of change of the output calculated by the model ($L_f h(x) := \nabla h(x) f(x)$ is the Lie derivative of the output map).

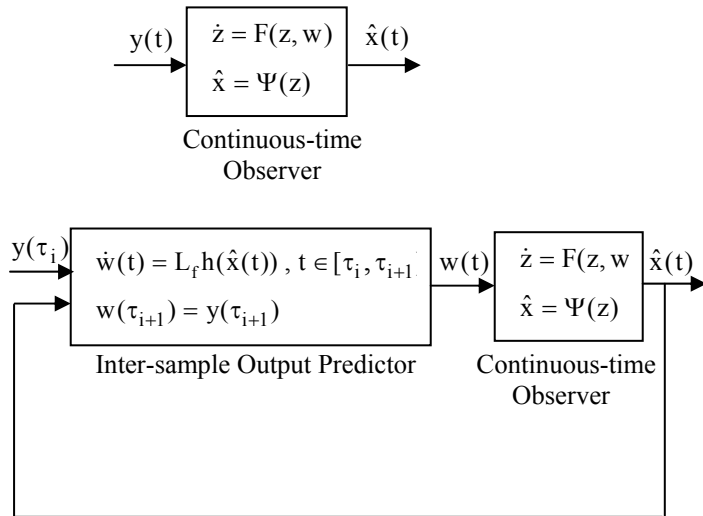


Figure 1: Continuous-time observer (2) (top) versus sampled-data observer (3) (bottom).

It is important to point out that the entire system (1) with (3) is a hybrid system, which does not satisfy the classical semigroup property. However, the weak semigroup property holds (see [15,16]) and consequently it can be analyzed using the recent results in [15,16,17].

The main result of the present paper is that the properties of the observer (2) under continuous measurement are inherited by the observer (3) under sampled measurements, as long as the sampling period is not too large.

Notations Throughout this paper we adopt the following notations:

- * Let $A \subseteq \mathfrak{R}^n$ be an open set. By $C^0(A; \Omega)$, we denote the class of continuous functions on A , which take values in $\Omega \subseteq \mathfrak{R}^k$. By $C^l(A; \Omega)$, where $l \in \{1, 2, \dots\}$, we denote the class of continuous functions on A with continuous derivatives of order l , which take values in Ω .
- * For a vector $x \in \mathfrak{R}^n$ we denote by $|x|$ its usual Euclidean norm and by x' its transpose. By $|A| := \sup\{|Ax|; x \in \mathfrak{R}^n, |x| = 1\}$ we denote the induced norm of a matrix $A \in \mathfrak{R}^{m \times n}$ and I denotes the identity matrix. By $B = \text{diag}(b_1, \dots, b_n)$ we denote the diagonal matrix $B \in \mathfrak{R}^{n \times n}$ with b_1, \dots, b_n in its diagonal.

- * By \mathfrak{R}^+ we denote the set of non-negative real numbers.
- * We denote by K^+ the class of positive C^0 functions defined on \mathfrak{R}^+ . We say that a non-decreasing continuous function $\gamma: \mathfrak{R}^+ \rightarrow \mathfrak{R}^+$ is of class \mathcal{N} if $\gamma(0) = 0$. We say that a function $\rho: \mathfrak{R}^+ \rightarrow \mathfrak{R}^+$ is positive definite if $\rho(0) = 0$ and $\rho(s) > 0$ for all $s > 0$. By K we denote the set of positive definite, increasing and continuous functions. We say that a positive definite, increasing and continuous function $\rho: \mathfrak{R}^+ \rightarrow \mathfrak{R}^+$ is of class K_∞ if $\lim_{s \rightarrow +\infty} \rho(s) = +\infty$. By KL we denote the set of all continuous functions $\sigma = \sigma(s, t): \mathfrak{R}^+ \times \mathfrak{R}^+ \rightarrow \mathfrak{R}^+$ with the properties: (i) for each $t \geq 0$ the mapping $\sigma(\cdot, t)$ is of class K ; (ii) for each $s \geq 0$, the mapping $\sigma(s, \cdot)$ is non-increasing with $\lim_{t \rightarrow +\infty} \sigma(s, t) = 0$.

- * Let $D \subseteq \mathfrak{R}^l$ be a non-empty set and $I \subseteq \mathfrak{R}^+$ an interval. By $L_{loc}^\infty(I; D)$ we denote the class of all Lebesgue measurable and locally bounded mappings $d: \mathfrak{R}^+ \rightarrow D$. Notice that by $\sup_{\tau \in [0, t]} |d(\tau)|$ we do not mean the essential supremum of $d: \mathfrak{R}^+ \rightarrow D$ on $[0, t]$ but the actual supremum of $d: \mathfrak{R}^+ \rightarrow D$ on $[0, t]$.

- * Let $f \in C^1(\mathfrak{R}^n; \mathfrak{R}^n)$, $h \in C^1(\mathfrak{R}^n; \mathfrak{R})$. By $L_f h(x) := \nabla h(x) f(x)$ we denote the Lie derivative of the function $h \in C^1(\mathfrak{R}^n; \mathfrak{R})$ along the vector field $f \in C^1(\mathfrak{R}^n; \mathfrak{R}^n)$.

II. MAIN ASSUMPTIONS AND NOTIONS

In the present work we study systems of the form (1) under the following hypotheses:

- (H)** System (1) is Robustly Forward Complete (see [15]), i.e., there exist functions $\mu \in K^+$ and $a \in K_\infty$ such that for every $x_0 \in \mathfrak{R}^n$ the solution $x(t)$ of (1) with initial condition $x(0) = x_0$ satisfies

$$|x(t)| \leq \mu(t) a(|x_0|), \quad \forall t \geq 0 \quad (4)$$

The following definition of the notion of robust observer for system (1) with respect to measurement errors is crucial to the development of the main results of the present work.

Definition 2.1: Consider the following system

$$\begin{aligned} \dot{z} &= F(z, y), \quad z \in \mathfrak{R}^k \\ \hat{x} &= \Psi(z), \quad \hat{x} \in \mathfrak{R}^n \end{aligned} \quad (5)$$

where $F \in C^1(\mathfrak{R}^k \times \mathfrak{R}; \mathfrak{R}^k)$, $\Psi \in C^1(\mathfrak{R}^k; \mathfrak{R}^n)$ with $F(0,0) = 0$, $\Psi(0) = 0$. System (5) is called a **robust observer** for system (1) with respect to measurement errors, if the following conditions are met:

i) there exist functions $\sigma \in KL$, $\gamma, p \in \mathcal{N}$, $\mu \in K^+$ and $a \in K_\infty$ such that for every $(x_0, z_0) \in \mathfrak{R}^n \times \mathfrak{R}^k$ and $v \in L_{loc}^\infty(\mathfrak{R}^+; \mathfrak{R})$, the solution $(x(t), z(t))$ of

$$\begin{aligned} \dot{\hat{x}} &= f(x) \\ \dot{z} &= F(z, h(x) + v) \\ \hat{x} &= \Psi(z) \end{aligned} \quad (6)$$

with initial condition $(x(0), z(0)) = (x_0, z_0)$ corresponding to $v \in L_{loc}^\infty(\mathfrak{R}^+; \mathfrak{R})$ satisfies the following estimates:

$$|\hat{x}(t) - x(t)| \leq \sigma(|x_0| + |z_0|, t) + \sup_{0 \leq \tau \leq t} \gamma(|v(\tau)|), \quad \forall t \geq 0 \quad (7)$$

$$|z(t)| \leq \mu(t) \left[a(|x_0| + |z_0|) + \sup_{0 \leq \tau \leq t} p(|v(\tau)|) \right], \quad \forall t \geq 0 \quad (8)$$

ii) for every $x_0 \in \mathfrak{R}^n$ there exists $z_0 \in \mathfrak{R}^k$ such that the solution $(x(t), z(t))$ of (6) with initial condition $(x(0), z(0)) = (x_0, z_0)$ corresponding to $v \equiv 0$, satisfies $x(t) = \Psi(z(t))$ for all $t \geq 0$.

Remark 2.2: If system (5) is a robust observer for system (1) with respect to measurement errors, then system (6) with output $Y = \Psi(z) - x$ satisfies the Uniform Input-to-Output Stability property from the input $v \in L_{loc}^\infty(\mathfrak{R}^+; \mathfrak{R})$ with gain $\gamma \in \mathcal{N}$ (see [17]).

We next define the corresponding notion of robust sampled-data observer. Notice that contrary to usual observers for which the output signal $y(t)$ of system (1) is available on-line, a sampled-data observer uses only the output values $y(\tau_i)$ at certain time instances $\pi = \{\tau_i\}_{i=0}^\infty$ with $0 < \tau_{i+1} - \tau_i \leq r$ for all $i = 0, 1, \dots$. The number $r > 0$ is called the *upper diameter* of the sampling partition.

Definition 2.3: The system

$$\begin{aligned} \dot{z}(t) &= g(z(t), z(\tau_i), y(\tau_i)), \quad t \in [\tau_i, \tau_{i+1}) \\ z(\tau_{i+1}) &= G \left(\lim_{t \rightarrow \tau_{i+1}^-} z(t), y(\tau_{i+1}) \right) \\ \hat{x}(t) &= \Psi(z(t)) \end{aligned} \quad (9)$$

where $g \in C^1(\mathfrak{R}^k \times \mathfrak{R}^k \times \mathfrak{R}; \mathfrak{R}^k)$, $G \in C^0(\mathfrak{R}^k \times \mathfrak{R}; \mathfrak{R}^k)$, $\Psi \in C^1(\mathfrak{R}^k; \mathfrak{R}^n)$ with $g(0,0,0) = 0$, $G(0,0) = 0$, $\Psi(0) = 0$, is called a **robust sampled-data observer** for (1) with respect to measurement errors, if the following conditions are met:

i) there exist functions $\sigma \in KL$, $\gamma, p \in \mathcal{N}$, $\mu \in K^+$ and $a \in K_\infty$ such that for every $(x_0, z_0, d, v) \in \mathfrak{R}^n \times \mathfrak{R}^k \times L_{loc}^\infty(\mathfrak{R}^+; \mathfrak{R}^+) \times L_{loc}^\infty(\mathfrak{R}^+; \mathfrak{R})$, the solution $(x(t), z(t))$ of

$$\begin{aligned} \dot{x}(t) &= f(x(t)) \\ \dot{z}(t) &= g(z(t), z(\tau_i), y(\tau_i) + v(\tau_i)), \quad t \in [\tau_i, \tau_{i+1}) \\ z(\tau_{i+1}) &= G \left(\lim_{t \rightarrow \tau_{i+1}^-} z(t), y(\tau_{i+1}) + v(\tau_{i+1}) \right) \\ \tau_{i+1} &= \tau_i + r \exp(-d(\tau_i)) \\ \hat{x}(t) &= \Psi(z(t)) \end{aligned} \quad (10)$$

with initial condition $(x(0), z(0)) = (x_0, z_0)$ corresponding to $d \in L_{loc}^\infty(\mathfrak{R}^+; \mathfrak{R}^+)$, $v \in L_{loc}^\infty(\mathfrak{R}^+; \mathfrak{R})$ satisfies the following estimates:

$$|\hat{x}(t) - x(t)| \leq \sigma(|x_0| + |z_0|, t) + \sup_{0 \leq \tau \leq t} \gamma(|v(\tau)|), \quad \forall t \geq 0 \quad (11)$$

$$|z(t)| \leq \mu(t) \left[a(|x_0| + |z_0|) + \sup_{0 \leq \tau \leq t} p(|v(\tau)|) \right], \quad \forall t \geq 0 \quad (12)$$

ii) for every $x_0 \in \mathfrak{R}^n$ there exists $z_0 \in \mathfrak{R}^k$ such that for all $d \in L_{loc}^\infty(\mathfrak{R}^+; \mathfrak{R}^+)$ the solution $(x(t), z(t))$ of (10) with initial condition $(x(0), z(0)) = (x_0, z_0)$ corresponding to $d \in L_{loc}^\infty(\mathfrak{R}^+; \mathfrak{R}^+)$ and $v \equiv 0$, satisfies $x(t) = \Psi(z(t))$ for all $t \geq 0$.

Remark 2.4: For each $(t_0, x_0, z_0) \in \mathfrak{R}^+ \times \mathfrak{R}^n \times \mathfrak{R}^k$ and for each $d \in L_{loc}^\infty(\mathfrak{R}^+; \mathfrak{R}^+)$, $v \in L_{loc}^\infty(\mathfrak{R}^+; \mathfrak{R})$ the solution $(x(t), z(t))$ of (10) with initial condition $(x(0), z(0)) = (x_0, z_0)$ corresponding to $d \in L_{loc}^\infty(\mathfrak{R}^+; \mathfrak{R}^+)$, $v \in L_{loc}^\infty(\mathfrak{R}^+; \mathfrak{R})$ is produced by the following algorithm:

Step i :

- 1) Given τ_i and $d \in L_{loc}^\infty(\mathfrak{R}^+; \mathfrak{R}^+)$, calculate τ_{i+1} using the equation $\tau_{i+1} = \tau_i + r \exp(-d(\tau_i))$,
- 2) Compute the state trajectory $(x(t), z(t))$, $t \in [\tau_i, \tau_{i+1})$ as the solution of the differential equation $\dot{x}(t) = f(x(t))$ and $\dot{z}(t) = g(z(t), z(\tau_i), h(x(\tau_i)) + v(\tau_i))$,
- 3) Calculate $z(\tau_{i+1})$ using the equation $z(\tau_{i+1}) = G \left(\lim_{t \rightarrow \tau_{i+1}^-} z(t), h(x(\tau_{i+1})) + v(\tau_{i+1}) \right)$.

For $i = 0$ we take $\tau_0 = t_0$ and $x(\tau_0) = x_0$ (initial condition). Hybrid systems of the form (10) were studied in [15,16,17], where the weak semigroup property for such systems was exploited. Taking into account hypothesis (H)

for system (1), regularity properties of the right hand-sides of (9) and using the results of [15], we may conclude that

(i) system (10) has the **“Boundedness-Implies-Continuation” (BIC)** property, i.e., for each $(t_0, x_0, z_0, d, v) \in \mathfrak{R}^+ \times \mathfrak{R}^n \times \mathfrak{R}^k \times L_{loc}^\infty(\mathfrak{R}^+; \mathfrak{R}^+) \times L_{loc}^\infty(\mathfrak{R}^+; \mathfrak{R})$, there exists $t_{\max} \in (t_0, +\infty]$ such that the solution $(x(t), z(t))$ of (10) with initial condition $(x(t_0), z(t_0)) = (x_0, z_0)$ corresponding to $d \in L_{loc}^\infty(\mathfrak{R}^+; \mathfrak{R}^+)$, $v \in L_{loc}^\infty(\mathfrak{R}^+; \mathfrak{R})$ exists for all $t \in [t_0, t_{\max})$. In addition, if $t_{\max} < +\infty$ then for every $C > 0$ there exists $t \in [t_0, t_{\max})$ with $|z(t)| > C$.

(ii) $0 \in \mathfrak{R}^n \times \mathfrak{R}^k$ is a **robust equilibrium point** from the input $(d, v) \in L_{loc}^\infty(\mathfrak{R}^+; \mathfrak{R}^+) \times L_{loc}^\infty(\mathfrak{R}^+; \mathfrak{R})$, i.e., for every $\varepsilon > 0$, $T \in \mathfrak{R}^+$ there exists $\delta := \delta(\varepsilon, T) > 0$ such that for all $(t_0, x_0, z_0, d, v) \in \mathfrak{R}^+ \times \mathfrak{R}^n \times \mathfrak{R}^k \times L_{loc}^\infty(\mathfrak{R}^+; \mathfrak{R}^+) \times L_{loc}^\infty(\mathfrak{R}^+; \mathfrak{R})$, with $|x_0| + |z_0| + \sup_{t \geq 0} d(t) + \sup_{t \geq 0} |v(t)| < \delta$ it holds that the solution $(x(t), z(t))$ of (10) with initial condition $(x(t_0), z(t_0)) = (x_0, z_0)$ corresponding to $(d, v) \in L_{loc}^\infty(\mathfrak{R}^+; \mathfrak{R}^+) \times L_{loc}^\infty(\mathfrak{R}^+; \mathfrak{R})$ exists for all $t \in [t_0, t_0 + T]$ and $\sup\{|(x(t), z(t))|; t \in [t_0, t_0 + T], t_0 \in [0, T]\} < \varepsilon$

(iii) system (10) is **autonomous**, i.e., for each $(t_0, x_0, z_0, d, v) \in \mathfrak{R}^+ \times \mathfrak{R}^n \times \mathfrak{R}^k \times L_{loc}^\infty(\mathfrak{R}^+; \mathfrak{R}^+) \times L_{loc}^\infty(\mathfrak{R}^+; \mathfrak{R})$, $t \geq t_0$ and for each $\theta \in (-\infty, t_0]$ it holds that the solution $(x(t), z(t))$ of (10) with initial condition $(x(t_0), z(t_0)) = (x_0, z_0)$ corresponding to $d \in L_{loc}^\infty(\mathfrak{R}^+; \mathfrak{R}^+)$, $v \in L_{loc}^\infty(\mathfrak{R}^+; \mathfrak{R})$ coincides with $(\tilde{x}(t - \theta), \tilde{z}(t - \theta))$ where $(\tilde{x}(t), \tilde{z}(t))$ is the solution of (10) with initial condition $(\tilde{x}(t_0 - \theta), \tilde{z}(t_0 - \theta)) = (x_0, z_0)$ corresponding to $P_\theta d \in L_{loc}^\infty(\mathfrak{R}^+; \mathfrak{R}^+)$ and $P_\theta v \in L_{loc}^\infty(\mathfrak{R}^+; \mathfrak{R})$, where $(P_\theta d)(t) = d(t + \theta)$ and $(P_\theta v)(t) = v(t + \theta)$ for all $t + \theta \geq 0$.

Remark 2.5: The reader should notice that the sampling period is allowed to be time-varying. The factor $\exp(-d(\tau_i)) \leq 1$, with $d(t) \geq 0$ some non-negative function of time, is an uncertainty factor in the end-point of the sampling interval. Proving stability for any non-negative input $d \in L_{loc}^\infty(\mathfrak{R}^+; \mathfrak{R}^+)$ will guarantee stability for all sampling schedules with $\tau_{i+1} - \tau_i \leq r$ (robustness to perturbations of the sampling schedule). To understand the importance of robustness to perturbations of the sampling schedule, consider the following situation. Suppose that hardware limitations restrict the sampling period to be $1s$. If we manage to design a sampled-data observer with $r \geq 2s$, then the application of the sampled-data observer will

guarantee convergence of the state estimates even if we “miss measurements” or if we have “delayed measurements” (for example, due to improper operation of the sensor). In such a case robustness to perturbations of the sampling schedule becomes critical. The introduction of the factor $\exp(-d(\tau_i)) \leq 1$ is a mathematical way of introducing perturbations to the sampling schedule; however, it is not unique. Other ways of introducing perturbations of the sampling schedule can be considered.

III. MAIN RESULTS

We are now in a position to state our main result.

Theorem 3.1: Consider system (1) under hypothesis (H) and suppose that system (5) is a robust observer for system (1) with respect to measurement errors. Moreover, suppose that there exists a constant $K \geq 0$ and a function $\bar{\sigma} \in KL$ such that for every $(x_0, z_0) \in \mathfrak{R}^n \times \mathfrak{R}^k$ and $v \in L_{loc}^\infty(\mathfrak{R}^+; \mathfrak{R})$, the solution $(x(t), z(t))$ of (6) satisfies the following estimate for all $t \geq 0$:

$$\begin{aligned} & |L_f h(\Psi(z(t))) - L_f h(x(t))| \\ & \leq \bar{\sigma}(|x_0| + |z_0|, t) + K \sup_{0 \leq \tau \leq t} |v(\tau)| \end{aligned} \quad (13)$$

Finally, suppose that $rK < 1$, where $r > 0$ is the upper diameter of the sampling partition and $K \geq 0$ is the constant involved in estimate (13). Then (3) is a robust sampled-data observer for system (1) with respect to measurement errors.

Remark 3.2: The reader should notice the structural differences between the continuous time observer (5) and the sampled-data observer (3), which are shown in Figure 1. The sampled-data observer uses the estimate of the state $\hat{x}(t)$ and the measurement $y(\tau_i)$ in order to generate an additional signal $w(t)$: the signal $w(t)$ will approximate the output signal $y(t)$ and actually replaces the output signal $y(t)$ in the observer.

The proof of Theorem 3.1 utilizes the recent Small-Gain Theorem for hybrid systems in [17] and is omitted due to space limitations.

IV. APPLICATIONS

In this section we present the application of Theorem 3.1 to triangular globally Lipschitz systems, leading to concrete sampled-data observer designs.

Consider the system

$$\begin{aligned}
\dot{x}_1 &= f_1(x_1) + x_2 \\
&\vdots \\
\dot{x}_{n-1} &= f_{n-1}(x_1, \dots, x_{n-1}) + x_n \\
\dot{x}_n &= f_n(x_1, \dots, x_n) \\
y &= x_1
\end{aligned} \tag{14}$$

where $f_i : \mathfrak{R}^i \rightarrow \mathfrak{R}$ ($i=1, \dots, n$) with $f_i(0) = 0$ ($i=1, \dots, n$) are globally Lipschitz functions, i.e., there exists a constant $L \geq 0$ such that the following inequalities hold for $i=1, \dots, n$:

$$\begin{aligned}
|f_i(x_1, \dots, x_i) - f_i(z_1, \dots, z_i)| &\leq L|(x_1 - z_1, \dots, x_i - z_i)|, \\
\forall (x_1, \dots, x_i) \in \mathfrak{R}^i, \forall (z_1, \dots, z_i) \in \mathfrak{R}^i
\end{aligned} \tag{15}$$

The reader should notice that all linear observable systems can be written in the form (14) with $f_i : \mathfrak{R}^i \rightarrow \mathfrak{R}$ ($i=1, \dots, n$) being linear functions. Notice that systems of the form (14) are Robustly Forward Complete and satisfy hypothesis (H), since for every $x_0 \in \mathfrak{R}^n$ the solution of (14) with initial condition $x(0) = x_0$ satisfies the estimate:

$$|x(t)| \leq \exp(ct)|x_0|, \quad \forall t \geq 0 \tag{16}$$

where $c := nL + n - 1$. Inequality (16) is obtained by evaluating the derivative of the function $W(x) = \frac{1}{2}|x|^2$ along the solutions of (14) and using inequalities (15).

A high-gain observer design is described in [12]: first a vector $k = (k_1, \dots, k_n)' \in \mathfrak{R}^n$ is found so that the matrix $(A + kc') \in \mathfrak{R}^{n \times n}$ is Hurwitz, where $c := (1, 0, \dots, 0)' \in \mathfrak{R}^n$ and $A \in \mathfrak{R}^{n \times n}$ is the matrix $A = \{a_{i,j} : i=1, \dots, n, j=1, \dots, n\}$ with $a_{i,i+1} = 1$ for $i=1, \dots, n-1$ and $a_{i,j} = 0$ if otherwise. The existence of the required vector $k = (k_1, \dots, k_n)' \in \mathfrak{R}^n$ is guaranteed since the pair of matrices (A, c) is observable. The proposed observer is of the form:

$$\begin{aligned}
\dot{z}_i &= f_i(z_1, \dots, z_i) + z_{i+1} + \theta^i k_i (c'z - y), \quad i=1, \dots, n-1 \\
\dot{z}_n &= f_n(z_1, \dots, z_n) + \theta^n k_n (c'z - y) \\
\hat{x} &= z \\
z &= (z_1, \dots, z_n)' \in \mathfrak{R}^n
\end{aligned} \tag{17}$$

where $\theta \geq 1$ is a constant sufficiently large. The proof is based on the quadratic error Lyapunov function $V(e) := e' \Delta_\theta^{-1} P \Delta_\theta^{-1} e$, where $e := z - x$, $\Delta_\theta := \text{diag}(\theta, \theta^2, \dots, \theta^n)$ and $P \in \mathfrak{R}^{n \times n}$ is a symmetric positive definite matrix that satisfies $P(A + kc') + (A' + ck')P + 2\mu I \leq 0$ for certain constant

$\mu > 0$ (see [12] for details). After some computations, we guarantee that for all $(x_0, z_0, v) \in \mathfrak{R}^n \times \mathfrak{R}^n \times L_{loc}^\infty(\mathfrak{R}^+; \mathfrak{R})$

and $\theta \geq \max\left(1, \frac{2|P|L\sqrt{n}}{\mu}\right)$ the solution of (14) with

$$\begin{aligned}
\dot{z}_i &= f_i(z_1, \dots, z_i) + z_{i+1} + \theta^i k_i (c'z - c'x + v), \quad i=1, \dots, n-1 \\
\dot{z}_n &= f_n(z_1, \dots, z_n) + \theta^n k_n (c'z - c'x + v) \\
\hat{x} &= z \\
z &= (z_1, \dots, z_n)' \in \mathfrak{R}^n
\end{aligned} \tag{18}$$

and initial condition $(x(0), z(0)) = (x_0, z_0) \in \mathfrak{R}^n \times \mathfrak{R}^n$ corresponding to $v \in L_{loc}^\infty(\mathfrak{R}^+; \mathfrak{R})$ satisfies the following estimates for all $t \geq 0$ and $i=1, \dots, n$:

$$|\hat{x}(t) - x(t)| \leq \theta^{n-1} \sqrt{\frac{K_2}{K_1}} \exp\left(-\frac{\theta\mu}{4|P|}\right) |z_0 - x_0| \tag{19}$$

$$+ 2\theta^{n-1} \frac{|P||k|}{\mu} \sqrt{\frac{K_2}{K_1}} \sup_{0 \leq \tau \leq t} |v(\tau)|$$

$$|z_i(t) - x_i(t)| \leq \theta^{i-1} \sqrt{\frac{K_2}{K_1}} \exp\left(-\frac{\theta\mu}{4|P|}\right) |z_0 - x_0| \tag{20}$$

$$+ 2\theta^{i-1} \frac{|P||k|}{\mu} \sqrt{\frac{K_2}{K_1}} \sup_{0 \leq \tau \leq t} |v(\tau)|$$

where $K_1, K_2 > 0$ are constants such that $K_1|x|^2 \leq x'Px \leq K_2|x|^2$ for all $x \in \mathfrak{R}^n$. It follows from (19) and (16) that system (17) is a robust observer for system (14) with respect to measurement errors. Moreover, using inequality (15) for $i=1$ and (20) we obtain that for all $(x_0, z_0, v) \in \mathfrak{R}^n \times \mathfrak{R}^n \times L_{loc}^\infty(\mathfrak{R}^+; \mathfrak{R})$ the solution of (14) with (18) and initial condition $(x(0), z(0)) = (x_0, z_0) \in \mathfrak{R}^n \times \mathfrak{R}^n$ corresponding to $v \in L_{loc}^\infty(\mathfrak{R}^+; \mathfrak{R})$ satisfies the estimate for all $t \geq 0$:

$$\begin{aligned}
&|f_1(z_1(t)) + z_2(t) - f_1(x_1(t)) - x_2(t)| \\
&\leq (L + \theta) \sqrt{\frac{K_2}{K_1}} \exp\left(-\frac{\theta\mu}{4|P|}\right) |z_0 - x_0| \\
&+ 2(L + \theta) \frac{|P||k|}{\mu} \sqrt{\frac{K_2}{K_1}} \sup_{0 \leq \tau \leq t} |v(\tau)|
\end{aligned} \tag{21}$$

It follows from Theorem 3.1 and inequality (21) that the following system

$$\begin{aligned}
\dot{z}_i(t) &= f_i(z_1(t), \dots, z_i(t)) + z_{i+1}(t) + \theta^i k_i (c'z(t) - w(t)), \quad i=1, \dots, n-1 \\
\dot{z}_n(t) &= f_n(z_1(t), \dots, z_n(t)) + \theta^n k_n (c'z(t) - w(t)) \\
\dot{w}(t) &= f_1(z_1(t)) + z_2(t), \quad t \in [\tau_i, \tau_{i+1}) \\
w(\tau_{i+1}) &= y(\tau_{i+1}) \\
\hat{x} &= z \\
z &= (z_1, \dots, z_n)' \in \mathfrak{R}^n
\end{aligned} \tag{22}$$

is a robust sampled-data observer for system (14) with respect to measurement errors provided that the upper diameter of the sampling partition $r > 0$ satisfies the inequality:

$$2r(L + \theta) \frac{|P|k|}{\mu} \sqrt{\frac{K_2}{K_1}} < 1 \tag{23}$$

Notice that since $\theta \geq \max\left(1, \frac{2|P|L\sqrt{n}}{\mu}\right)$, it follows from

(23) that the upper diameter of the sampling partition $r > 0$ must necessarily be less than

$$\frac{\mu^2}{2|P|k|(L\mu + \max(\mu, 2|P|L\sqrt{n}))} \sqrt{\frac{K_1}{K_2}}$$

V. CONCLUDING REMARKS

The present work developed a design method for nonlinear sampled-data observers based on an available continuous-time design, coupled with an inter-sample output predictor.

In addition to being intuitively meaningful, key attractive features of the proposed sampled-data observer include that

- 1) it provides checkable sufficient conditions for robustness with respect to measurement errors.
- 2) it provides an explicit formula for estimating the maximum allowable sampling period.
- 3) it provides explicit bounds for the estimation error between sampling instants.
- 4) it provides robustness with respect to perturbations of the sampling schedule.

REFERENCES

- [1] Aeyels, D., "On the Number of Samples Necessary to Achieve Observability", *Systems and Control Letters*, 1(2), 1981, 92-94.
- [2] Alamir, M., "Optimization Based Nonlinear Observers Revisited", *International Journal of Control*, 72(13), 1999, 1204-1217.
- [3] Alessandri, A., M. Baglietto, T. Parisini and R. Zoppoli, "A Neural State Estimator with Bounded Errors for Nonlinear Systems", *IEEE Transactions on Automatic Control*, 44(11), 1999, 2028-2042.
- [4] Arcak, M. and D. Nesić, "A Framework for Nonlinear Sampled-Data Observer Design via Approximate Discrete-Time Models and Emulation", *Automatica*, 40, 2004, 1931-1938.
- [5] Astrom, K. J. and B. Wittenmark, "Computer-Controlled Systems, Theory and Design", Prentice Hall, New Jersey, 1997.
- [6] Biyik, E. and M. Arcak, "A Hybrid Redesign of Newton Observers in the Absence of an Exact Discrete-Time Model", *Systems and Control Letters*, 55, 2006, 429-436.
- [7] Besancon, G., H. Hammouri and S. Benamor, "State Equivalence of Discrete-Time Nonlinear Control Systems to State Affine Form up to Input/Output Injection", *Systems and Control Letters*, 33, 1998, 1-10.
- [8] Boutayeb, M. and M. Darouach, "A Reduced-Order Observer for Nonlinear Discrete-Time Systems", *Systems and Control Letters*, 39, 2000, 141-151.
- [9] Califano, C., S. Monaco and D. Normand-Cyrot, "On the Observer Design in Discrete-Time", *Systems and Control Letters*, 49, 2003, 255-265.
- [10] Ciccarela, G., M. Dalla Mora and A. Germani, "A Robust Observer for Discrete-Time Nonlinear Systems", *Systems and Control Letters*, 24, 1995, 291-300.
- [11] El Assoudi, A., E. H. El Yaagoubi and H. Hammouri, "Nonlinear Observer Based on the Euler Discretization", *International Journal of Control*, 75(11), 2002, 784-791.
- [12] Gauthier, J. P. and I. Kupka, "Deterministic Observation Theory and Applications", Cambridge University Press, 2001.
- [13] Kang, W., "Moving Horizon Numerical Observers of Nonlinear Control Systems", *IEEE Transactions on Automatic Control*, 51(2), 2006, 344-350.
- [14] Karafyllis, I. and C. Kravaris, "On the Observer Problem for Discrete-Time Control Systems", *IEEE Transactions on Automatic Control*, 52(1), 2007, 12-25.
- [15] Karafyllis, I., "A System-Theoretic Framework for a Wide Class of Systems I: Applications to Numerical Analysis", *Journal of Mathematical Analysis and Applications*, 328(2), 2007, 876-899.
- [16] Karafyllis, I., "A System-Theoretic Framework for a Wide Class of Systems II: Input-to-Output Stability", *Journal of Mathematical Analysis and Applications*, 328(1), 2007, 466-486.
- [17] Karafyllis, I. and Z.-P. Jiang, "A Small-Gain Theorem for a Wide Class of Feedback Systems with Control Applications", *SIAM Journal Control and Optimization*, 46(4), 2007, 1483-1517.
- [18] Kazantzis, N. and C. Kravaris, "Nonlinear Observer Design Using Lyapunov's Auxiliary Theorem", *Systems and Control Letters*, 34, 1998, 241-247.
- [19] Kazantzis, N. and C. Kravaris, "Discrete-Time Nonlinear Observer Design Using Functional Equations", *Systems and Control Letters*, 42, 2001, 81-94.
- [20] Krener, A.J. and A. Isidori, "Linearization by Output Injection and Nonlinear Observers", *Systems and Control Letters*, 3, 1983, 47-52.
- [21] Lee, W. and K. Nam, "Observer Design for Autonomous Discrete-Time Nonlinear Systems", *Systems and Control Letters*, 17, 1991, 49-58.
- [22] Lin, W. and C. I. Byrnes, "Remarks On Linearization of Discrete-Time Autonomous Systems and Nonlinear Observer Design", *Systems and Control Letters*, 25, 1995, 31-40.
- [23] Moraal, P.E. and J.W. Grizzle, "Observer Design for Nonlinear Systems with Discrete-Time Measurements", *IEEE Transactions on Automatic Control*, 40, 1995, 395-404.
- [24] Rao, C. V., J. B. Rawlings and D. Q. Mayne, "Constrained State Estimation for Nonlinear Discrete-Time Systems: Stability and Moving-Horizon Approximations", *IEEE Transactions on Automatic Control*, 48(2), 2003, 246-258.
- [25] Respondek, W., A. Pogromsky and H. Nijmeijer, "Time Scaling for Observer Design with Linearizable Error Dynamics", *Automatica*, 40, 2004, 277-285.
- [26] Sontag, E.D., "Mathematical Control Theory", 2nd Edition, Springer-Verlag, New York, 1998.
- [27] Tsiniias, J., "Further Results on the Observer Design Problem", *Systems and Control Letters*, 14, 1990, 411-418.
- [28] Xiao, M., N. Kazantzis, C. Kravaris and A.J. Krener, "Nonlinear Discrete-Time Observer Design with Linearizable Error Dynamics", *IEEE Transactions on Automatic Control*, 48(4), 2003, 622-626.
- [29] Zimmer, G., "State Observation by on-line Minimization", *International Journal of Control*, 60(4), 1994, 595-606.