Proceedings of the
47th IEEE Conference on Decision and Control
Cancun, Mexico, Dec. 9-11, 2008

Model Reference Adaptive H_∞ Control for Flexible Arms by Finite Dimensional Controllers

Yoshihiko Miyasato

***Abstract*— The problem of constructing model reference adaptive H_∞ control for flexible arms is considered in this manuscript. Control schemes of flexible arms are mixed parameter systems composed of distributed parameter systems of hyperbolic type (flexible arms) and lumped parameter systems (motor control systems). Owing to infinite dimensional modes of distributed parameter systems, control of those complex systems is a difficult problem. A stabilizing control signal is added to regulate the effect of infinite dimensional modes, and it is derived as a solution of certain H_∞ control problem where the effect of infinite dimensional modes are considered as external disturbances to the process.**

I. INTRODUCTION

Model reference adaptive control (MRAC) problems have been studied based on stability analysis via Lyapunov functions [1], [2], and so much attention has not been paid on control performance such as optimal property or transient performance. Recently, stable controller designs for nonlinear and adaptive control systems are investigated from the view point of inverse optimality [3], [4]. In those research works, the resulting control systems are shown to be optimal to certain meaningful cost functionals, and stability of the overall systems is also assured. Those approaches are extended to the design of inverse optimal H_∞ adaptive control systems, and various adaptive control systems are derived from those strategies together with additional control performances such as robustness to uncertain time-varying elements of system parameters [5], [6].

In the present manuscript, those approaches are applied to the problem of constructing model reference adaptive H_∞ control for flexible arms based on the notion of inverse optimality. Control schemes of flexible arms are mixed parameter systems composed of distributed parameter systems (DPS) of hyperbolic type (flexible arms) and lumped parameter systems (motor control systems) [7], [8]. Owing to infinite dimensional modes of distributed parameter systems, control of those complex systems is a difficult problem. The proposed control strategy is composed of finite dimensional compensators, and is derived as a solution of certain H_∞ control problem where spill-overs are regarded as external disturbance to the process. Among several research works in the field of adaptive control for DPS [9]–[18], the proposed methodology provides systematic procedures of constructing finite dimensional adaptive controllers for several kinds of DPS. Especially, the present paper shows one design tool of those approaches; that is, a design of finite dimensional adaptive controllers for flexible arms.

Y. Miyasato is with Department of Mathematical Analysis and Statistical Inference, The Institute of Statistical Mathematics, 106-8569 Tokyo, Japan miyasato@ism.ac.jp

II. PROBLEM STATEMENT

Let Ω be a bounded open domain in a finite dimensional Euclidian space, and let $L^2(\Omega)$ denote the Hilbert space of all square integrable functions with the inner product

$$(u, v) = \int_\Omega u(x)v^*(x)dx, \tag{1}$$

where v^* is a complex conjugate of v. We consider a uniform flexible arm of finite length, one end of which (the origin) is attached to the shaft of a control motor that rotates the flexible arm in the horizontal plane. It is described as a single-input, single-output distributed parameter system of hyperbolic type in $L^2(\Omega)$ as follows [7], [8]:

$$\frac{d^2}{dt^2}u(t) + 2\alpha A\frac{d}{dt}u(t) + Au(t) = b\,\ddot{\theta}(t), \tag{2}$$

$$y(t) = (c,\, u(t)) \equiv Cu(t), \tag{3}$$

where $u(t)$ $(\in\ L^2(\Omega))$ is a state (the distributed bending displacement of the arm) and $y(t)$ (an output) is a scalar function on $t \in [0, \infty)$, and is a spatial average of the bending displacement. A is an unbounded operator, and b and c $(\in\ L^2(\Omega))$ are an input influence function and a sensor influence function, respectively. α is a small damping constant $(0 < \alpha \ll 1)$. Ω corresponds to the domain of the flexible arm itself. $\theta(t)$ (a scalar function on $t \in [0, \infty)$) is a rotation angle of the motor, and the motor-driving term is represented by the following equations:

$$\bar{\theta}(t) = \theta(t) - \theta_d(t), \tag{4}$$

$$\frac{d}{dt}\bar{\theta}(t) = \dot{\theta}(t) - \dot{\theta}_d(t), \tag{5}$$

$$\frac{d^2}{dt^2}\theta(t) = f(t), \tag{6}$$

where $\theta_d(t)$ (a differentiable scalar function on $t \in [0, \infty)$) is a reference rotation angle, which θ should follow, and $f(t)$ (a scalar function on $t \in [0, \infty)$) is a driving force of the motor.

We assume that the operator A is a self-adjoint, positive definite, and unbounded operator with compact resolvent whose eigenvalues λ_i

$$0 < \lambda_1 < \lambda_2 < \cdots < \lambda_i < \cdots, \quad (\lim_{i\to\infty} \lambda_i = \infty), \tag{7}$$

are simple. The domain $\mathcal{D}(A)$ is dense in $L^2(\Omega)$. The normalized eigenfunctions of A are denoted by ϕ_i such that

$$A\phi_i = \lambda_i\phi_i, \quad (i = 1, 2, \cdots). \tag{8}$$

978-1-4244-3124-3/08/$25.00 ©2008 IEEE

The set ϕ_i $(i = 1, 2, \cdots)$ forms a complete orthonormal system in $L^2(\Omega)$.

For the flexible arm (2), (3), only the output $y(t)$, and $\frac{d}{dt}y(t)$ are assumed to be measurable, but the state $u(t)$ and the systems parameters in A, b, c, and α are unknown. Furthermore, for the motor-driving term, $\theta(t)$, $\theta_d(t)$ and time derivatives of those, and $f(t)$ is assumed to be available for measurement.

The control problem of the paper is to determine a suitable control input $f(t)$ such that the output $y(t)$ of the flexible arm (spatial average of distributed bending displacement) and the tracking error of the rotation angle of the motor $\bar{\theta}(t)$ converge to a small residual region, by utilizing finite dimensional controllers.

Hereafter, by combining (2) and (6), the flexible arm is represented by

$$\frac{d^2}{dt^2}u(t) + 2\alpha A\frac{d}{dt}u(t) + Au(t) = bf(t). \tag{9}$$

Remark Although the problem statement in the paper, seems to be close to the previous study [19], convergence of $\bar{\theta}(t)$ to a small residual region is also necessary in the present situation. That is the important difference from [19].

III. MATHEMATICAL PRELIMINARY

The next assumption is introduced.

Assumption 1 *α and λ_i satisfy the following conditions.*

$$\alpha^2\lambda_i^2 - \lambda_i \neq 0, \quad (i \geq 1), \quad \alpha\lambda_1 < \frac{1}{2\alpha}. \tag{10}$$

Based on Assumption 1, $g(A)$ is defined by

$$g(\lambda) \equiv \left(\alpha^2\lambda^2 - \lambda\right)^{1/2}, \tag{11}$$

$$g(A) \equiv \sum_{i=1}^{\infty} g(\lambda_i)(\cdot,\, \phi_i)\phi_i. \tag{12}$$

Since $g(\lambda_i) \sim \alpha\lambda_i$ as $i \to \infty$, it follows that $g(A)$ is a unbounded operator and that $\mathcal{D}(g(A)) = \mathcal{D}(A)$. Furthermore, (10) shows that $g(A)^{-1}$ is a bounded operator. By utilizing $g(A)$, the solution of (9) is given by the following lemma.

Lemma 1 [7] *The next evolution equations in $L^2(\Omega)$ are considered,*

$$\frac{d}{dt}\xi(t) = A^+\xi(t) + g(A)^{-1}bf(t), \tag{13}$$

$$\frac{d}{dt}\eta(t) = A^-\eta(t) - g(A)^{-1}bf(t), \tag{14}$$

$$A^{\pm} \equiv -\alpha A \pm g(A). \tag{15}$$

Then, the unique solution $u(t)$ of (9) is described as follows:

$$u(t) = \frac{\xi(t) + \eta(t)}{2}, \tag{16}$$

where initial conditions $\xi(0)$, $\eta(0)$ of (13), (14) are determined such that

$$\xi(0) = u(0) + g(A)^{-1}\left\{\frac{d}{dt}u(0) + \alpha Au(0)\right\}, \tag{17}$$

$$\eta(0) = u(0) - g(A)^{-1}\left\{\frac{d}{dt}u(0) + \alpha Au(0)\right\}. \tag{18}$$

The operator $A^{\pm}$ have eigenfunctions ϕ_i and corresponding eigenvalues $\mu_i^{\pm}$ as follows:

$$A^{\pm}\phi_i = \mu_i^{\pm}\phi_i, \tag{19}$$

$$\mu_i^+ = -\alpha\lambda_i + g(\lambda_i), \qquad \left(\lim_{i\to\infty}\mu_i^+ = -\frac{1}{2\alpha}\right), \tag{20}$$

$$\mu_i^- = -\alpha\lambda_i - g(\lambda_i), \qquad \left(\lim_{i\to\infty}\mu_i^+ = -\infty\right). \tag{21}$$

Then, $\xi(t)$, $\eta(t)$ and $u(t)$ are rewritten into the following eigenfunction expansion forms:

$$\xi(t) = \sum_{i=1}^{\infty}\xi_i(t)\phi_i, \tag{22}$$

$$\eta(t) = \sum_{i=1}^{\infty}\eta_i(t)\phi_i, \tag{23}$$

$$\frac{d}{dt}\xi_i(t) = \mu_i^+\xi_i(t) + g(\lambda_i)^{-1}b_if(t), \tag{24}$$

$$\frac{d}{dt}\eta_i(t) = \mu_i^-\eta_i(t) - g(\lambda_i)^{-1}b_if(t), \tag{25}$$

$$u(t) = \sum_{i=1}^{\infty}\frac{\xi_i(t) + \eta_i(t)}{2}\phi_i, \tag{26}$$

where $\xi_i(t) = (\xi(t), \phi_i)$, $\eta_i(t) = (\eta(t), \phi_i)$, $b_i = (b, \phi_i)$.

IV. SYSTEM REPRESENTATION OF FLEXIBLE ARMS

In the present section, an input-output representation of the flexible arm (9), (3) is derived. First, let $\tilde{\lambda}_N\ (> 0)$ be a given damping constant. We take an integer N such that

$$0 < \tilde{\lambda}_N < -\Re(\mu_{N+1}^{\pm}), \qquad 0 < \tilde{\lambda}_N < \frac{1}{2\alpha}. \tag{27}$$

Then, it follows that

$$\tilde{\lambda}_N < -\Re(\mu_i^{\pm}), \qquad (i \geq N+1). \tag{28}$$

By utilizing N, we define orthogonal projection operators.

$$P_N \cdot\, = \sum_{i=1}^{N}(\,\cdot,\, \phi_i)\phi_i, \tag{29}$$

$$Q_N \cdot\, = (I - P_N)\cdot = \sum_{i=N+1}^{\infty}(\,\cdot,\, \phi_i)\phi_i. \tag{30}$$

Then, $u(t)$ and $y(t)$ of (9), (3) are expressed by

$$u(t) = P_Nu(t) + Q_Nu(t) \equiv u_N(t) + \tilde{u}_N(t), \tag{31}$$

$$y(t) = C\{u_N(t) + \tilde{u}_N(t)\} \equiv y_N(t) + \tilde{y}_N(t), \tag{32}$$

and those are expanded into the following forms.

$$u_N(t) = \sum_{i=1}^{N}\frac{\xi_i(t) + \eta_i(t)}{2}\phi_i, \tag{33}$$

$$y_N(t) = \sum_{i=1}^{N}\frac{\xi_i(t) + \eta_i(t)}{2}c_i, \tag{34}$$

$$\tilde{u}_N(t) = \sum_{i=N+1}^{\infty}\frac{\xi_i(t) + \eta_i(t)}{2}\phi_i, \tag{35}$$

$$\tilde{y}_N(t) = \sum_{i=N+1}^{\infty}\frac{\xi_i(t) + \eta_i(t)}{2}c_i, \tag{36}$$

where $c_i \equiv (c, \phi_i) = C\phi_i$. The controlled process (9), (3) is divided into two subsystems $[S_1]$:$(u_N(t), y_N(t))$ and $[S_2]$:$(\tilde{u}_N(t), \tilde{y}_N(t))$, respectively. $[S_1]$ is a finite dimensional $(2N)$ system, and is represented in the state space form.

$[S_1]$

$$\frac{d}{dt}\bar{u}_N(t) = \bar{A}_N\bar{u}_N(t) + \bar{b}_N f(t) \tag{37}$$

$$y_N(t) = \bar{C}_N\bar{u}_N(t), \tag{38}$$

$$u_N(t) = \bar{u}_N(t)^T\bar{\phi}_N, \tag{39}$$

where

$$\bar{u}_N(t) = [\xi_1(t),\, \eta_1(t),\, \cdots,\, \xi_N(t),\, \eta_N(t)]^T \quad (\in \mathbf{C}^{2N}), \tag{40}$$

$$\bar{\phi}_N = [\phi_1,\, \phi_1,\, \cdots,\, \phi_N,\, \phi_N]^T, \tag{41}$$

$$\bar{A}_N = \mathrm{diag}\,(\mu_1^+,\, \mu_1^-,\, \cdots,\, \mu_N^+,\, \mu_N^-)\,(\in \mathbf{C}^{2N\times 2N}), \tag{42}$$

$$\begin{aligned}\bar{b}_N = [&g(\lambda_1)^{-1}b_1,\, -g(\lambda_1)^{-1}b_1,\, \cdots,\\ &g(\lambda_N)^{-1}b_N,\, -g(\lambda_N)^{-1}b_N]^T\,(\in \mathbf{C}^{2N}),\end{aligned} \tag{43}$$

$$\bar{C}_N = [c_1/2,\, c_1/2,\, \cdots,\, c_N/2,\, c_N/2]\,(\in \mathbf{C}^{1\times 2N}). \tag{44}$$

For the subsystem $[S_1]$, we assume that

Assumption 2 *The finite dimensional subsystem $[S_1]$ $(\bar{C}_N, \bar{A}_N, \bar{b}_N)$ is completely controllable and observable, that is,*

$$c_i \neq 0, \quad b_i \neq 0, \quad (1 \leq i \leq N). \tag{45}$$

Then, on Assumption 2, we can construct a finite dimensional observer for $[S_1]$, which is denoted by $[S_1']$.

$[S_1']$

$$\frac{d}{dt}\hat{\bar{u}}_N(t) \;=\; \bar{A}_{NK}\hat{\bar{u}}_N(t) + \bar{b}_N f(t) + \bar{K}_N y_N(t), \tag{46}$$

where $\bar{A}_{NK}$ $(\in \mathbf{C}^{2N\times 2N})$ is a stable matrix defined by

$$\bar{A}_{NK} \;=\; \bar{A}_N - \bar{K}_N\bar{C}_N, \tag{47}$$

and $\bar{K}_N$ $(\in \mathbf{C}^{2N\times 1})$ is an observer gain matrix selected properly such that the following relations hold for $\lambda_f > 0$.

$$\|\exp(\bar{A}_{NK}t)\|_{\mathbf{C}^{2N}} \leq \text{const.}\exp(-\lambda_f t), \tag{48}$$

$$\|\bar{u}_N(t) - \hat{\bar{u}}_N(t)\|_{\mathbf{C}^{2N}} \sim \exp(-\lambda_f t) \to 0. \tag{49}$$

Remark Assumption 2 states that only $[S_1]$ should be controllable and observable, and that $[S_2]$ need not to be. The effect of $[S_2]$ is included in the residual term $\delta(t)$ (62) and is evaluated by Lemma 2, which will be shown hereafter.

Remark Decomposition of DPS into finite dimensional parts and infinite dimensional ones, has been discussed in many previous related works (for example, see [20]). In those works, finite dimensional parts describe primal structures of overall systems. However, in the present paper, $[S_1]$ need not to possess primal property of the process, that is, $[S_1]$ need not to be a good approximation of the overall system. Hence, N need not to be a large integer.

Now, we introduce a Hurwitz polynomial $h(s) = s^2 + a_1 s + a_2$, and get the input-output representation.

$$\begin{aligned}&\frac{d^2}{dt^2}y(t) + a_1\frac{d}{dt}y(t) + a_2 y(t)\\ &= (\bar{C}_N\bar{A}_{NK}^2 + \bar{C}_N\bar{K}_N\bar{C}_N\bar{A}_{NK} + a_1\bar{C}_N\bar{A}_{NK})\hat{\bar{u}}_N(t)\\ &\quad + \{\bar{C}_N\bar{A}_N\bar{K}_N + (\bar{C}_N\bar{K}_N)^2 + a_1\bar{C}_N\bar{K}_N + a_2\}y_N(t)\\ &\quad + \sum_{i=N+1}^{\infty}\frac{c_i}{2}\{h(\mu_i^+)\xi_i(t) + h(\mu_i^-)\eta_i(t)\}\\ &\quad + \psi_0 f(t) + \epsilon(t),\end{aligned} \tag{50}$$

where

$$\psi_0 \equiv Cb = \int_\Omega c(x)b(x)dx = \sum_{i=1}^{\infty} c_i b_i. \tag{51}$$

Hereafter, all exponentially decaying terms are denoted by $\epsilon(t)$. Here we introduce $f_f(t)$.

$$\frac{d}{dt}f_f(t) \;=\; -a_0 f_f(t) + f(t), \tag{52}$$

where a_0 is a positive constant. By utilizing $f_f(t)$, $\hat{\bar{u}}_N(t)$ is rewritten into the form

$$\begin{aligned}\hat{\bar{u}}_N(t) &= \exp\bar{A}_{NK}t\cdot\{\hat{\bar{u}}_N(0) - \bar{b}_N f_f(0)\} + \bar{b}_N f_f(t)\\ &\quad + (\bar{A}_{NK} + a_0 I)\int_0^t\{\exp\bar{A}_{NK}(t-\tau)\}\cdot\bar{b}_N f_f(\tau)\tau\\ &\quad + \int_0^t\{\exp\bar{A}_{NK}(t-\tau)\}\cdot\bar{K}_N\{y(\tau) - \tilde{y}_N(\tau)\}d\tau,\end{aligned} \tag{53}$$

where partial integration is applied. The substitution of (53) and $y_N(t) = y(t) - \tilde{y}_N(t)$ into (50) yields

$$\begin{aligned}&\frac{d^2}{dt^2}y(t) + a_1\frac{d}{dt}y(t) + a_2 y(t)\\ &= (\bar{C}_N\bar{A}_{NK}^2 + \bar{C}_N\bar{K}_N\bar{C}_N\bar{A}_{NK} + a_1\bar{C}_N\bar{A}_{NK})\cdot\\ &\quad\cdot\Big\{(\bar{A}_{NK} + a_0 I)\int_0^t\{\exp\bar{A}_{NK}(t-\tau)\}\cdot\bar{b}_N f_f(\tau)\tau\\ &\quad + \int_0^t\{\exp\bar{A}_{NK}(t-\tau)\}\cdot\bar{K}_N y(\tau)d\tau + \bar{b}_N f_f(t)\Big\}\\ &\quad + \{\bar{C}_N\bar{A}_N\bar{K}_N + (\bar{C}_N\bar{K}_N)^2 + a_1\bar{C}_N\bar{K}_N + a_2\}y(t)\\ &\quad - (\bar{C}_N\bar{A}_{NK}^2 + \bar{C}_N\bar{K}_N\bar{C}_N\bar{A}_{NK} + a_1\bar{C}_N\bar{A}_{NK})\cdot\\ &\qquad\cdot\int_0^t\{\exp\bar{A}_{NK}(t-\tau)\}\cdot\bar{K}_N\tilde{y}_N(\tau)d\tau\\ &\quad - \{\bar{C}_N\bar{A}_N\bar{K}_N + (\bar{C}_N\bar{K}_N)^2 + a_1\bar{C}_N\bar{K}_N + a_2\}\tilde{y}_N(t)\\ &\quad + \sum_{i=N+1}^{\infty}\frac{c_i}{2}\{h(\mu_i^+)\xi_i(t) + h(\mu_i^-)\eta_i(t)\}\\ &\quad + \psi_0 f(t) + \epsilon(t).\end{aligned} \tag{54}$$

Here we define finite dimensional state variable filters ($2N$ dimension) such as

$$\frac{d}{dt}\bar{v}_1(t) \;=\; \bar{F}_N\bar{v}_1(t) + \bar{g}_0 f_f(t), \tag{55}$$

$$\frac{d}{dt}\bar{v}_2(t) \;=\; \bar{F}_N\bar{v}_2(t) + \bar{g}_0 y(t), \tag{56}$$

where $(\bar{F}_N, \bar{g}_0)$ $(\bar{F}_N \in \mathbf{R}^{2N\times 2N}, \bar{g}_0 \in \mathbf{R}^{2N})$ is controllable, and $\bar{F}_N$ is chosen such that the next relation holds.

$$\det(sI - \bar{F}_N) \;=\; \det(sI - \bar{A}_{NK}). \tag{57}$$

Since $(\bar{C}_N, \bar{A}_N)$ is observable, there exists $\bar{K}_N$ satisfying (57) for an arbitrary stable matrix $\bar{F}_N$ $(\in \mathbf{R}^{2N\times 2N})$. Then, since $(\bar{F}_N, \bar{g}_0)$ is controllable, there exists θ_1, θ_2 $(\in \mathbf{R}^{2N})$ satisfying the following relation [1], [2].

$$\begin{aligned}&(\bar{C}_N\bar{A}_{NK}^2 + \bar{C}_N\bar{K}_N\bar{C}_N\bar{A}_{NK} + a_1\bar{C}_N\bar{A}_{NK})\cdot\\ &\quad\cdot\left\{(\bar{A}_{NK}+a_0I)\int_0^t\{\exp\bar{A}_{NK}(t-\tau)\}\cdot\bar{b}_Nf_f(\tau)\tau\right.\\ &\qquad\left.+\int_0^t\{\exp\bar{A}_{NK}(t-\tau)\}\cdot\bar{K}_Ny(\tau)d\tau\right\}\\ &\quad=\psi_1^T\bar{v}_1(t)+\psi_2^T\bar{v}_2(t)+\epsilon(t).\end{aligned}\tag{58}$$

The substitution of (58) into (54) yields

$$\begin{aligned}&\frac{d^2}{dt^2}y(t)+a_1\frac{d}{dt}y(t)+a_2y(t)\\ &\quad=\psi_1^T\bar{v}_1(t)+\psi_2^T\bar{v}_2(t)+\psi_3f_f(t)+\psi_4y(t)\\ &\qquad+\psi_0f(t)+\delta(t)+\epsilon(t),\end{aligned}\tag{59}$$

$$\psi_3 = (\bar{C}_N\bar{A}_{NK}^2 + \bar{C}_N\bar{K}_N\bar{C}_N\bar{A}_{NK} + a_1\bar{C}_N\bar{A}_{NK})\bar{b}_N,\tag{60}$$

$$\psi_4 = \{\bar{C}_N\bar{A}_N\bar{K}_N + (\bar{C}_N\bar{K}_N)^2 + a_1\bar{C}_N\bar{K}_N + a_2\},\tag{61}$$

$$\begin{aligned}\delta(t) &= -(\bar{C}_N\bar{A}_{NK}^2 + \bar{C}_N\bar{K}_N\bar{C}_N\bar{A}_{NK} + a_1\bar{C}_N\bar{A}_{NK})\cdot\\ &\qquad\cdot\int_0^t\{\exp\bar{A}_{NK}(t-\tau)\}\cdot\bar{K}_N\tilde{y}_N(\tau)d\tau\\ &\quad-\{\bar{C}_N\bar{A}_N\bar{K}_N+(\bar{C}_N\bar{K}_N)^2+a_1\bar{C}_N\bar{K}_N+a_2\}\tilde{y}_N(t)\\ &\quad+\sum_{i=N+1}^{\infty}\frac{c_i}{2}\{h(\mu_i^+)\xi_i(t)+h(\mu_i^-)\eta_i(t)\}.\end{aligned}\tag{62}$$

Therefore, the input-output representation of the process is given by (59), and is composed of two terms, $\psi_1^T\bar{v}_1(t)+\psi_2^T\bar{v}_2(t)+\psi_3f_f(t)+\psi_4y(t)+\psi_0f(t)$ and $\delta(t)$. The former half is constructed by finite dimensional systems, and is considered a primal part for controller design. On the contrary, the latter $\delta(t)$ is owing to the infinite dimensional system $[S_2]$, and corresponds to a spillover term. It is seen as a residual part for design of control systems.

In the rest of the present section, the residual part $\delta(t)$ is to be evaluated. For that purpose, we introduce next state variable filters whose dimensions are 1.

$$\frac{d}{dt}w_1(t) = -\tilde{\lambda}_Nw_1(t)+|f_f(t)|,\tag{63}$$

$$\frac{d}{dt}w_2(t) = -\lambda_fw_2(t)+w_1(t).\tag{64}$$

We assume that the sensor influence function c (output function) and the input influence function b are smooth in the following meaning.

Assumption 3 *The following inequalities hold.*

$$\sum_{i=1}^{\infty}|\lambda_i^kc_ib_i|<\infty,\quad\sum_{i=1}^{\infty}\lambda_i^{2k}c_i^2<\infty,\quad(k=1,2).\tag{65}$$

Then, $\delta(t)$ of (59) is evaluated by Lemma 2.

Lemma 2 *On Assumption 3, $\delta(t)$ is evaluated as follows:*

$$|\delta(t)|\le g_\delta(t)^Td_\delta+|\epsilon(t)|,\tag{66}$$

$$g_\delta=\left[\begin{array}{ccc}|f_f(t)|, & w_1(t), & w_2(t)\end{array}\right]^T,\tag{67}$$

$$d_\delta=\left[\begin{array}{ccc}M_1, & M_2, & M_3\end{array}\right]^T,\tag{68}$$

$$0<M_1\sim M_3<\infty,$$

$$\epsilon(t)\sim \mathrm{e}^{-\tilde{\lambda}_Nt},\ \mathrm{e}^{-\lambda_ft},\ \mathrm{e}^{-a_0t}\to 0.$$

V. ADAPTIVE H_∞ CONTROL

In the present section, the proposed adaptive H_∞ control systems are constructed by considering the system representations in the previous section, and by using finite dimensional controllers.

First, $h(s)$ is chosen such that

$$h(s)=(s+\lambda_0)^2,\quad(\lambda_0>0),\tag{69}$$

and $Y(t)$ and $\bar{\Theta}(t)$ are introduced.

$$Y(t)=\frac{d}{dt}y(t)+\lambda_0y(t),\tag{70}$$

$$\bar{\Theta}(t)=\frac{d}{dt}\bar{\theta}(t)+\lambda_0\bar{\theta}(t).\tag{71}$$

Furthermore, an augmented output $z(t)$ is defined as follows:

$$z(t)=C_YY(t)+C_{\bar{\Theta}}\bar{\Theta}(t),\tag{72}$$

where C_Y and $C_{\bar{\Theta}}$ are design parameters. By utilizing the system representation (59), the next relation is derived.

$$\begin{aligned}&\frac{d}{dt}z(t)+\lambda_0z(t)\\ &\quad=\Psi^T\omega(t)+\bar{\psi}_0f(t)+C_Y\{\delta(t)+\epsilon(t)\}+r(t)\\ &\quad=\bar{\psi}_0[p\{\Psi^T\omega(t)+r(t)\}+f(t)]\\ &\qquad+C_Y\{\delta(t)+\epsilon(t)\},\end{aligned}\tag{73}$$

$$\Psi=[C_Y\psi_1^T,\ C_Y\psi_2^T,\ C_Y\psi_3,\ C_Y\psi_4]^T,\tag{74}$$

$$\omega(t)=[\bar{v}_1(t)^T,\ \bar{v}_2(t)^T,\ f_f(t),\ y(t)]^T,\tag{75}$$

$$r(t)=C_{\bar{\Theta}}\{-\ddot{\theta}_d(t)+2\lambda_0\dot{\bar{\theta}}(t)+\lambda_0^2\bar{\theta}(t)\},\tag{76}$$

$$\bar{\psi}_0=C_Y\psi_0+C_{\bar{\Theta}},\tag{77}$$

$$p=1/\bar{\psi}_0.\tag{78}$$

The next assumptions are introduced.

Assumption 4

$$\psi_0\ne 0,\quad\bar{\psi}_0\ne 0,\tag{79}$$

and $\mathrm{sgn}(\bar{\psi}_0)$ *is known. In the following, it is assumed that* $\bar{\psi}_0>0$ *without loss of generality.*

Assumption 5 *There exist M_{f0} and M_{f1} such that*

$$|f_f(t)|\le M_{f0}+M_{f1}\sup_{0\le\tau\le t}\left\{|z(\tau)|,\ \left|\frac{d}{d\tau}z(\tau)\right|\right\},\tag{80}$$

$$(0\le M_{f0}<\infty,\ \ 0<M_{f1}<\infty).$$

Remark Assumption 4 states that the relative degree of the total process is 2, and Assumption 5 asserts that the total process has a stable inverse.

For unknown systems parameters Ψ, $\bar{\psi}_0$, p, we assume that

Assumption 6 *Positive constants M_Ψ, $\bar{\psi}_{0\max}$, $\bar{\psi}_{0\min}$ $p_{\max}$, $p_{\min}$ satisfying*

$$\|\Psi\| \le M_\Psi < \infty, \tag{81}$$
$$0 < \bar{\psi}_{0\min} \le \bar{\psi}_0 \le \bar{\psi}_{0\max} < \infty, \tag{82}$$
$$0 < p_{\min} \le p \le p_{\max} < \infty, \tag{83}$$

are known a priori.

The control input $f(t)$ is synthesized as follows:

$$\begin{aligned} f(t) &= -\hat{p}(t)\left\{\hat{\Psi}(t)^T\omega(t) + r(t)\right\} + v_1(t) \\ &\equiv -\hat{p}(t)v_0(t) + v_1(t). \end{aligned} \tag{84}$$

$\hat{\Psi}(t)$ and $\hat{p}(t)$ are current estimates of Ψ and p respectively. The projection type adaptive laws, where tuning parameters $\hat{\psi}$ are constrained to certain closed regions $\mathcal{S}$, are defined by

$$\dot{\hat{\psi}} = \Pr(\Gamma\phi e) \equiv \begin{cases} \Gamma\phi e & \text{Case } 1 \\ \Gamma\phi e - \Gamma\frac{\nabla g \nabla g^T}{\nabla g^T \Gamma \nabla g}\Gamma\phi e & \text{Case } 2, \end{cases} \tag{85}$$

where $\Gamma = \Gamma^T > 0$, and
Case 1 : $\hat{\psi} \in \mathcal{S}^o$, or $\hat{\psi} \in \partial\mathcal{S}$ & $(\Gamma\phi e)^T\nabla g \le 0$,
Case 2 : otherwise, $\mathcal{S} = \{\hat{\psi} : g(\hat{\psi}) \le 0\}$,
$\mathcal{S}^o$ = interior of $\mathcal{S}$, $\partial\mathcal{S}$ = boundary of $\mathcal{S}$.
Then, $\hat{\Psi}(t)$, $\hat{p}(t)$, $\hat{\psi}_0(t)$ are tuned by

$$\dot{\hat{\Psi}}(t) = \Pr\{G_1\omega(t)z(t)\}, \quad (G_1 = G_1^T > 0), \tag{86}$$
$$\dot{\hat{p}}(t) = \Pr\{G_2 v_0(t)z(t)\}, \quad (G_2 > 0), \tag{87}$$
$$\dot{\hat{\bar{\psi}}}_0(t) = \Pr\{G_3 v_1(t)z(t)\}, \quad (G_3 > 0), \tag{88}$$

where each constraints are given such that

$$\begin{aligned} &g_\Psi(\hat{\Psi}) = \|\hat{\Psi}\|^2 - M_\Psi^2, \\ &g_p(\hat{p}) = \left(\hat{p} - \frac{p_{\min}+p_{\max}}{2}\right)^2 - \left(\frac{p_{\max}-p_{\min}}{2}\right)^2, \\ &g_{\bar{\psi}_0}(\hat{\bar{\psi}}_0) = \left(\hat{\bar{\psi}}_0 - \frac{\bar{\psi}_{0\min}+\bar{\psi}_{0\max}}{2}\right)^2 \\ &\qquad - \left(\frac{\bar{\psi}_{0\max}-\bar{\psi}_{0\min}}{2}\right)^2. \end{aligned} \tag{89}$$

Those tuning parameters $\hat{\Psi}(t)$, $\hat{p}(t)$, $\hat{\bar{\psi}}_0(t)$ are made uniformly bounded by those projection type adaptive laws [2].

A positive function $V(t)$ is defined by

$$\begin{aligned} V(t) &= \frac{1}{2}z(t)^2 + \frac{1}{2}\{\hat{\Psi}(t)-\Psi\}^T G_1^{-1}\{\hat{\Psi}(t)-\Psi\} \\ &+ \frac{\bar{\psi}_0}{2}\{\hat{p}(t)-p\}^2/G_2 + \frac{1}{2}\{\hat{\bar{\psi}}_0(t)-\bar{\psi}_0\}^2/G_3. \end{aligned} \tag{90}$$

We take the time derivative of $V(t)$ along the trajectories of $z(t)$, $\hat{\Psi}(t)$, $\hat{p}(t)$, $\hat{\bar{\psi}}_0$ as follows:

$$\begin{aligned} \dot{V}(t) \le& -\lambda_0 z(t)^2 + \hat{\bar{\psi}}_0(t)v_1(t)z(t) \\ &+ C_Y\{\delta(t)+\epsilon(t)\}z(t). \end{aligned} \tag{91}$$

From the evaluation of $\dot{V}$, the following virtual system is introduced.

$$\begin{aligned} \dot{z} &= -\lambda_0 z + \hat{\bar{\psi}}_0 v_1 + g_\delta^T|C_Y|d_\delta + |C_Y\epsilon| \\ &\equiv f_e + g_{11}d_1 + g_{12}d_2 + g_2 v_1, \end{aligned} \tag{92}$$

$$\begin{aligned} &f_z = -\lambda_0 z, \ \ g_{11} = g_\delta^T, \ \ g_{12} = 1, \ \ g_2 = \hat{\bar{\psi}}_0, \\ &d_1 = |C_Y|d_\delta, \ \ d_2 = |C_Y\epsilon|. \end{aligned} \tag{93}$$

We are to stabilize the virtual system via v_1, where $d_1 = |C_Y|d_\delta$, $d_2 = |C_Y\epsilon|$ are regarded as exogenous disturbances to the process. For that purpose, we consider the Hamilton-Jacobi-Isaacs equation (HJI equation) and its solution V_0.

$$\frac{\partial V_0}{\partial z}f_z + \frac{1}{4}\left(\frac{\|g_{11}\|^2}{\gamma_1^2} + \frac{g_{12}^2}{\gamma_2^2} - \frac{g_2^2}{r}\right)\left(\frac{\partial V_0}{\partial z}\right)^2 + qz^2 \le 0, \tag{94}$$

$$V_0(t) = \frac{1}{2}z(t)^2. \tag{95}$$

q and r are positive functions to be determined from the inequality (94) based on inverse optimality [3], [4] for the given solution $V_0(t)$ (95) and the positive constant γ_1, γ_2. The substitution of (95) into (94) yields

$$-\lambda_0 z^2 + \left\{\frac{\|g_\delta\|^2}{\gamma_1^2} + \frac{1}{\gamma_2^2} - \frac{\hat{\bar{\psi}}_0^2}{r}\right\}\frac{z^2}{4} + qz^2 \le 0. \tag{96}$$

Once the positive functions q and r are obtained from (96), the control input is given by

$$v_1 = -\frac{1}{2r}g_2\frac{\partial V_0}{\partial z} = -\frac{1}{2r}\hat{\bar{\psi}}_0 z. \tag{97}$$

Then, we have the next theorems.

Theorem 1 *The adaptive control system ((84), (86), (87), (88), (97)) is uniformly bounded.*

Theorem 2 *In the proposed adaptive control scheme ((84), (86), (87), (88), (97)), v_1 is a sub-optimal control input which minimizes the upper bound on the following cost functional J.*

$$\begin{aligned} J(t) \equiv& \sup_{d_\delta,\epsilon\in\mathcal{L}_2}\left[\int_0^t \{qz(\tau)^2 + rv_1(\tau)^2\}d\tau + V(t)\right. \\ &\left. -\gamma_1^2\int_0^t \|C_Y d_\delta\|^2 d\tau - \gamma_2^2\int_0^t C_Y^2\epsilon^2 d\tau\right]. \end{aligned} \tag{98}$$

Also we have the next inequality.

$$\begin{aligned} &\int_0^t \{qz(\tau)^2 + rv_1(\tau)^2\}d\tau + V(t) \\ &\le V(0) + \gamma_1^2\int_0^t \|C_Y d_\delta\|^2 d\tau + \gamma_2^2\int_0^t C_Y^2\epsilon^2 d\tau. \end{aligned} \tag{99}$$

Remark The control scheme (84) is composed of two parts; one is $-\hat{p}(t)\left\{\hat{\Psi}(t)^T\omega(t) + r(t)\right\} = -\hat{p}(t)v_0(t)$, and the other is $v_1(t)$. The former half $-\hat{p}(t)v_0(t)$ is a conventional finite dimensional adaptive controller for the finite dimensional subsystem $[S_1]$. The latter one v_1 is added to regulate the effect of the neglected infinite dimensional modes in $[S_2]$ or spill-over terms $\delta(t)$. It should be noted that the overall control scheme is finite dimensional.

Up to now, the general forms of the control schemes were provided by (96), (97), and Theorem 1 and Theorem 2. Next, q and r are solved, and the explicit control structures are given by assuming specified forms to q and r.

Solution I From (96), r can be chosen such that

$$\frac{1}{r} = \frac{k_1 + k_2\|g_\delta\|^2}{r_0} \Leftrightarrow r = \frac{r_0}{k_1 + k_2\|g_\delta\|^2}, \quad (k_1,\, k_2,\, r_0 > 0), \tag{100}$$

where k_1, k_2, $r_0\,(>\ 0)$ are design parameters. Then, we obtain the corresponding q and related conditions such that

$$q \le \left\{ \lambda_0 + \frac{\hat{\bar{\psi}}_0^2 k_1}{4r_0} - \frac{1}{4\gamma_2^2} \right\} + \left(\frac{\hat{\bar{\psi}}_0^2 k_2}{4r_0} - \frac{1}{4\gamma_1^2} \right) \|g_\delta\|^2, \quad (101)$$

$$\lambda_0 + \frac{\bar{\psi}_{0\,\min}^2 k_1}{4r_0} - \frac{1}{4\gamma_2^2} > 0, \quad \frac{\bar{\psi}_{0\,\min}^2 k_2}{4r_0} - \frac{1}{4\gamma_1^2} \ge 0. \quad (102)$$

And, we get the explicit description of the control input

$$v_1 = -\frac{\hat{\bar{\psi}}_0}{2r} z = -\frac{\hat{\bar{\psi}}_0 \left[k_1 + k_2 \|g_\delta\|^2 \right]}{2r_0} z. \quad (103)$$

Solution II Next, we obtain q and r by setting

$$q = ar_1 + \frac{\hat{\bar{\psi}}_0^2}{4r_0}, \quad \frac{1}{r} = \frac{1}{r_0} + \frac{1}{r_1}, \quad (104)$$

where a, $r_0\,(0 < a, r_0 < \infty)$ are positive constants, which prescribe the ratio between r and q. Then, for equality condition of (96), we obtain r_1 and q such as

$$r_1 = \frac{-G + \sqrt{G^2 + a\hat{\bar{\psi}}_0^2}}{2a} = \frac{\hat{\bar{\psi}}_0^2}{2\left\{ \sqrt{G^2 + a\hat{\bar{\psi}}_0^2} + G \right\}}, \quad (105)$$

$$q = \frac{-G + \sqrt{G^2 + a\hat{\bar{\psi}}_0^2}}{2} + \frac{\hat{\bar{\psi}}_0^2}{4r_0}, \quad (106)$$

and the explicit description of the control input is given by

$$v_1 = -\frac{\hat{\bar{\psi}}_0}{2r} z = -\left\{ \frac{1}{\hat{\bar{\psi}}_0} \left(\sqrt{G^2 + a\hat{\bar{\psi}}_0} + G \right) + \frac{\hat{\bar{\psi}}_0}{2r_0} \right\} z. \quad (107)$$

Theorem 3 *Those two solutions of v_1 (Solution I and Solution II), can make the residual regions of z arbitrarily small by proper choices of design parameters k_1, k_2, r_0, a, γ_1, γ_2 (sufficiently large k_1, k_2, a, and sufficiently small r_0, γ_1, γ_2).*

From Theorem 3, it is seen that the augmented output signal $z(t)$, which is composed of the bending displacement of the arm (u, y), and the tracking error of the motor angle $\bar{\theta}$, converge to an arbitrary small residual region. Hence, when Assumption 5 is satisfied and $\ddot{\theta}_d \to 0$, it is thought that $Y(t) \simeq \mathcal{O}(z(t)) \to 0$, $\bar{\Theta}(t) \simeq \mathcal{O}(z(t)) \to 0$, $y(t) \simeq \mathcal{O}(z(t)) \to 0$, $\bar{\theta}(t) \simeq \mathcal{O}(z(t)) \to 0$ as $z(t) \simeq 0$.

Remark Solution I and Solution II include high-gain feedback structures with nonlinear damping, and those seem to be similar to certain classes of sliding mode control strategies in [21], [22]. However, attenuation problems of spillover terms are not discussed in those previous works.

VI. CONCLUDING REMARKS

A design method of adaptive control for flexible arms via finite dimensional controllers is provided based on the notion of inverse optimality. The adaptive controller is composed of two $2N$ dimensional state variable filters, and the index N together with the damping ration $\tilde{\lambda}_N$ are determined such that (27) holds. It is shown that the adaptive control systems are uniformly bounded, and that the tracking error of motor angles $\bar{\theta}$ and the spatial average of distributed bending displacement of the arm $y(t)$ converge to a small residual region. The proposed control scheme can be also applied to DPS of parabolic type and hyperbolic type [19], [23], or DPS with more complicated structures such as input nonlinearities. Parts of those will be presented in the future.

REFERENCES

[1] K.S. Narendra and A.M. Annaswamy, *Stable Adaptive Systems*, Prentice-Hall, 1989.

[2] P.A. Ioannou and J. Sun, *Robust Adaptive Control,* PTR Prentice-Hall, 1996.

[3] M. Krstić and H. Deng, *Stabilization of Nonlinear Uncertain Systems*, Springer, 1998.

[4] Y. Miyasato, "Redesign of Adaptive Control Systems Based on the Notion of Optimality", *Proc. 38th IEEE Conf. Dec. . Contr.*, pp.3315-3320, 1999.

[5] Y. Miyasato, "Adaptive nonlinear H_∞ control for processes with bounded variations of parameters – general relative degree case – ", *Proc. 39th IEEE Conf. Dec. Contr.*, pp.1453-1458, 2000.

[6] Y. Miyasato, "Adaptive nonlinear H_∞ control for processes with bounded variations of parameters – general forms and general relative degree case –", *Proc. ALCOSP 2001*, pp.419-424, 2002.

[7] Y. Sakawa, "Feedback Control of Second Order Evolution Equations with Damping", *SIAM J. Contr. Optim.*, vol.22, pp.343-361, 1984.

[8] Y. Sakawa, F. Matsuno and S. Fukushima, "Modeling and feedback control of a flexible arm", *J. Robotic Systems*, vol.2, pp.453-472, 1985.

[9] Y. Miyasato, "Model Reference Adaptive Control for Distributed Parameter Systems of Parabolic Type", *Trans. Soc. Instr. Contr. Engin.*, vol.21, pp.1163-1170, 1985.

[10] T. Kobayashi, "Finite-Dimensional Adaptive Control for Infinite-Dimensional Systems", *Int. J. Contr.*, vol.48, pp.289-302, 1988.

[11] Y. Miyasato, "Model Reference Adaptive Control for Distributed Parameter Systems of Parabolic Type by Finite-Dimensional Controller", *Proc. 29th IEEE Conf. Dec. Contr.*, pp.1459-1464, 1990.

[12] Y. Miyasato, "Model Reference Adaptive Control for Distributed Parameter Systems of hyperbolic Type by Finite-Dimensional Controllers", *Recent Advances in Math. Theory of Syst., Contr., Networks and Signal Processing II*, pp.413-418, 1992.

[13] M. Böhm, M.A. Demetriou, S. Reich, and I.G. Rosen, "Model Reference Adaptive Control of Distributed Parameter Systems", *SIAM J. Contr. Optim.*, vol.36, pp.33-81, 1998.

[14] M. Krstić, "On Global Stabilization of Burgers' Equation by Boundary Control", *Sys. Contr. Lett.*, vol. 37, pp 123-141, 1999.

[15] T. Kobayashi, "Adaptive Stabilization of the Kuramoto-Sivashinsky Equation", *Int. J. Syst. Sci.*, vol.33, pp.175-180, 2002.

[16] K.S. Hong and J. Bentsman, "Direct Adaptive Control of Parabolic Systems: Algorithm Synthesis and Convergence and Stability Analysis", *IEEE Trans. Autom. Contr.*, vol.39, pp.2018-2033, 1994.

[17] B. King and N. Hovakimyan, "An Adaptive Approach to Control of Distributed Parameter Systems", *Proc. . 42nd IEEE Conf. Dec. Contr.*, pp.5715-5720, 2003.

[18] M. Krstić, A. Balogh and A. Smyshlyaev, "Backstepping Boundary Controller and Observer for the Undamped Shear Beam", *Proc. 17th Int. Symp. Math. Theory of Networks and Syst.*, pp.1389-1394, 2006.

[19] Y. Miyasato, "Model Reference Adaptive H_∞ Control for Distributed Parameter Systems of Hyperbolic Type by Finite-Dimensional Controller", *Proc. 45th IEEE Conf. Dec. Contr.*, pp.459-464, 2006.

[20] S. Dubljevic, D. Christofides and I.G. Kevrekidis, "Distributed Nonlinear Control of Diffusion-Reaction Process", *Proc. Amer. Contr. Conf. 2003*, pp.1341-1348, 2003.

[21] Y. Orlov, "Sliding Mode Observer-based Synthesis of State Derivative-Free Model Reference Adaptive Control of Distributed Parameter Systems", *ASME J. Dynamic Syst., Measurement, and Contr.*, vol.122, pp.726-731, 2000.

[22] Y. Orlov, Y, Lou and P.D. Christofides, "Robust Stabilization of Infinite-Dimensional Systems Using Sliding-Mode Output Feedback Control", *Int. J. Contr.*, vol.77, pp.1115-1136, 2004.

[23] Y. Miyasato, "Model reference adaptive H_∞ control for distributed parameter systems of parabolic type by finite dimensional controllers", *Proc.17th Int. Symp. Math. Theory of Networks and Syst.*, pp.1140-1148, 2006.