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A Lyapunov Approach to Integral Input-to-State Stability of Cascaded Systems with External Signals

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***Abstract*— This paper deals with the stability of cascade interconnection of integral input-to-state (iISS) time-varying systems. A new technique is introduced for the purpose of constructing smooth Lyapunov functions of cascaded systems explicity. From the construction, sufficient conditions for internal stability and stability with respect to external signals are derived. One of the proposed conditions is a generalized version of trade-off between slower convergence of the driving system and steeper input growth of the driven system. It is demonstrated that the trade-off is no more necessary if the speed of convergence of the driven system is not radially vanishing. The results are related to trajectory-based approaches in the literature and small-gain techniques for feedback interconnection. The difference between the feedback case and the cascade case is also viewed from the requirement on convergence speed of autonomous parts.**

I. Introduction

The development of stability and stabilization theory for cascaded systems has been playing significant roles in nonlinear systems control[15], [13], [10], [7]. The stability and stabilizability are often related to growth rate conditions on functions describing the interaction between systems (See [11], references and literature review therein). It is well known that the cascade of input-to-state stable (ISS) systems is ISS since a growth rate condition can be always satisfied[17]. Arcak et al.[2] have considered a time-invariant cascade interconnected system in which an integral input-to-state stable (iISS) system is driven by a globally asymptotically stable (GAS) system. Their result has had a great impact on the analysis and synthesis of nonlinear cascades since iISS is less restrictive than ISS. They employed a trajectory-based approach to derive a sufficient condition for GAS of the time-invariant cascade. It is proved that the iISS gain of driven system needs to be steep satisfactorily in the direction toward the equilibrium if the convergence of the driving system is slow. This trade-off between convergence rate and growth rate of iISS gain has unified several results in the literature

As for feedback interconnection, stability conditions for interconnections of iISS systems have been derived in [4]. The development is based on explicit construction of smooth Lyapunov functions of feedback systems. The relation between the cascade result [2] and the feedback result [4] has never been discussed in the literature yet. One of obstacles is that Arcak et al.[2] gave no interpretation in terms of constructing Lyapunov functions of the whole system. Removing this obstacle is one of the main motivations for this paper.

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An advantage of the trade-off condition in [2] is that we do not always have to assume local exponential stability (LES) of the driving system. Authors of [2] also explain the effectiveness of the trade-off condition by using examples of cascades which are not GAS when the trade-off between the convergence rate and the growth of gain is not fulfilled. However, in order to discuss the necessity of the trade-off condition for stability of the cascade, the effect of autonomous parts in systems should be taken into account carefully. One of the main motivations for this paper stems from this point. It would be also interesting and important to study the time-varying cascade consisting of an iISS driving system and an iISS driven system affected by external disturbances, which has not been accomplished in [2].

The purposes of this paper are

(G1) to derive counterparts of [2] by constructing a Lyapunov function explicitly for time-varying systems;
(G2) to establish stability with respect to external inputs;
(G3) to shows that every cascade system of some class can be stable even if the driven system does not satisfy the growth restriction of the interconnection term according with the speed of convergence of the driving system.

This paper accomplishes all these points through explicit construction of Lyapunov functions of the whole cascaded systems by employing a Lyapunov formulation introduced in [4]. This paper thoroughly improves the approach by specializing in the cascade systems and proposing a new type of Lyapunov function. This paper also corrects an error in [4]. As for (G2) alone, a growth rate condition for iISS of cascaded time-invariant systems has been derived in [3] recently without constructing a Lyapunov function of the cascades. The growth rate result can be also covered by the constructive Lyapunov approach in this paper except in a special circumstance.

Throughout this paper, $\gamma \in \mathscr{P}_0$ denotes that γ is a continuous function from $\mathbb{R}_+$ to $\mathbb{R}_+$ satisfying $\gamma(0)=0$. The set of $\gamma \in \mathscr{P}_0$ satisfying $\gamma(s)>0$ for all $s\in\mathbb{R}_+\setminus\{0\}$ is denoted by $\mathscr{P}$. A function is said to belong to class $\mathscr{K}$ if it is in $\mathscr{P}$ and increasing. A class $\mathscr{K}$ function is said to be of class $\mathscr{K}_\infty$ if it tends to infinity as its argument approaches infinity. For a function $h\in\mathscr{P}$, we write $h\in\mathscr{O}(>L)$ with a non-negative real number L if there exists a positive real number $K>L$ such that $\limsup_{s\to 0+} h(s)/s^K<\infty$. We write $h\in\mathscr{O}(L)$ when $K=L$. Note that $\mathscr{O}(L)\subset\mathscr{O}(S)$ holds for $L>S$. The symbols $\vee$ and $\wedge$ denote logical sum and logical product, respectively. A system is said to be GAS if it has a globally asymptotically stable equilibrium at the origin of the state space. In this manner, UGAS stands for uniformly global asymptotic stability in the case of time-varying systems.

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II. Illustrations

In order to illustrate motivations, we consider the following cascade consisting of two subsystems:

$$\dot{x}_1 = -x_1 + x_1 x_2 \tag{1}$$
$$\dot{x}_2 = -x_2^3 \tag{2}$$

The x_1-system is not ISS since the constant input $x_2 > 1$ results in unbounded trajectories. Only iISS property holds. The x_2-system is not LES, which violates the assumption of Corollary 2 and Corollary 3 in [2]. This cascade does not satisfy the coupled condition on convergence rate and gain growth required by Theorem 1 in [2] either because the input x_2 appears in the x_1-system in a first order fashion. Indeed, the driving system (2) and the input term of the driven system (1) are the same as Example 1 in [2] which exhibits unbounded solutions. Thus, the stability of the cascade system (1)-(2) cannot be determined by the characterizations proposed in [2]. Nevertheless, due to the presence of the radially non-vanishing convergent term $-x_1$ in the x_1-system, the cascade (1)-(2) is GAS. To see this, define $V_1(x_1) = \frac{1}{2}\log(x_1^2+1)$ and $V_2(x_2) = \frac{1}{2}x_2^2$, and the dissipation inequalities

$$\frac{\partial V_1}{\partial x_1} f_1 \le -\frac{|x_1|^2}{|x_1|^2+1} + |x_2|, \quad \frac{\partial V_2}{\partial x_2} f_2 \le -|x_2|^4 \tag{3}$$

are satisfied by Σ_1 and Σ_2. It can be verified that

$$V(x) = d\int_0^{V_1} \left(\frac{e^{2s}-1}{e^{2s}}\right)^4 ds + \frac{1}{3}|x_2|^3, \quad 0<d<1 \tag{4}$$

is a Lyapunov function establishing GAS of $x = [x_1, x_2]^T = 0$. This claim is precisely formulated into Corollary 1 in this paper for general systems.

Introducing external inputs to (1)-(2), we consider

$$\dot{x}_1 = -x_1 + x_1 x_2 + r_1^3 \tag{5}$$
$$\dot{x}_2 = -x_2^3 + r_2 \tag{6}$$

The time-derivative of $V(x)$ in (4) along the trajectories of (5) and (6) is obtained as

$$\dot{V}(x) \le -\frac{d}{5}\left(\frac{x_1^2}{x_1^2+1}\right) - \frac{1}{5}(3-d)|x_2|^5 + \frac{d}{2}|r_1|^3 + \frac{3}{5}|r_2|^{\frac{5}{3}} \tag{7}$$

Therefore, the cascade system is iISS with respect to input (r_1, r_2) and state (x_1, x_2), and $V(x)$ is an iISS Lyapunov function. In fact, Theorem 1 in this paper demonstrates that iISS property of cascade systems can be established whenever dissipation inequalities of individual systems in the absence of external inputs are given by (3).

The next example of cascade is the following:

$$\dot{x}_1 = -\frac{x_1}{x_1^2+1} + x_2^2 + r_1, \quad x_1(0), x_2(0) \in \mathbb{R}_+ \tag{8}$$
$$\dot{x}_2 = -\frac{2x_2^4}{x_2^4+1} + \frac{r_2}{r_2+1}, \quad r_1(t), r_2(t) \in \mathbb{R}_+, \forall t \in \mathbb{R}_+ \tag{9}$$

which evolves in the positive orthant $\mathbb{R}_+^2$ for all $t \in \mathbb{R}_+$. The x_1-system is not ISS since the term of convergence is vanishing toward zero in the radial direction. There exists $M > 0$ for each pair of constants $x_2, r_1 > 0$ such that every trajectory of the x_1-system starting from $x_2(0) > M$ diverges.

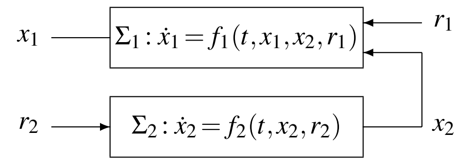

Fig. 1. Cascade system Σ

The system is only iISS with respect to input (x_2, r_1) and state x_1. Meanwhile, the x_2-system is ISS with respect to input r_2 and state x_2. Although the convergence rate of x_2-system near the origin is much slower than LES, we can still not only verify GAS of the cascade in the absence of the external inputs r_1 and r_2, but also establish stability with respect to (r_1, r_2). To this end, taking $V_1(x_1) = x_1$ and $V_2(x_2) = x_2$, we obtain

$$\frac{\partial V_1}{\partial x_1} f_1 \le -\frac{x_1}{x_1^2+1} + x_2^2 + r_1, \quad \frac{\partial V_2}{\partial x_2} f_2 \le -\frac{2x_2^4}{x_2^4+1} + \frac{r_2}{r_2+1}$$

Since these functions fulfill the assumptions of Theorem 3 with $k = 2$, we can conclude that the interconnected system is iISS with respect to input (r_1, r_2) and state (x_1, x_2). Theorem 3 also provides the following iISS Lyapunov function.

$$V(x_1, x_2) = \frac{1}{2}\log(x_1^2+1) + \frac{1}{5}x_2^5 + x_2 \tag{10}$$

It is worth mentioning that this iISS property cannot be verified directly by the result in [3].

III. Definition of Cascade Systems

Consider the nonlinear interconnected system Σ shown in Fig.1. The subsystems Σ_1 is driven by Σ_2. Both the subsystems are allowed to be time-varying. The state vector of Σ is $x = [x_1^T, x_2^T]^T \in \mathbb{R}^n$. The signals r_1 and r_2 are packed into $r = [r_1^T, r_2^T]^T \in \mathbb{R}^k$. The following sets are considered.

Definition 1: Given $\alpha_1, \alpha_2 \in \mathscr{P}$, $\sigma_1 \in \mathscr{K}$ and $\sigma_{r1}, \sigma_{r2} \in \mathscr{P}_0$, we write $\Sigma_1 \in \mathscr{S}_1(n_1, \alpha_1, \sigma_1, \sigma_{r1})$ and $\Sigma_2 \in \mathscr{S}_2(n_2, \alpha_2, \sigma_{r2})$ if Σ_1 and Σ_2 are described by

$$\dot{x}_1 = f_1(t, x_1, x_2, r_1), \ x_i \in \mathbb{R}^{n_i}, r_i \in \mathbb{R}^{k_i} \tag{11}$$
$$\dot{x}_2 = f_2(t, x_2, r_2), \tag{12}$$
$$f_i(t, 0, \cdots, 0) = 0, \qquad t \in \mathbb{R}_+, \quad i = 1, 2 \tag{13}$$
$$f_i \text{ is locally Lipschitz in } (x, r_i) \text{ uniformly in } t \text{ and piecewise continuous in } t \tag{14}$$

which admit the existence of $\mathbf{C}^1$ functions V_i: $\mathbb{R}_+ \times \mathbb{R}^{n_i} \to \mathbb{R}$ and $\underline{\alpha}_i$, $\bar{\alpha}_i \in \mathscr{K}_\infty$, $i = 1, 2$, such that

$$\underline{\alpha}_i(|x_i|) \le V_i(t, x_i) \le \bar{\alpha}_i(|x_i|), \quad i = 1, 2 \tag{15}$$
$$\frac{\partial V_1}{\partial t} + \frac{\partial V_1}{\partial x_1} f_1 \le -\alpha_1(|x_1|) + \sigma_1(|x_2|) + \sigma_{r1}(|r_1|) \tag{16}$$
$$\frac{\partial V_2}{\partial t} + \frac{\partial V_2}{\partial x_2} f_2 \le -\alpha_2(|x_2|) + \sigma_{r2}(|r_2|) \tag{17}$$

hold for all $x \in \mathbb{R}^n$, $r \in \mathbb{R}^k$ and $t \in \mathbb{R}_+$.

The Lipschitzness imposed on f_i guarantees the existence of a unique maximal solution of Σ for locally essentially bounded $r_i(t)$. The inequalities (16) and (17) are often referred to as "dissipation inequalities", and their right hand sides are called supply rates. The individual system Σ_i fulfilling the above definition is said to be integral input-to-state stable (iISS)[16]. The function V_i is called a $\mathbf{C}^1$ iISS

Lyapunov function[1]. Under a stronger assumption $\alpha_i \in \mathcal{K}_\infty$, the system Σ_i is said to be input-to-state stable (ISS)[14], and the function V_i is a $\mathbf{C}^1$ ISS Lyapunov function[18]. The trajectory-based definition of ISS and iISS may be seen more often than the Lyapunov-based definition this paper adopts. The two types of definition are equivalent in the sense of the existence of ISS (iISS) Lyapunov functions[18], [16]. By definition, an ISS system is always iISS. The converse does not hold. In this paper, the convergence speed of the system Σ_i is said to be radially vanishing if $\liminf_{s\to\infty} \alpha_i(s) = 0$.

IV. Radially Non-Vanishing Case

The goals (G2) and (G3) are achieved by the following.

Theorem 1: Let n_i be a positive integer for $i = 1,2$. Assume that

$$\alpha_i \in \mathcal{K}, \quad i = 1,2 \tag{18}$$

holds. Then, the following hold true.

(i) If one of

$$\alpha_2 \in \mathcal{K}_\infty \tag{19}$$

$$\sigma_1 \notin \mathcal{K}_\infty \tag{20}$$

$$\infty > \lim_{s\to\infty} \alpha_2(s) > \sup_{s\in\mathbb{R}_+} \sigma_{r2}(s) \tag{21}$$

is satisfied, the cascade system Σ is iISS with respect to input r and state x for all $\Sigma_1 \in \mathcal{S}_1(n_1, \alpha_1, \sigma_1, \sigma_{r1})$ and $\Sigma_2 \in \mathcal{S}_2(n_2, \alpha_2, \sigma_{r2})$. Furthermore, an iISS Lyapunov function of Σ is

$$V(t,x) = \int_0^{V_1(t,x_1)} \lambda_1(s)ds + \int_0^{V_2(t,x_2)} \lambda_2(s)ds \tag{22}$$

given with

$$\lambda_1(s) = \left[\delta\alpha_2 \circ \bar{\alpha}_2^{-1} \circ \underline{\alpha}_2 \circ \hat{\sigma}_1^{-1} \circ \frac{1}{\tau_1}\hat{\alpha}_1 \circ \bar{\alpha}_1^{-1}(s)\right]\left[\psi \circ \frac{1}{\tau_1}\hat{\alpha}_1 \circ \bar{\alpha}_1^{-1}(s)\right] \tag{23}$$

$$\lambda_2(s) = \hat{\sigma}_1 \circ \underline{\alpha}_2^{-1}(s)\left[\psi \circ \hat{\sigma}_1 \circ \underline{\alpha}_2^{-1}(s)\right] \tag{24}$$

$$\tau_1 > 1 \tag{25}$$

where
ψ is any continuous function
δ is any real number
} satisfying

$$\lambda_1(s), \lambda_2(s) : \text{non-decreasing}, \ \forall s \in \mathbb{R}_+ \tag{26}$$

$$0 < \psi(s) < \infty, \ s \in (0,L), \quad L := \lim_{s\to\infty} \hat{\sigma}_1(s)$$

and one of

$$(19) \ \wedge \ 0 < \delta < 1 \tag{27}$$

$$(20) \ \wedge \ 0 < \delta < 1 \ \wedge \ \lim_{s\to L} \psi(s) < \infty \tag{28}$$

$$(21) \ \wedge \ 0 < \delta < 1 - \frac{\sup_{s\in\mathbb{R}_+} \sigma_{r2}(s)}{\lim_{s\to\infty} \alpha_2(s)} \tag{29}$$

$\hat{\alpha}_1$ and $\hat{\sigma}_1$ are any class $\mathcal{K}$ functions satisfying

$$\lim_{s\to\infty} \hat{\sigma}_1(s) \geq \lim_{s\to\infty} \hat{\alpha}_1(s) \tag{30}$$

$$\hat{\alpha}_1(s) \leq \alpha_1(s), \ \forall s \in \mathbb{R}_+ \tag{31}$$

$$\hat{\sigma}_1(s) := \sigma_1(s), \ \forall s \in \mathbb{R}_+ \tag{32}$$

(ii) If

$$\alpha_i \in \mathcal{K}_\infty, \quad i = 1,2 \tag{33}$$

is satisfied, the cascade system Σ is ISS with respect to input r and state x for all $\Sigma_1 \in \mathcal{S}_1(n_1, \alpha_1, \sigma_1, \sigma_{r1})$ and $\Sigma_2 \in \mathcal{S}_2(n_2, \alpha_2, \sigma_{r2})$. Furthermore, an ISS Lyapunov function of Σ is (22) given with (23)-(27), (30) and

$$\hat{\alpha}_1(s) := \alpha_1(s), \ \forall s \in \mathbb{R}_+ \tag{34}$$

$$\hat{\sigma}_1(s) \geq \sigma_1(s), \ \forall s \in \mathbb{R}_+ \tag{35}$$

The assumption (18) implies that the speed of convergence of Σ_1 and Σ_2 is radially non-vanishing.

We next address (G3) alone. When we do not deal with external signals, all assumptions in Theorem 1 except (36) can be removed as follows:

Corollary 1: Let n_i be a positive integer for $i = 1,2$. If

$$\alpha_1 \in \mathcal{K} \tag{36}$$

holds, the cascade system Σ is UGAS for $r_i(t) \equiv 0$, $i = 1,2$, for all $\Sigma_1 \in \mathcal{S}_1(n_1, \alpha_1, \sigma_1, \sigma_{r1})$ and $\Sigma_2 \in \mathcal{S}_2(n_2, \alpha_2, \sigma_{r2})$.

The main result of [2] imposes restriction on growth of the interconnection term. The above corollary removes the restriction completely when the convergence speed of the driven system is not vanishing in the radial direction.

V. Radially Vanishing Case

If the convergence term of the driven system is allowed to vanish as its state tends to infinity, we are able to achieve (G2) as follows.

Theorem 2: Let n_i be a positive integer for $i = 1,2$. Assume that

$$\alpha_2 \in \mathcal{K} \tag{37}$$

holds. If there exists $c_2 > 0$ and $k \geq 1$ such that

$$c_2\sigma_1 \circ \underline{\alpha}_2^{-1}(s) \leq [\alpha_2 \circ \bar{\alpha}_2^{-1}(s)]^k, \quad \forall s \in \mathbb{R}_+ \tag{38}$$

is satisfied, the cascade system Σ is iISS with respect to input r and state x for all $\Sigma_1 \in \mathcal{S}_1(n_1, \alpha_1, \sigma_1, \sigma_{r1})$ and $\Sigma_2 \in \mathcal{S}_2(n_2, \alpha_2, \sigma_{r2})$. Furthermore, an iISS Lyapunov function of Σ is (22) given with

$$\lambda_1(s) = h, \ \lambda_2(s) = [\alpha_2 \circ \bar{\alpha}_2^{-1}(s)]^{k-1}, \ 0 < h < c_2/k \tag{39}$$

We can relax (38) of Theorem 2 by employing another type of Lyapunov function as follows:

Theorem 3: Let n_i be a positive integer for $i = 1,2$. Assume that (37) holds and there exists $k \geq 1$ such that

$$\int_1^\infty [\min_{w\in[\bar{\alpha}_1^{-1}(s), \underline{\alpha}_1^{-1}(s)]} \alpha_1(w)]^{k-1} ds = \infty \tag{40}$$

$$\lim_{s\to 0+} \frac{[\sigma_1 \circ \underline{\alpha}_2^{-1}(s)]^k}{\alpha_2 \circ \bar{\alpha}_2^{-1}(s)} < \infty \tag{41}$$

hold. If one of (19), (20) and (21) is satisfied, the cascade system Σ is iISS with respect to input r and state x for all $\Sigma_1 \in \mathcal{S}_1(n_1, \alpha_1, \sigma_1, \sigma_{r1})$ and $\Sigma_2 \in \mathcal{S}_2(n_2, \alpha_2, \sigma_{r2})$. Furthermore, an iISS Lyapunov function of Σ is (22) given with

$$\lambda_1(s) = [\min_{w\in[\bar{\alpha}_1^{-1}(s), \underline{\alpha}_1^{-1}(s)]} \alpha_1(w)]^{k-1} \tag{42}$$

$$\lambda_2(s) = h \max_{w\in[0,s]} \frac{[\sigma_1 \circ \underline{\alpha}_2^{-1}(w)]^k}{\alpha_2 \circ \bar{\alpha}_2^{-1}(w)} \tag{43}$$

$$h > \begin{cases} \dfrac{\lim_{s\to\infty} \alpha_2(s)}{k(\lim_{s\to\infty} \alpha_2(s) - \sup_{s\in\mathbb{R}_+} \sigma_{r2}(s))} & \text{if } \alpha_2 \notin \mathcal{K}_\infty \wedge \sigma_1 \in \mathcal{K}_\infty \\ \dfrac{1}{k} & \text{otherwise} \end{cases} \tag{44}$$

Moreover, (41) and (43) can be replaced by

$$\int_0^1 \frac{[\sigma_1 \circ \underline{\alpha}_2^{-1}(s)]^k}{\alpha_2 \circ \bar{\alpha}_2^{-1}(s)} ds < \infty \tag{45}$$

$$\lambda_2(s) = \begin{cases} h\dfrac{[\sigma_1 \circ \underline{\alpha}_2^{-1}(s)]^k}{\alpha_2 \circ \bar{\alpha}_2^{-1}(s)} & ,s \in [0,1) \\ h \max\limits_{w \in [1,s]} \dfrac{[\sigma_1 \circ \underline{\alpha}_2^{-1}(w)]^k}{\alpha_2 \circ \bar{\alpha}_2^{-1}(w)} & ,s \in [1,\infty) \end{cases} \tag{46}$$

respectively, if $\sigma_{r2}(s) \equiv 0$ or $r_2(t) \equiv 0$.

The assumption

$$(40) \wedge (41) \wedge \{(19) \vee (20) \vee (21)\} \tag{47}$$

is milder than (38). In fact, the condition (40) always holds for $k = 1$, and the assumption (38) for a $k \geq 1$ implies that (41) holds for any $k \geq 1$. In addition, the assumption (38) is never fulfilled unless the condition $\{(19) \vee (20) \vee (21)\}$ holds. Thus, Theorem 3 is better than Theorem 2 although the assumption looks complicated.

The following corollary addresses (G1) which is a consequence of Theorem 3.

Corollary 2: Let n_i be a positive integer for $i=1,2$. If

$$\int_0^1 \frac{[\sigma_1 \circ \underline{\alpha}_2^{-1}(s)]}{\alpha_2 \circ \bar{\alpha}_2^{-1}(s)} ds < \infty \tag{48}$$

holds, the cascade system Σ is UGAS for $r_i(t) \equiv 0$, $i = 1,2$, for all $\Sigma_1 \in \mathscr{S}_1(n_1, \alpha_1, \sigma_1, \sigma_{r1})$ and $\Sigma_2 \in \mathscr{S}_2(n_2, \alpha_2, \sigma_{r2})$.

The condition (48) constrains the growth of interconnection term in the driven system to be slow enough to cope with a low speed of convergence of the driving system near the equilibrium.

Theorem 2 is essentially the same as Corollary 3 (i) of [4]. This paper, however, has the advantage of providing a Lyapunov function explicitly for the cascade system. Theorem 3 relaxes the restrictive assumption (38) by correcting and thoroughly improving Corollary 3 (ii)-(v) of [4]. First of all, Corollary 3 in [4] has merely missed including the assumptions $\alpha_i \in \mathscr{K}$, $i = 1,2$ in (ii)-(v). This paper has dealt with the case of $\alpha_i \in \mathscr{K}$, $i = 1,2$ by a different Lyapunov function in Theorem 1. Theorem 1 is notably less restrictive than the results which can be derived from the Lyapunov function employed in [4]. Theorem 3 covers $\alpha_1 \notin \mathscr{K}$ and allows $-\alpha_1(s)$ to be vanishing as s tends to ∞.

Theorem 3 generalizes a similar result in [3] and covers time-varying systems. The condition (40) is fulfilled if we restrict k to 1. The remaining condition (41) for $k=1$ is the growth order restriction used in [3]. Theorem 3 has an advantage of explicitly providing Lyapunov function of the whole system over [3]. The growth order restriction (41) involves the $\mathscr{K}_\infty$ bounds on individual iISS Lyapunov functions, while the $\mathscr{K}_\infty$ bounds are not involved in a result of [3]. It is natural since this paper constructs Lyapunov functions of the whole system. It is worth adding that this paper does not show iISS Lyapunov functions in the case of $\alpha_2(\infty)=0$ although a Lyapunov functions for GAS is derived in Corollary 2. The question of how to construct an iISS Lyapunov function in the case of $\alpha_2(\infty)=0$ remains open, while the iISS property is proved in [3] without Lyapunov functions.

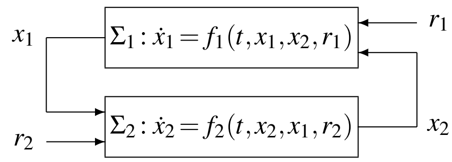

Fig. 2. Feedback system Σ_F

VI. Difference between cascade and feedback

Consider the feedback system shown in Fig.2, where $\Sigma_1 \in \mathscr{S}_1(n_1, \alpha_1, \sigma_1, \sigma_{r1})$ and $\Sigma_2 \in \mathscr{S}_2(n_2, \alpha_2, \sigma_2, \sigma_{r2})$ defined with an additional term $+\sigma_2(|x_1|)$ in (17). A special case of the feedback is cascade connection. Indeed, if we assume $\sigma_2(s) \equiv 0$ in Fig.2, the system is identical with Fig.1. This fact surely implies that stability of cascade systems can be verified by means of the small-gain technique for ISS systems and iISS systems derived in [9] and [4], respectively. For example, if Σ_1 and Σ_2 are individually ISS, the small-gain constraint on the pair of Σ_1 and Σ_2 is always met by zero loop-gain since the closed-loop is broken. In the case of iISS subsystems, the cascade is not always iISS since some pairs of Σ_1 and Σ_2 do not have finite loop-gain associated with the small-gain constraint. Thus, it is important to be aware that requiring finite loop-gain may not be a tight stability condition for cascaded systems. In order to avoid such conservativeness, the previous sections have proposed stability conditions specialized in cascade connection. This section spotlights this issue from the speed of convergence of individual systems.

As long as we derive stability from supply rates of constituent systems, the convergence rate of each autonomous term needs to be radially increasing if the interconnection forms a closed loop. The following demonstrates this fact.

Theorem 4: Consider the feedback interconnection shown in Fig.2. Let n_i be a positive integer for each $i=1,2$. Assume that functions $\alpha_i, \sigma_i, \sigma_{ri} : \mathbb{R}_+ \to \mathbb{R}_+$ are $\mathbf{C}^1$, and satisfy

$$\alpha_i \in \mathscr{O}(>1), \quad \sigma_i, \sigma_{ri} \in \mathscr{O}(>0), \quad i=1,2 \tag{49}$$

Then, the feedback system Σ_F with $r_i(t) \equiv 0$, $i=1,2$, is GAS for all $\Sigma_i \in \mathscr{S}_i(n_i, \alpha_i, \sigma_i, \sigma_{ri})$, $i=1,2$, only if

$$\liminf_{s\to\infty} \alpha_i(s) > 0 \tag{50}$$

hold for $i=1,2$.

The smoothness of functions and (49) are only for proving the necessity condition among subsystems having unique maximal solutions. If systems are defined on the positive(or negative) orthant $\mathbb{R}_+^{n_i}$, the assumption $\alpha_i \in \mathscr{O}(>1)$ can be relaxed into $\alpha \in \mathscr{O}(1)$. Theorem 4 implies that both the subsystems Σ_1 and Σ_2 accept α_1, $\alpha_2 \in \mathscr{K}$ whenever their feedback interconnection is GAS, iISS or ISS.

The necessity of $\alpha_1, \alpha_2 \in \mathscr{K}$ does not hold any more in the cascade case. There are pairs of supply rates from which we can derive stability of their cascade even if α_i are only positive definite. A simple example is the cascade connection of $\Sigma_1 \in \mathscr{S}_1(n_1, \alpha_1, \sigma_1, \sigma_{r1})$ and $\Sigma_2 \in \mathscr{S}_2(n_2, \alpha_2, \sigma_{r2})$ satisfying

$$\liminf_{s\to\infty} \alpha_1(s) = 0, \quad \sigma_1 = c\alpha_2, \quad c > 0 \tag{51}$$

since $V(t,x) = V_1(t,x_1) + 2cV_2(t,x_2)$ is an iISS Lyapunov function. In order to obtain a stability condition which is less restrictive than $\sigma_1 = c\alpha_2$, Arcak et al.[2] has proved based on a trajectory approach that a cascade connection can be GAS if the speed α_2 of convergence of the driving system near the equilibrium is fast enough to compensate the growth of input nonlinearity σ_1 of the driven system. For pursuing the improvement of their result, this paper have placed emphasis on the construction of Lyapunov functions. This paper has investigated not only UGAS, but also the stability with respect to external signals, i.e., iISS. The results in this paper are also far beyond the application of small-gain technique in [4] to cascaded systems since all the results expect Theorem 1 and Corollary 1 allow the driven system to have radially vanishing convergence rate. In the GAS case, both the driving and the driven systems are allowed to have radially vanishing convergence rate.

VII. Concluding remarks

This paper has investigated stability of cascade interconnection of subsystems which are not necessary ISS. The interconnection of iISS and GAS subsystems are not always GAS. The problem of establishing GAS was addressed in [2] which derived a trade-off condition between slower convergence of the driving system and steeper input growth of the driven system. The purpose of this paper is to study a similar problem further in order to pursue advanced stability conditions from the perspectives of external signals and construction of Lyapunov functions. This paper has shown that a smooth Lyapunov function for GAS can be constructed explicitly if the trade-off condition holds. This result has also been extended to stability with respect to external signals, i.e., iISS. The results are applicable to time-varying systems. Another result has demonstrated that the trade-off between the convergence and the input growth rate is no more necessary if the convergence rate of the driven system is not radially vanishing. In addition, this paper has discussed the difference between feedback interconnection and cascade interconnection, and the work of this paper has been related to small-gain techniques. It is explained that tools specialized in cascade interconnection can allow the convergence speed of individual subsystems to be radially vanishing.

Finally, it is mentioned that this paper does not show iISS Lyapunov functions in the case of $\alpha_2(\infty)=0$ in contrast to the GAS Lyapunov function in Corollary 2. Recently, the iISS property in the case of $\alpha_2(\infty)=0$ has been proved by [3] without driving a Lyapunov function of the cascade. It would be interesting to continue this direction of research to develop a way to construct a Lyapunov function explicitly.

Appendix

A sketch of proof of Theorem 1

(i) Case (19): Replace σ_{ri} by $\bar{\sigma}_{ri}\in\mathcal{K}$ satisfying

$$\sigma_{ri}(s)\le\bar{\sigma}_{ri}(s),\quad \forall s\in\mathbb{R}_+,\ i=1,2 \tag{52}$$

Let τ_1, τ_{r1} and τ_{r2} be positive real numbers satisfying $\tau_1>1$, $\tau_{r1}>1$, $1>\frac{1}{\tau_1}+\frac{1}{\tau_{r1}}$, $\tau_{r2}>1$, $1-\delta>\frac{1}{\tau_{r2}}$. The existence of such τ_{r2} is ensured by (27). Define

$$\begin{aligned}\theta_1(s)&=\bar{\alpha}_1\circ\hat{\alpha}_1^{-1}\circ\tau_1\hat{\sigma}_1(s),\quad s\in[0,Y_1)\\ \theta_{ri}(s)&=\bar{\alpha}_i\circ\hat{\alpha}_i^{-1}\circ\tau_{ri}\bar{\sigma}_{ri}(s),\ s\in[0,Y_{ri}),\ i=1,2\\ Y_1&=\lim_{s\to\infty}\hat{\sigma}_1^{-1}\circ\tfrac{1}{\tau_1}\hat{\alpha}_1(s),\quad Y_{r2}=\infty\\ Y_{r1}&=\begin{cases}\infty\ , & \text{if } \lim_{s\to\infty}\hat{\alpha}_1(s)\ge\lim_{s\to\infty}\tau_{r1}\bar{\sigma}_{r1}\\ \lim_{s\to\infty}\bar{\sigma}_{r1}^{-1}\circ\frac{1}{\tau_{r1}}\hat{\alpha}_1(s) & \text{, otherwise}\end{cases}\end{aligned}$$

Due to (23)-(27), we can define continuous functions

$$\begin{aligned}\lambda_{\theta 1}(s)&=\begin{cases}\lambda_1\circ\theta_1(s) & ,\quad s\in[0,Y_1)\\ \lim_{s\to\infty}\lambda_1(s) & ,\quad s\in[Y_1,\infty)\end{cases}\\ \lambda_{\theta r1}(s)&=\begin{cases}\lambda_1\circ\theta_{r1}(s) & ,\quad s\in[0,Y_{r1})\\ \lim_{s\to\infty}\lambda_1(s) & ,\quad s\in[Y_{r1},\infty)\end{cases}\\ \lambda_{\theta r2}(s)&=\lambda_2\circ\theta_{r2}(s)\ ,\quad s\in\mathbb{R}_+\end{aligned}$$

which are non-decreasing. We can verify the following:

$$\begin{aligned}&\lambda_1(V_1)\{-\alpha_1(|x_1|)+\sigma_1(|x_2|)+\sigma_{r1}(|r_1|)\}\le\\ &\quad-\left(1-\tfrac{1}{\tau_1}-\tfrac{1}{\tau_{r1}}\right)\lambda_1(\underline{\alpha}_1(|x_1|))\hat{\alpha}_1(|x_1|)\\ &\quad+\lambda_{\theta 1}(|x_2|)\hat{\sigma}_1(|x_2|)+\lambda_{\theta r1}(|r_1|)\bar{\sigma}_{r1}(|r_1|)\end{aligned}\tag{53}$$

$$\begin{aligned}&\lambda_2(V_2)\{-\alpha_2(|x_2|)+\sigma_{r2}(|r_2|)\}\le\\ &\quad-\left(1-\tfrac{1}{\tau_{r2}}\right)\lambda_2(\underline{\alpha}_2(|x_2|))\alpha_2(|x_2|)+\lambda_{\theta r2}(|r_2|)\bar{\sigma}_{r2}(|r_2|)\end{aligned}\tag{54}$$

Then, λ_1 and λ_2 in (23) and (24) satisfy

$$\begin{aligned}&\sum_{i=1}^{2}\lambda_i(V_i)\{-\alpha_i(|x_i|)+\sigma_i(|x_{3-i}|)+\sigma_{ri}(|r_i|)\}\\ &\qquad\le-\sum_{i=1}^{2}\alpha_{o,i}(|x_i|)+\sigma_{o,i}(|r_i|)\end{aligned}\tag{55}$$

$$\alpha_{o,1}(s)=\left(1-\tfrac{1}{\tau_1}-\tfrac{1}{\tau_{r1}}\right)\lambda_1(\underline{\alpha}_1(s))\hat{\alpha}_1(s)\tag{56}$$

$$\alpha_{o,2}(s)=\left(1-\tfrac{1}{\tau_{r2}}-\delta\right)\lambda_2(\underline{\alpha}_2(s))\alpha_2(s)\tag{57}$$

$$\sigma_{o,i}(s)=\lambda_{\theta ri}(s)\bar{\sigma}_{ri}(s),\quad i=1,2\tag{58}$$

The property (55) implies that $V(t,x)$ defined by (22) is an iISS Lyapunov function.

Case (20): The property $\hat{\alpha}_1(\infty)<\infty$ follows from $\sigma_1(\infty)<\infty$ and (30)-(32). The property (28) guarantees $\lambda_i(\infty)<\infty$, $i=1,2$, which allows us to replace $\lambda_{\theta ri}(s)$ by $\lambda_i(\infty)$ for $i=1,2$. Choose $\sigma_{o,i}=\lambda_i(\infty)\bar{\sigma}_{ri}\in\mathcal{K}$, and replace (54) and (57) by

$$\begin{aligned}&\lambda_2(V_2)\{-\alpha_2(|x_2|)+\sigma_{r2}(|r_2|)\}\\ &\qquad\le-\lambda_2(\underline{\alpha}_2(|x_2|))\alpha_2(|x_2|)+\lambda_2(\infty)\bar{\sigma}_{r2}\\ &\alpha_{o,2}(s)=(1-\delta)\lambda_2(\underline{\alpha}_2(s))\alpha_2(s)\end{aligned}$$

Case (21): Let $\bar{\sigma}_{r2}$ be a class $\mathcal{K}$ function satisfying

$$\sup_{s\in\mathbb{R}_+}\sigma_{r2}(s)\le\lim_{s\to\infty}\bar{\sigma}_{r2}(s)<(1-\delta)\lim_{s\to\infty}\alpha_2(s)\tag{59}$$

and (52). The existence is guaranteed by (29). Define

$$\tau_{r2}=\lim_{s\to\infty}\alpha_2(s)/\lim_{s\to\infty}\bar{\sigma}_{r2}(s)>1\tag{60}$$

The property (59) yields $1-\delta>\frac{1}{\tau_{r2}}$.

(ii) Under (33) and (34), we have $\alpha_{o,1},\alpha_{o,2}\in\mathcal{K}_\infty$.

A key to proofs of Corollary 1 and Corollary 2

If $\alpha_2\notin\mathcal{K}$, consider $\hat{V}_2(t,x_2)=\int_0^{V_2(t,x_2)}\theta\circ\underline{\alpha}_2^{-1}(s)ds$ with $\theta(s)=1$, $\forall s\in[0,\bar{\alpha}_2^{-1}(1)]$ and $\inf_{s\in\mathbb{R}_+}\theta(s)>0$. There exists a continuous function θ on $\mathbb{R}_+$ satisfying $\dot{\hat{V}}_2\le-\hat{\alpha}_2(|x_2|)$ along the trajectories of Σ_2 for some $\hat{\alpha}_2\in\mathcal{K}$.

A sketch of proof of Theorem 3

Due to $k \geq 1$, (19), (20), (21) and (44), we have $0 < 1/k < h < \infty$. The properties (40) and (41) guarantee that V defined by (22), (42) and (43) is positive definite and radially unbounded. Let $\bar{\sigma}_{r1}, \bar{\sigma}_{r2} \in \mathscr{K}$ be (52). Suppose $k > 1$. Pick $\zeta > (k-1)^{\frac{k-1}{k}}$. Young's inequality and (42) yield

$$\begin{aligned}&\lambda_1(V_1)\{-\alpha_1(|x_1|)+\sigma_1(|x_2|)+\sigma_{r1}(|r_1|)\}\\ &\leq -\left\{\tfrac{1}{k}-\tfrac{k-1}{k}\left(\tfrac{1}{\zeta}\right)^{\frac{k}{k-1}}\right\}[\min_{w\in[\bar{\alpha}_1^{-1}(V_1),\underline{\alpha}_1^{-1}(V_1)]}\alpha_1(w)]^k\\ &\quad+\tfrac{1}{k}[\sigma_1\circ\underline{\alpha}_2^{-1}(V_2)]^k+\tfrac{1}{k}[\zeta\bar{\sigma}_{r1}(|r_1|)]^k \end{aligned} \tag{61}$$

In the case of $k = 1$, we obtain

$$\begin{aligned}&\lambda_1(V_1)\{-\alpha_1(|x_1|)+\sigma_1(|x_2|)+\sigma_{r1}(|r_1|)\}\\ &\leq -\alpha_1(|x_1|)+\sigma_1\circ\underline{\alpha}_2^{-1}(V_2)+\bar{\sigma}_{r1}(|r_1|)\end{aligned} \tag{62}$$

If $\lim_{s\to\infty}\sigma_1(s) < \infty$, $\alpha_2 \in \mathscr{K}$ and (43) imply that, for each $k \geq 1$, there exists $h_{r2} > 0$ satisfying

$$\begin{aligned}&\lambda_2(V_2)\{-\alpha_2(|x_2|)+\sigma_{r2}(|r_2|)\}\\ &\leq -h[\sigma_1\circ\underline{\alpha}_2^{-1}(V_2)]^k+h_{r2}\bar{\sigma}_{r2}(|r_2|)\}\end{aligned} \tag{63}$$

Next, suppose $\sigma_1(\infty) = \infty$. There exists $\tau_{r2} > 1$ satisfying

$$1-\tfrac{1}{\tau_{r2}}-\tfrac{1}{hk} > 0 \tag{64}$$

If $\lim_{s\to\infty}\alpha_2(s) < \infty$, pick $\bar{\sigma}_{r2} \in \mathscr{K}$ satisfying (52) and

$$\sup_{s\in\mathbb{R}_+}\sigma_{r2}(s) \leq \lim_{s\to\infty}\bar{\sigma}_{r2}(s) < \left(1-\tfrac{1}{hk}\right)\lim_{s\to\infty}\alpha_2(s)$$

The existence is ensured by (44). Then, $\tau_{r2} > 1$ in (60) fulfills (64). Define a non-decreasing function

$$\lambda_{\theta r2}(s) = \lambda_2\circ\bar{\alpha}_2\circ\alpha_2^{-1}\circ\tau_{r2}\bar{\sigma}_{r2}(s), \quad s\in\mathbb{R}_+$$

From (43) and $\alpha_2 \in \mathscr{K}$, for each $k \geq 1$, we obtain

$$\begin{aligned}&\lambda_2(V_2)\{-\alpha_2(|x_2|)+\sigma_{r2}(|r_2|)\} \leq\\ &-h\left(1-\tfrac{1}{\tau_{r2}}\right)[\sigma_1\circ\underline{\alpha}_2^{-1}(V_2)]^k+\lambda_{\theta r2}(|r_2|)\bar{\sigma}_{r2}(|r_2|)\end{aligned} \tag{65}$$

in the case of $\lim_{s\to\infty}\sigma_1(s) = \infty$. Using (61), (62), (63), (65) and some $\alpha_{o,1} \in \mathscr{P}$, $\sigma_{o,2} \in \mathscr{K}$, $\eta_2, \xi_1 > 0$, we arrive at

$$\begin{aligned}&\lambda_1(V_1)\{-\alpha_1(|x_1|)+\sigma_1(|x_2|)+\sigma_{r1}(|r_1|)\}+\\ &\lambda_2(V_2)\{-\alpha_2(|x_2|)+\sigma_{r2}(|r_2|)\} \leq -\alpha_{o,1}(V_1)\\ &\quad-\eta_2[\sigma_1\circ\underline{\alpha}_2^{-1}(V_2)]^k+\xi_1[\bar{\sigma}_{r1}(|r_1|)]^k+\sigma_{o,2}(|r_2|)\end{aligned}$$

Hence, V is an iISS Lyapunov function of Σ. If $\sigma_{r2}(s) \equiv 0$ or $r_2(t) \equiv 0$, τ_{r2} and $\lambda_{\theta r2}$ vanish. The pair (45)-(46) ensures $\lim_{V_2\to 0}\lambda_2(V_2)\alpha_2(\bar{\alpha}_2^{-1}(V_2)) = 0$ and that V is positive definite and radially unbound.

A sketch of proof of Theorem 4

Consider the feedback system Σ_F with $r_i(t) \equiv 0$, $i = 1,2$. Suppose that $\liminf_{s\to\infty}\alpha_i(s) = 0$ holds for at least one of $i = 1,2$. Due to $\sigma_1, \sigma_2 \in \mathscr{K}$, there exist $l_i > 0$ and $\delta_i > 0$ for $i = 1,2$ such that

$$|x_i| = l_i,\ |x_{3-i}| \geq l_{3-i} \Rightarrow (1+\delta_i)\alpha_i(|x_i|) < \sigma_i(|x_{3-i}|)$$

hold. Using Lemma 1 of [5] , choose a pair $f_1(x_1,u_1,r_1)$, $f_2(x_2,u_2,r_2)$: $\mathbb{R}^{n_i}\times\mathbb{R}^{m_i}\times\mathbb{R}^{k_i}\to\mathbb{R}$ for which there exist $\mathbf{C}^1$ functions V_1, V_2: $\mathbb{R}^{n_i}\to\mathbb{R}$ and $\underline{\alpha}_1$, $\bar{\alpha}_1$, $\underline{\alpha}_2$, $\bar{\alpha}_2 \in \mathscr{K}_\infty$ such that $\Sigma_i \in \mathscr{S}_i(n_i,\alpha_i,\sigma_i,\sigma_{ri})$, $\underline{\alpha}_i(|x_i|) = V_i(x_i) = \bar{\alpha}_i(|x_i|)$ and

$$(1+\delta_i)\alpha_i(|x_i|) < \sigma_i(|x_{3-i}|) \ \Rightarrow\ \tfrac{\partial V_i}{\partial x_i}f_i > \delta_i\alpha_i(|x_i|)$$

hold with $r_i(t) \equiv 0$ for $i = 1,2$. These systems Σ_i, $i = 1,2$, defined with $\dot{x}_i = f_i$ satisfy

$$|x_i| = l_i,\ |x_{3-i}| \geq l_{3-i} \Rightarrow \tfrac{\partial V_i}{\partial x_i}f_i > \delta_i\alpha_i(|x_i|) \tag{66}$$

The pair of (66), $i = 1,2$, implies that trajectories starting from $(x_1(0),x_2(0)) \in \{(x_1,x_2)\in\mathbb{R}^{n_1}\times\mathbb{R}^{n_2} : V_i(x_i) \geq V_i(l_i), i = 1,2\}$ stay there for all $t \in \mathbb{R}_+$. This invariance property implies that Σ_F is not GAS.

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