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Control Lyapunov Functionals and Robust Stabilization of Nonlinear Time-Delay Systems

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Abstract—**In this work, we provide necessary and sufficient Lyapunov-like conditions for the existence of a stabilizing feedback law for uncertain control systems described by Retarded Functional Differential Equations. A methodology for the construction of Control Lyapunov Functionals for uncertain triangular nonlinear time-delay systems is provided. Moreover, the method leads to the explicit design of robust nonlinear controllers for the class of time-delay nonlinear systems with a triangular structure.**

I. Introduction

THE purpose of this paper is to provide a methodology for the construction of Control Lyapunov Functionals for uncertain nonlinear systems described by Retarded Functional Differential Equations of the form:

$$\begin{aligned} &\dot{x}_i(t) = f_i(t, d(t), T_r(t)x_1, ..., T_r(t)x_i) \\ &+ g_i(t, d(t), T_r(t)x_1, ..., T_r(t)x_i)x_{i+1}(t) \quad , \quad i = 1, ..., n-1 \\ &\dot{x}_n(t) = f_n(t, d(t), T_r(t)x) + g_n(t, d(t), T_r(t)x)u(t) \\ &x(t) = (x_1(t), ..., x_n(t)) \in \Re^n \, , d(t) \in D \, , u(t) \in \Re \, , t \geq 0 \end{aligned} \tag{1}$$

where $T_r(t)x$ we denote the "r-history" of x at time t, i.e., $T_r(t)x := x(t+\theta)\,;\, \theta \in [-r,0]$. Systems of the form (1) have been studied in [2,3,4,7,8,9,10]. More specifically, in [3,4] the global stabilization problem for autonomous and disturbance-free systems of the form (1) was studied using Control Lyapunov Razumikhin Functions, while in [8] stabilization with delayed feedbacks was studied under certain growth conditions using Lyapunov functions (see also [7]). In [9] the semiglobal stabilization problem for partially linear delay systems was studied and backstepping methods based on Lyapunov functionals under certain conditions were provided in [2,10] (see also [11]).

In the present work it is shown that the construction of a stabilizing feedback law for (1) proceeds in parallel with the construction of a State Robust Control Lyapunov Functional. Moreover, sufficient conditions for the existence and design of a stabilizing feedback law $u(t) = k(x(t))$, which is independent of the delay are given.

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The notion of the State Robust Control Lyapunov Functional (SRCLF) used in the present work generalizes the notion of the Robust Control Lyapunov Function introduced in [1] for finite-dimensional systems. More specifically, the proposed notion of the SRCLF can be applied to uncertain systems described by RFDEs of the form:

$$\begin{aligned} &\dot{x}(t) = f(t, d(t), T_r(t)x, u(t)) \, , t \geq t_0 \\ &x(t) \in \Re^n \, , \; d(t) \in D \, , u(t) \in U \end{aligned} \tag{2}$$

where $r > 0$ is a constant, $f : \Re^+ \times D \times C^0([-r,0];\Re^n) \times U \to \Re^n$ satisfies $f(t,d,0,0) = 0$ for all $(t,d) \in \Re^+ \times D$, $D \subseteq \Re^l$ is a non-empty compact set, $U \subseteq \Re^m$ is a closed convex set with $0 \in U$ and $T_r(t)x = x(t+\theta)\,;\, \theta \in [-r,0]$. It is shown that the existence of a State Robust Control Lyapunov Functional is a necessary and sufficient condition for the existence of a stabilizing feedback for (2).

Notations Throughout this paper we adopt the following notations:

* Let $A \subseteq \Re^n$ be a set. By $C^0(A;\Omega)$, we denote the class of continuous functions on A, which take values in Ω. By $C^k(A;\Omega)$, where $k \geq 1$ is an integer, we denote the class of differentiable functions on A with continuous derivatives up to order k, which take values in Ω. By $C^\infty(A;\Omega)$, we denote the class of differentiable functions on A having continuous derivatives of all orders, which take values in Ω, i.e., $C^\infty(A;\Omega) = \bigcap_{k \geq 1} C^k(A;\Omega)$.
* A continuous mapping $A \times B \ni (z,x) \to k(z,x) \in \Re^m$, where $B \subseteq \mathcal{X}$, $A \subseteq \mathcal{Y}$ and $\mathcal{X}, \mathcal{Y}$ are normed linear spaces, is called completely locally Lipschitz with respect to $x \in B$ if for every closed and bounded set $S \subseteq A \times B$ it holds that $\sup\left\{ \frac{|k(z,x) - k(z,y)|}{\|x-y\|_{\mathcal{X}}} : (z,x) \in S, (z,y) \in S, x \neq y \right\} < +\infty$. If the normed linear spaces $\mathcal{X}, \mathcal{Y}$ are finite-dimensional spaces then we simply say that the continuous mapping $A \times B \ni (z,x) \to k(z,x) \in \Re^m$ is locally Lipschitz with respect to $x \in B$ if for every compact set $S \subseteq A \times B$ it holds that $\sup\left\{ \frac{|k(z,x) - k(z,y)|}{|x-y|} : (z,x) \in S, (z,y) \in S, x \neq y \right\} < +\infty$.

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* For a vector $x \in \Re^n$ we denote by $|x|$ its usual Euclidean norm and by x' its transpose. For $x \in C^0([-r,0];\Re^n)$ we define $\|x\|_r := \max_{\theta \in [-r,0]} |x(\theta)|$.
* $\Re^+$ denotes the set of non-negative real numbers.
* $\mathcal{E}$ denotes the class of non-negative C^0 functions $\mu : \Re^+ \to \Re^+$, for which it holds: $\int_0^{+\infty} \mu(t)dt < +\infty$ and $\lim_{t \to +\infty} \mu(t) = 0$.
* We denote by K^+ the class of positive C^0 functions defined on $\Re^+$. We say that a function $\rho : \Re^+ \to \Re^+$ is positive definite if $\rho(0) = 0$ and $\rho(s) > 0$ for all $s > 0$. We say that a positive definite, increasing and continuous function $\rho : \Re^+ \to \Re^+$ is of class K_∞ if $\lim_{s \to +\infty} \rho(s) = +\infty$.
* Let $D \subseteq \Re^l$ be a non-empty set. By M_D we denote the class of all Lebesgue measurable and locally essentially bounded mappings $d : \Re^+ \to D$.
* Let $x : [a-r, b) \to \Re^n$ be a continuous mapping with $b > a > -\infty$ and $r > 0$. By $T_r(t)x$ we denote the "r-history" of x at time $t \in [a,b)$, i.e., $T_r(t)x := x(t+\theta)\,;\, \theta \in [-r,0]$. Notice that $T_r(t)x \in C^0([-r,0];\Re^n)$.

II. CONTROL LYAPUNOV FUNCTIONALS

We consider control systems of the form (2) under the following hypotheses:

(S1) The mapping $(x,u,d) \to f(t,d,x,u)$ is continuous for each fixed $t \ge 0$ and such that for every bounded $I \subseteq \Re^+$ and for every bounded $S \subset C^0([-r,0];\Re^n) \times U$, there exists a constant $L \ge 0$ such that:

$$\big(x(0) - y(0)\big)' \big(f(t,d,x,u) - f(t,d,y,u)\big) \le L \|x - y\|_r^2$$
$$\forall t \in I\,,\, \forall (x,u,y,u) \in S \times S\,,\, \forall d \in D$$

(S2) For every bounded $\Omega \subset \Re^+ \times D \times C^0([-r,0];\Re^n) \times U$ the image set $f(\Omega) \subset \Re^n$ is bounded.

(S3) There exists a countable set $A \subset \Re^+$, which is either finite or $A = \{t_k\,;\, k = 1,...,\infty\}$ with $t_{k+1} > t_k > 0$ for all $k = 1,2,...$ and $\lim t_k = +\infty$, such that mapping $(t,x,u,d) \in (\Re^+ \setminus A) \times C^0([-r,0];\Re^n) \times U \times D \to f(t,d,x,u)$ is continuous. Moreover, for each fixed $(t_0,x,u,d) \in \Re^+ \times C^0([-r,0];\Re^n) \times U \times D$, we have $\lim_{t \to t_0^+} f(t,d,x,u) = f(t_0,d,x,u)$.

(S4) For every $\varepsilon > 0$, $t \in \Re^+$, there exists $\delta := \delta(\varepsilon,t) > 0$ such that

$$\sup\left\{ |f(\tau,d,x,u)|\,;\, \tau \in \Re^+\,,\, d \in D, u \in U\,,\, |\tau - t| + \|x\|_r + |u| < \delta \right\} < \varepsilon$$
.

(S5) The mapping $u \to f(t,d,x,u)$ is Lipschitz on bounded sets, in the sense that for every bounded $I \subseteq \Re^+$ and for every bounded $S \subset C^0([-r,0];\Re^n) \times U$, there exists a constant $L_U \ge 0$ such that:

$$|f(t,d,x,u) - f(t,d,x,v)| \le L_U |u - v|$$
$$\forall t \in I\,,\, \forall (x,u,x,v) \in S \times S\,,\, \forall d \in D$$

(S6) The set $D \subset \Re^l$ is compact and $U \subseteq \Re^m$ is a closed convex set.

Let $x \in C^0\big([-r,0];\Re^n\big)$ and $V : \Re^+ \times C^0\big([-r,0];\Re^n\big) \to \Re$ be a locally bounded functional. By $E_h(x;v)$, where $0 \le h < r$ and $v \in \Re^n$ we denote the following operator:

$$E_h(x;v) := \begin{cases} x(0) + (\theta + h)v & \text{for} \quad -h < \theta \le 0 \\ x(\theta + h) & \text{for} \quad -r \le \theta \le -h \end{cases} \tag{3}$$

and we define

$$V^0(t,x;v) := \limsup_{\substack{h \to 0^+ \\ y \to 0,\, y \in C^0([-r,0];\Re^n)}} \frac{V(t+h, E_h(x;v) + hy) - V(t,x)}{h} \tag{4}$$

An important class of functionals is presented next (see [5] for more details).

Definition 2.1: *We say that a continuous functional $V : \Re^+ \times C^0([-r,0];\Re^n) \to \Re^+$, is "almost Lipschitz on bounded sets", if there exist non-decreasing functions $L_V : \Re^+ \to \Re^+$, $P : \Re^+ \to \Re^+$, $G : \Re^+ \to [1,+\infty)$ such that for all $R \ge 0$, the following properties hold:*

(P1) *For every $x, y \in \{x \in C^0([-r,0];\Re^n)\,;\, \|x\|_r \le R\}$, it holds that:*

$$|V(t,y) - V(t,x)| \le L_V(R)\, \|y - x\|_r\,,\ \forall t \in [0,R]$$

(i.e., the mapping $\Re^+ \times C^0([-r,0];\Re^n) \ni (t,x) \to V(t,x) \in \Re^+$ is completely locally Lipschitz with respect to $x \in C^0([-r,0];\Re^n)$)

(P2) *For every absolutely continuous function $x:[-r,0] \to \Re^n$ with $\|x\|_r \le R$ and essentially bounded derivative, it holds that:*

$$\left|V(t+h,x)-V(t,x)\right| \le hP(R)\left(1+\sup_{-r\le\tau\le 0}\left|\dot{x}(\tau)\right|\right),$$
$$\text{for all } t\in[0,R] \text{ and } 0 \le h \le \frac{1}{G\left(R+\sup_{-r\le\tau\le 0}\left|\dot{x}(\tau)\right|\right)}$$

The reader should notice that for functionals $V:\Re^+ \times C^0([-r,0];\Re^n) \to \Re^+$, which are almost Lipschitz on bounded sets we obtain the following simplification for the derivative $V^0(t,x;v)$ defined by (4) for all $(t,x,v) \in \Re^+ \times C^0\left([-r,0];\Re^n\right) \times \Re^n$:

$$V^0(t,x;v) = \limsup_{h\to 0^+} \frac{V(t+h,E_h(x;v))-V(t,x)}{h}$$

We next give the definition of the Output Robust Control Lyapunov Functional for system (2).

Definition 2.2: *We say that (2) admits a* ***State Robust Control Lyapunov Functional (SRCLF)*** *if there exists an almost Lipschitz on bounded sets functional $V:\Re^+ \times C^0([-r,0];\Re^n) \to \Re^+$ (called the Output Control Lyapunov Functional), which satisfies the following properties:*

(i) *There exist functions $a_1, a_2 \in K_\infty$, $\beta \in K^+$ such that the following inequality holds for all $(t,x) \in \Re^+ \times C^0\left([-r,0];\Re^n\right)$*

$$a_1\left(\|x\|_r\right) \le V(t,x) \le a_2\left(\beta(t)\|x\|_r\right) \tag{5}$$

(ii) *There exists a function $\Psi:\Re^+ \times \Re^q \times U \to \Re \cup \{+\infty\}$ with $\Psi(t,0,0)=0$ for all $t \ge 0$ such that for each $u \in U$ the mapping $(t,\varphi) \to \Psi(t,\varphi,u)$ is upper semi-continuous, a function $q \in \mathcal{E}$, a continuous mapping $\Re^+ \times C^0([-r,0];\Re^n) \ni (t,x) \to \Phi(t,x) \in \Re^p$ being completely locally Lipschitz with respect to $x \in C^0([-r,0];\Re^n)$ with $\Phi(t,0)=0$ for all $t \ge 0$ and a C^0 positive definite function $\rho:\Re^+ \to \Re^+$ such that the following inequality holds:*

$$\inf_{u\in U} \Psi(t,\varphi,u) \le q(t),$$
$$\forall t \ge 0, \forall \varphi = (\varphi_1,...,\varphi_p)' \in \Re^p \tag{6}$$

Moreover, for every finite set $\{u_1,u_2,...,u_N\} \subset U$ and for every $\lambda_i \in [0,1]$ ($i=1,...,N$) with $\sum_{i=1}^N \lambda_i = 1$, it holds that:

$$\sup_{d\in D} V^0\left(t,x; f\left(t,d,x,\sum_{i=1}^N \lambda_i u_i\right)\right) \le -\rho\left(V(t,x)\right),$$
$$+\max\left\{\Psi(t,\Phi(t,x),u_i)\,,\, i=1,...,N\right\}$$
$$\forall (t,x) \in \Re^+ \times C^0\left([-r,0];\Re^n\right) \tag{7}$$

If in addition to the above there exist $a \in K_\infty$, $\gamma \in K^+$ such that for every $(t,\varphi) \in \Re^+ \times \Re^q$ there exists $u \in U$ with $|u| \le a\left(\gamma(t)|\varphi|\right)$ such that

$$\Psi(t,\varphi,u) \le q(t) \tag{8}$$

then we say that $V:\Re^+ \times C^0([-r,0];\Re^n) \to \Re^+$ satisfies the "small-control" property.

The feedback stabilization problem for (2) is the problem of existence/design of a continuous mapping $\Re^+ \times C^0([-r,0];\Re^n) \ni (t,x) \to k(t,x) \in U$ being completely locally Lipschitz with respect to $x \in C^0([-r,0];\Re^n)$ with $k(t,0)=0$ for all $t \ge 0$, such that $0 \in C^0([-r,0];\Re^n)$ is Robustly Globally Asymptotically Stable (RGAS) for the closed-loop system (2) with

$$u = k(t, T_r(t)x) \tag{9}$$

in the sense that the following properties hold for the solution $x(t,t_0,x_0,d)$ of the closed-loop system (2) with (9) initiated from $x_0 \in C^0([-r,0];\Re^n)$ at time $t_0 \ge 0$ and corresponding to input $d \in M_D$:

P1(Stability) *For every $\varepsilon > 0$, $T \ge 0$ it holds that*

$$\sup\left\{\|x(t,t_0,x_0,d)\|_r \,;\, t \ge t_0 \,,\, \|x_0\|_r \le \varepsilon \,,\, t_0 \in [0,T] \,,\, d \in M_D\right\} < +\infty$$

and there exists a $\delta := \delta(\varepsilon,T) > 0$ such that

$\|x_0\|_r \le \delta$, $t_0 \in [0,T] \Rightarrow \|x(t,t_0,x_0,d)\|_r \le \varepsilon$, $\forall t \ge t_0$, $\forall d \in M_D$

P2(Attractivity on bounded sets of initial data) *For every $\varepsilon > 0$, $T \ge 0$ and $R \ge 0$, there exists a $\tau := \tau(\varepsilon, R, T) \ge 0$, such that*
$\|x_0\|_r \le R$, $t_0 \ge 0 \Rightarrow \|x(t,t_0,x_0,d)\|_r \le \varepsilon$, $\forall t \ge t_0 + \tau$, $\forall d \in M_D$

Moreover, if δ and τ involved in properties P1 and P2 above are independent of $T \ge 0$ and

$$\sup\left\{\|x(t,t_0,x_0,d)\|_r \; ; t \ge t_0 \, , \; \|x_0\|_r \le \varepsilon \, , t_0 \ge 0 \, , d \in M_D \right\} < +\infty$$

then we say that $0 \in C^0([-r,0];\Re^n)$ is Uniformly Robustly Globally Asymptotically Stable (URGAS) for the closed-loop system (2) with (9).

We next present results for systems of the form (2) (see [6]) which show that the existence of a SRCLF is a necessary and sufficient condition for the existence of a continuous mapping $\Re^+ \times C^0([-r,0];\Re^n) \ni (t,x) \to k(t,x) \in U$ being completely locally Lipschitz with respect to $x \in C^0([-r,0];\Re^n)$ with $k(t,0) = 0$ for all $t \ge 0$, such that $0 \in C^0([-r,0];\Re^n)$ is (Uniformly) Robustly Globally Asymptotically Stable ((U)RGAS) for the closed-loop system (2) with (9).

Theorem 2.3: *Consider system (2) under hypotheses (S1-6). The following statements are equivalent:*

(a) *There exists a continuous mapping $\Re^+ \times C^0([-r,0];\Re^n) \ni (t,x) \to k(t,x) \in U$ being completely locally Lipschitz with respect to $x \in C^0([-r,0];\Re^n)$ with $k(t,0) = 0$ for all $t \ge 0$, such that the $0 \in C^0([-r,0];\Re^n)$ is RGAS for closed-loop system (2) with $u = k(t, T_r(t)x)$.*

(b) *System (2) admits a SRCLF, which satisfies the small control property with $q(t) \equiv 0$.*

(c) *System (2) admits a SRCLF.*

Theorem 2.4: *Consider system (2) under hypotheses (S1-6). The following statements are equivalent:*

(a) *System (2) admits a SRCLF, which satisfies the small-control property and inequalities (5), (8) with $\beta(t) \equiv 1$, $q(t) \equiv 0$. Moreover, there exist continuous mappings $\eta \in K^+$, $A \ni (t,\varphi) \to K(t,\varphi) \in U$ where $A = \bigcup_{t \ge 0} \{t\} \times \{\varphi \in \Re^p : |\varphi| < 4\eta(t)\}$ being locally Lipschitz with respect to φ with $K(t,0) = 0$ for all $t \ge 0$ and such that*

$$\Psi(t, \Phi(t,x), K(t, \Phi(t,x))) \le 0 \,,$$
$$\text{for all } (t,x) \in \Re^+ \times C^0([-r,0];\Re^n)$$
$$\text{with } |\Phi(t,x)| \le 2\eta(t) \tag{10}$$

where $\Phi = (\Phi_1,...,\Phi_p)' : \Re^+ \times C^0([-r,0];\Re^n) \to \Re^p$ and $\Psi : \Re^+ \times \Re^p \times U \to \Re \cup \{+\infty\}$ are the mappings involved in property (ii) of Definition 2.2.

(b) *There exists a continuous mapping $\Re^+ \times C^0([-r,0];\Re^n) \ni (t,x) \to k(t,x) \in U$ being completely locally Lipschitz with respect to $x \in C^0([-r,0];\Re^n)$ with $k(t,0) = 0$ for all $t \ge 0$, such that $0 \in C^0([-r,0];\Re^n)$ is URGAS for the closed-loop system (2) with $u = k(t, T_r(t)x)$.*

III. APPLICATIONS TO TRIANGULAR TIME-DELAY CONTROL SYSTEMS

Our main result concerning triangular time-delay control systems of the form (1) is stated next. It must be compared to Theorem 5.1 in [1], which deals with the triangular finite-dimensional case.

Theorem 3.1: *Consider system (1), where $r > 0$, $D \subset \Re^l$ is a compact set, the mappings $f_i : \Re^+ \times D \times C^0([-r,0];\Re^i) \to \Re$, $g_i : \Re^+ \times D \times C^0([-r,0];\Re^i) \to \Re$ ($i = 1,...,n$) are continuous with $f_i(t,d,0) = 0$ for all $(t,d) \in \Re^+ \times D$ and each $g_i : \Re^+ \times D \times C^0([-r,0];\Re^i) \to \Re$ ($i = 1,...,n$) is completely locally Lipschitz with respect to $x \in C^0([-r,0];\Re^i)$. Suppose that there exists a function $\varphi \in C^\infty(\Re^+;(0,+\infty))$ being non-decreasing, such that for every $i = 1,...,n$, it holds that:*

$$\frac{1}{\varphi\left(\|x\|_r\right)} \le g_i(t,d,x) \le \varphi\left(\|x\|_r\right),$$
$$\forall (t,x,d) \in \Re^+ \times C^0([-r,0];\Re^i) \times D \tag{11}$$

Moreover, suppose that for every $i = 1,...,n$, it holds that

$$\sup\left\{\frac{|f_i(t,d,x) - f_i(t,d,y)|}{\|x-y\|_r} : (t,d) \in \Re^+ \times D, x \in S, y \in S, x \ne y\right\} < +\infty,$$
$$\text{for every bounded } S \subset C^0([-r,0];\Re^i) \tag{12}$$

Then for every $\sigma > 0$ there exist functions $\mu_i \in C^\infty(\Re^i;(0,+\infty))$, $k_i \in C^\infty(\Re^i;\Re)$ ($i = 1,...,n$) with

$$k_1(\xi_1) := -\mu_1(\xi_1)\,\xi_1 \quad (13)$$

$$k_j(\xi_1,...,\xi_j) := -\mu_j(\xi_1,...,\xi_j)\big(\xi_j - k_{j-1}(\xi_1,...,\xi_{j-1})\big), \quad j = 2,...,n \quad (14)$$

such that the following functional:

$$V(x) := \max_{\theta\in[-r,0]} \exp(2\sigma\theta)\left(x_1^2(\theta) + \sum_{j=2}^{n} \left| x_j(\theta) - k_{j-1}(x_1(\theta),...,x_{j-1}(\theta)) \right|^2 \right) \quad (15)$$

is a State Robust Control Lyapunov Functional (SRCLF) for (1), which satisfies the "small-control" property. Moreover, the closed-loop system (1) with $u(t) = k_n(x(t))$ is URGAS. More specifically, the inequality $V^0(x;v) \le -2\sigma V(x)$ holds for all $(t,x,d) \in \Re^+ \times C^0([-r,0];\Re^n) \times D$ with

$$v = \begin{pmatrix} f_1(t,d,x_1) + g_1(t,d,x_1)x_2(0) \\ \vdots \\ f_n(t,d,x) + g_n(t,d,x)k_n(x(0)) \end{pmatrix} \in \Re^n .$$

Remark 3.2: The reader should notice that the feedback law $u(t) = k_n(x(t))$ is delay-independent. The proof of Theorem 3.1 shows that the functions $\mu_i \in C^\infty(\Re^i;(0,+\infty))$ ($i = 1,...,n$) are obtained by a procedure similar to the backstepping procedure used for finite-dimensional triangular control systems. Consequently, as in the finite-dimensional case, the feedback design and the construction of the State Robust Control Lyapunov Functional proceed in parallel.

The algorithm for the construction of the functions $\mu_i \in C^\infty(\Re^i;(0,+\infty))$, $k_i \in C^\infty(\Re^i;\Re)$ ($i = 1,...,n$) is described next. Notice that inequality (12) in conjunction with the fact that $f_i(t,d,0) = 0$ for all $(t,d) \in \Re^+ \times D$ ($i = 1,...,n$) implies the existence of a non-decreasing function $L \in C^\infty(\Re^+;(0,+\infty))$ such that for every $i = 1,...,n$, it holds:

$$|f_i(t,d,x)| \le L(\|x\|_r)\|x\|_r, \quad \forall (t,x,d) \in \Re^+ \times C^0([-r,0];\Re^i) \times D \quad (16)$$

Let $\sigma > 0$ be a given number. We define the functions $\mu_i \in C^\infty(\Re^i;(0,+\infty))$, $\gamma_i \in C^\infty(\Re^+;(0,+\infty))$, $b_i \in C^\infty(\Re^+;(0,+\infty))$ ($i = 1,...,n$) using the following algorithm:

Step $i = 1$: We define:

$$\mu_1(\xi_1) := \frac{\gamma_1(1+\xi_1^2) + n\sigma}{b_1(1+\xi_1^2)} \quad (17)$$

where

$$\gamma_1(s) := \exp(\sigma r)L\big(s\exp(\sigma r)\big) + \varphi\big(s\exp(\sigma r)\big) \quad (18)$$

$$b_1(s) := \frac{1}{\varphi\big(s\exp(\sigma r)\big)} \quad (19)$$

Step $i \ge 2$: Based on the knowledge of the functions $\mu_j \in C^\infty(\Re^j;(0,+\infty))$, $\gamma_j \in C^\infty(\Re^+;(0,+\infty))$, $b_j \in C^\infty(\Re^+;(0,+\infty))$ ($j = 1,...,i-1$) we will define the functions $\mu_i \in C^\infty(\Re^i;(0,+\infty))$, $\gamma_i \in C^\infty(\Re^+;(0,+\infty))$, $b_i \in C^\infty(\Re^+;(0,+\infty))$. First define

$$\delta_j(\xi_1,...,\xi_j) := \left|\nabla k_j(\xi_1,...,\xi_j)\right|\big(1 + \mu_1(\xi_1) + ... + \mu_j(\xi_1,...,\xi_i)\big), \quad j = 1,...,i-1 \quad (20)$$

and

$$\gamma_i(s) := i\exp(\sigma r)L\big(is\exp(\sigma r)B_i(is\exp(\sigma r))\big)B_i(is\exp(\sigma r)) + \varphi\big(is\exp(\sigma r)B_i(is\exp(\sigma r))\big) \quad (21)$$

$$b_i(s) := \frac{1}{\varphi\big(is\exp(\sigma r)B_i(is\exp(\sigma r))\big)} \quad (22)$$

where $B_i \in C^\infty(\Re^+;(0,+\infty))$ is a non-decreasing function that satisfies:

$$B_i(s) \ge \max\left\{ 1 + \sum_{j=1}^{i-1}\mu_j(\xi_1,...,\xi_j) : |\xi_1| + \sum_{j=2}^{i}\left|\xi_j - k_{j-1}(\xi_1,...,\xi_{j-1})\right| \le s \right\} \quad \text{for all } s \ge 0 \quad (23)$$

Let functions $\rho_j \in C^\infty(\Re^+;(0,+\infty))$ ($j = 1,...,i$) be such that the following inequalities hold for all $s \ge s' \ge 0$:

$$b_j(s) - b_j(s') + s\gamma_j(s) - s'\gamma_j(s') \le (s-s')\rho_j(s) \quad (24)$$

We define:

$$\mu_2(\xi_1,\xi_2) = b_2^{-1}(p)\big[(n-1)\sigma + \gamma_2(p) + \gamma_1(p)\delta_1(\xi_1)\big] + \frac{1}{\sigma}b_2^{-1}(p)\Big[\gamma_2^2(p) + \gamma_1^2(p)\delta_1^2(\xi_1) + \big(1+\mu_1^2(\xi_1)\xi_1^2\big)\rho_1^2(p)\Big] \quad (25)$$

for the case $i = 2$, where

$$p := 1 + \xi_1^2 + \left|\xi_2 - k_1(\xi_1)\right|^2 \tag{26}$$

and

$$\begin{aligned}
\mu_i(\xi_1,...\xi_i) &:= b_i^{-1}(p)\left[(n+1-i)\sigma + \gamma_i(p) + \left(\sum_{k=1}^{i-1}\gamma_k(p)\right)\delta_{i-1}(\xi_1,...\xi_{j-1})\right] \\
&+ \frac{5}{4\sigma}b_i^{-1}(p)\left[(i-1)\gamma_i^2(p) + \sum_{j=2}^{i-1}(\xi_j - k_{j-1}(\xi_1,...\xi_{j-1}))^2\rho_j^2(p)\mu_j^2(\xi_1,...\xi_j)\right] \\
&+ \frac{5}{4\sigma}b_i^{-1}(p)\left[(i-1)\left(\sum_{k=1}^{i-1}\gamma_k(p)\right)^2\delta_{i-1}^2(\xi_1,...\xi_{j-1}) + \left(\sum_{j=1}^{i-1}\rho_j^2(p)\right)\right] \\
&+ \frac{5}{4\sigma}b_i^{-1}(p)\left[\sum_{j=2}^{i-1}\left(\sum_{k=1}^{j-1}\rho_k(p)\right)^2\delta_{j-1}^2(\xi_1,...\xi_{j-1}) + \xi_1^2\rho_1^2(p)\mu_1^2(\xi_1)\right]
\end{aligned} \tag{27}$$

for the case $i > 2$, where

$$p := \frac{i}{2} + \left(\xi_1^2 + \sum_{j=2}^{i}\left|\xi_j - k_{j-1}(\xi_1,...,\xi_{j-1})\right|^2\right) \tag{28}$$

The proof of Theorem 3.1 is based on the following lemma. Its proof is omitted due to space limitations. The reader should notice that Lemma 3.3 in conjunction with definition (15) of the SRCLF for system (1) indicate one complication frequently encountered in the study of infinite-dimensional systems: although the differential equations (1) are affine in the control input $u \in \Re$, the derivative $V^0(x;v)$, where

$$v = \begin{pmatrix} f_1(t,d,x_1) + g_1(t,d,x_1)x_2(0) \\ \vdots \\ f_n(t,d,x) + g_n(t,d,x)u \end{pmatrix} \in \Re^n$$

is not affine in the control input $u \in \Re$.

Lemma 3.3: *Let $Q \in C^1(\Re^n;\Re^+)$, $\sigma > 0$ and consider the functional $V : C^0([-r,0];\Re^n) \to \Re^+$ defined by:*

$$V(x) := \max_{\theta\in[-r,0]} \exp(2\sigma\theta)Q(x(\theta)) \tag{29}$$

The functional $V : C^0([-r,0];\Re^n) \to \Re^+$ defined by (29), is Lipschitz on bounded sets of $C^0([-r,0];\Re^n)$ and satisfies:

$$\begin{gathered} V^0(x;v) \le -2\sigma V(x) \\ \text{for all } (x,v) \in C^0([-r,0];\Re^n)\times\Re^n \\ \text{with } Q(x(0)) < V(x) \end{gathered} \tag{30}$$

$$\begin{gathered} V^0(x;v) \le \max\{-2\sigma V(x), \nabla Q(x(0))v\} \\ \text{for all } (x,v) \in C^0([-r,0];\Re^n)\times\Re^n \\ \text{with } Q(x(0)) = V(x) \end{gathered} \tag{31}$$

IV. CONCLUSIONS

The case of uncertain control systems described by RFDEs of the form (2) is studied. It is shown that the existence of a SRCLF is a necessary and sufficient condition for the existence of a stabilizing feedback law. Special results are developed for the triangular case (1) of control systems described by RFDEs. It is shown that the construction of a stabilizing feedback law for (1) proceeds in parallel with the construction of a State Robust Control Lyapunov Functional. Moreover, sufficient conditions for the existence and design of a delay-free stabilizing feedback law are given.

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