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Stability and Robust Stabilization for Uncertain Discrete Stochastic Hybrid Singular Systems with Time-Delay

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Abstract— **The stochastic stability and robust stochastic stabilization for time-delay discrete Markovian jump singular systems with parameter uncertainties are discussed. Based on stochastic Lyapunov functional, a delay-dependent linear matrix inequalities (LMIs) condition for the time-delay discrete Markovian jump singular systems to be regular, causal and stochastically stable is given. With this condition, the problem of robust stochastic stabilization is solved. A numerical example to illustrate the effectiveness of the method is given in the paper.**

I. INTRODUCTION

In practice, many dynamical systems can not be represented by the class of linear time-invariant model since the dynamics of these systems is random with some features, for example, abrupt changes, breakdowns of components, changes in the interconnections of subsystems, etc. Such class of dynamical systems can be adequately described by the class of stochastic hybrid systems. A special class of hybrid systems referred to as Markovian jump systems, systems with random structures, has attracted a lot of researchers and many problems have been solved, such as stability, stabilization and H_∞ control problems, see [1-4]. On the other hand, time-delay is commonly encountered in various engineering systems and is frequently a source of instability and poor performance. The robust stability, robust stabilization and H_∞ control for uncertain time-delay discrete Markovian jump linear systems have been extensively investigated [1-4]. Commonly, the approaches to solve time-delay systems can be classified into two types: delay-dependent condition, which include information on the size of delays [1,5-7], and delay-independent condition, which are applicable to delays of arbitrary size [2-4]. Since the stability of systems depends explicitly on the time-delay, delay-independent conditions are more conservative, especially for small delays, while delay-dependent conditions ones are usually less conservative.

Singular systems, which are also referred to as implicit systems, descriptor systems, have comprehensive practical background, they are more general representation than state-space systems. Great progress has been made in the theory and applications on the class of systems since 1970s [8,9]. In recent years, much attention has been focussed on robust stability, robust stabilization and H_∞ control problems for singular systems [10-19]. Xu and Lam [10], and Ma and Cheng [11] gave some results on robust stability and robust stabilization for discrete singular systems. The H_∞ control problem for time-delay singular systems were investigated in [12-16]. For example, Xu, Lam and Yang [15] solved the robust stabilization and H_∞ control problems based on the delay-independent not strict linear matrix inequality condition, Ma et al [16] solved the robust H_∞ control problem based on state-control pairs transformation and the delay-dependent LMI. For Markovian jump singular systems, the stability for discrete Markovian jump singular systems was discussed in [17]. Boukas et al [18] and Boukas [19] discussed the stability and stabilizability and output feedback control for continue-time Markovian jump singular systems, respectively. To the best of our knowledge, the delay-dependent conditions for stochastic stability and robust stochastic stabilization problems for time-delay discrete Markovian jump singular systems have not been investigated in the literature.

In this paper, the stochastic stability and robust stochastic stabilization for time-delay discrete Markovian jump singular systems with parameter uncertainties are discussed. A delay-dependent linear matrix inequalities (LMIs) condition for the time-delay discrete Markovian jump singular systems to be regular, causal and stochastically stable is given. With this condition, the problem of robust stochastic stabilization is solved, and the delay-dependent LMIs condition is obtained. A numerical example to illustrate the effectiveness of the method is given in the paper.

Notations: Throughout this paper, for real symmetric matrices X and Y, the notation $X \geq Y$ (respectively, $X > Y$) means that the matrix $X - Y$ is semipositive definite (respectively, positive definite). I is the identity matrix with appropriate dimension, the superscript "T" represents the transpose, diag$\{\cdots\}$ denotes a block-diagonal matrix. $\|x\|$ refers to Euclidean norm of the vector x, that is $\|x\|^2 = x^T x$. $\mathcal{Z}$ denotes the set of non-negative integer numbers, and $\mathbf{E}\{\cdot\}$ denotes the mathematical expectation. $*$ denotes the matrix entries implied by symmetry of a matrix.

II. DESCRIPTION OF PROBLEM

The time-delay discrete Markovian jump singular system considered in this paper is described by the following

$$\begin{cases} Ex_{k+1} = A(k, r_k)x_k + A_d(k, r_k)x_{k-d} + B(k, r_k)u_k, \\ x_k = \phi(k), k = -d, \cdots, -1, 0, \end{cases} \tag{1}$$

where $k \in \mathcal{Z}$, $x_k \in R^n$ is the system state, $u_k \in R^p$ is the control input, $d > 0$ is an constant integer time-delay, $\phi(k)$ is a initial value at k. $\{r_k, k \in \mathcal{Z}\}$ is a Markov chain taking values in finite space $\mathcal{S} = \{1, 2, \cdots, N\}$, with transition

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probability from mode i at time k to mode j at time $k+1$, $k \in \mathcal{Z}$:

$$p_{ij} = \Pr\{r_{k+1} = j | r_k = i\} \tag{2}$$

with $p_{ij} \geq 0$ for $i, j \in \mathcal{S}$, and $\sum_{j=1}^{N} p_{ij} = 1$. The matrix $E \in R^{n \times n}$ is singular, and $\text{rank}(E) = r < n$. For each $i \in \mathcal{S}$, we have

$$\begin{array}{l} A(k,i) = A(i) + \delta A(k,i), A_d(k,i) = A_d(i) + \delta A_d(k,i), \\ B(k,i) = B(i) + \delta B(k,i) \end{array}$$

where $A(i), A_d(i), B(i)$ are known constant matrices with appropriate dimensions; $\delta A(k,i)$, $\delta A_d(k,i)$, $\delta B(k,i)$ are unknown matrices, denoting the uncertainties in the system.

In this paper, the uncertainties are norm-bounded and are assumed to be of the following form

$$\begin{array}{l} \left[\begin{array}{ccc} \delta A(k,i) & \delta A_d(k,i) & \delta B(k,i) \end{array} \right] \\ = E_1(i) \Delta(k,i) \left[F_a(i) \quad F_d(i) \quad F_b(i) \right], \end{array} \tag{3}$$

where $E_1(i)$, $F_a(i)$, $F_d(i)$, $F_b(i)$ are known constant matrices with appropriate dimensions, $\Delta(k,i) \in R^{q \times s}$ are unknown time-varying matrices function satisfying

$$\Delta^T(k,i)\Delta(k,i) \leq I, \tag{4}$$

Consider system (1) with $\delta A(k, r_k) = 0$, $\delta A_d(k, r_k) = 0$ and $u_k = 0$, i.e.

$$\begin{array}{l} Ex_{k+1} = A(r_k)x_k + A_d(r_k)x_{k-d}, \\ x_k = \phi(k), k = -d, \cdots, -1, 0. \end{array} \tag{5}$$

Definition 1 [17]. System $Ex_{k+1} = A(r_k)x_k$ (or the pair $(E, A(r_k))$) is said to be

(1) regular if $\det(zE - A(r_k)) \not\equiv 0$ for any $r_k = i$, $i \in \mathcal{S}$.

(2) causal if it is regular and degree $(\det(zE - A(r_k))) = \text{rank}(E)$ for any $r_k = i$, $i \in \mathcal{S}$.

Definition 2. (1) System (5) is said to be regular and causal, if the pair $(E, A(r_k))$ is regular, causal.

(2) System (5) is said to be stochastically stable, if for every initial state (ϕ, r_0), the following condition:

$$\mathbf{E}\{\sum_{k=0}^{\infty} \|x_k(\phi, r_0)\|^2 | \phi, r_0\} < \infty$$

is satisfied.

(3) System (1) with $u_k = 0$ is said to be robust stochastically stable, if it is stochastically stable for all uncertainties satisfying (3) and (4).

(4) System (1) is said to be regular, causal and robust stochastically stabilizable via state feedback if there exists a state feedback controller

$$u_k = K(r_k)x_k \tag{6}$$

with $K(i)$, when $r_k = i$, a constant matrix such that the resulting closed-loop system is regular, causal and stochastically stable for all uncertainties satisfying (3) and (4).

Remark 1 [18]. In the case when system (5) is regular and causal, then for any initial value $\phi(k)$, there exists a solution of system (5) for any $r_k = i$, $i \in \mathcal{S}$.

The purpose of this paper is to develop delay-dependent LMI conditions such that system (5) is regular, causal and stochastically stable, and design a state feedback controller of the form (6) such that the resulting closed-loop system is regular, causal and robust stochastically stable.

Lemma 1 [20]. Given matrices X, Y, Z with appropriate dimensions, and $Y > 0$. Then

$$-Z^T Y^{-1} Z \leq X^T Y X + X^T Z + Z^T X.$$

Lemma 2 [21]. Given a symmetric matrix Ω and matrices Γ, Ξ with appropriate dimensions, then $\Omega + \Gamma\Delta\Xi + \Xi^T\Delta^T\Gamma^T < 0$ for all Δ satisfying $\Delta^T\Delta \leq I$, if and only if there exists a scalar $\epsilon > 0$ such that $\Omega + \epsilon\Gamma\Gamma^T + \epsilon^{-1}\Xi^T\Xi < 0$.

III. MAIN RESULTS

In this section, first of all, we consider the regularity, causality and stochastic stability for system (5).

Theorem 1. System (5) is regular, causal and stochastically stable, if for each mode $i \in \mathcal{S}$, there exist matrices $X_i > 0$, $Z > 0$, $U > 0$, N_{i1}, N_{i2}, and symmetric matrix S_i satisfying the following set of coupled LMIs:

$$\Phi_i = \left[\begin{array}{ccc} \Phi_{i11} & \Phi_{i12} & dN_{i1} \\ * & \Phi_{i22} & dN_{i2} \\ * & * & -dZ \end{array} \right] < 0, \tag{7}$$

where

$$\begin{array}{rl} \Phi_{i11} & = A^T(i)\bar{X}_i A(i) - A^T(i)R^T\bar{S}_i R A(i) \\ & \quad -E^T X_i E + U + N_{i1}E + E^T N_{i1}^T \\ & \quad +d(A(i) - E)^T Z(A(i) - E), \\ \Phi_{i12} & = A^T(i)\bar{X}_i A_d(i) - A^T(i)R^T\bar{S}_i R A_d(i) \\ & \quad -N_{i1}E + E^T N_{i2}^T + d(A(i) - E)^T Z A_d(i), \\ \Phi_{i22} & = A_d^T(i)\bar{X}_i A_d(i) - A_d^T(i)R^T\bar{S}_i R A_d(i) - U \\ & \quad -N_{i2}E - E^T N_{i2}^T + dA_d^T(i) Z A_d(i), \\ \bar{X}_i & = \sum_{j=1}^{N} p_{ij}X_j, \quad \bar{S}_i = \sum_{j=1}^{N} p_{ij}S_j. \end{array} \tag{8}$$

$R \in R^{n \times n}$ is any constant matrix satisfying $RE = 0$, $\text{rank}(R) = n - r$.

Proof. First, let us prove that system (5) is regular and causal. Since E is singular and $\text{rank}(E) = r$, there exist two nonsingular matrices $M, N \in R^{n \times n}$ such that

$$MEN = \left[\begin{array}{cc} I_r & 0 \\ 0 & 0 \end{array} \right]. \tag{9}$$

Accordingly, let

$$\begin{array}{l} MA(i)N = \left[\begin{array}{cc} A_1(i) & A_2(i) \\ A_3(i) & A_4(i) \end{array} \right], \\ M^{-T}X_i M^{-1} = \left[\begin{array}{cc} X_{i1} & X_{i2} \\ X_{i2}^T & X_{i3} \end{array} \right], \\ N^T N_{i1} M^{-1} = \left[\begin{array}{cc} N_{i11} & N_{i12} \\ N_{i13} & N_{i14} \end{array} \right], \\ RM^{-1} = \left[\begin{array}{cc} R_1 & R_2 \end{array} \right]. \end{array} \tag{10}$$

So from $RE = 0$, $\text{rank}(R) = n - r$ yields that $R_1 = 0$ and $\text{rank}(R_2) = n - r$, $R_2 \in R^{n \times (n-r)}$, i.e.

$$RM^{-1} = \left[\begin{array}{cc} 0 & R_2 \end{array} \right]. \tag{11}$$

From (7), it follows that $\Phi_{i11} < 0$, and $X_i > 0$, $U > 0$, $Z > 0$ and $d > 0$, we get

$$-A^T(i)R^T\bar{S}_iRA(i)-E^TX_iE+N_{i1}E+E^TN_{i1}^T < 0. \quad (12)$$

Then

$$N^T(-A^T(i)R^T\bar{S}_iRA(i)-E^TX_iE+N_{i1}E+E^TN_{i1}^T)N < 0$$

it is equivalent to that

$$\begin{bmatrix} \star & \star \\ \star & -A_4^T(i)R_2^T\bar{S}_iR_2A_4(i) \end{bmatrix} < 0,$$

where $\star$ represents the matrix block we do not need, and $\bar{X}_{ik} = \sum_{j=1}^{N} p_{ij}X_{jk}$, $k = 1, 2, 3$. It follows that $-A_4^T(i)R_2^T\bar{S}_iR_2A_4(i) < 0$, and then $A_4(i)$ is nonsingular. From Definitions 1, 2 and (Dai,[9]), system (5) is regular and causal.

Next, to prove that system (5) is stochastically stable, rewrite system (5) as

$$\begin{array}{l} x_{k+1} = x_k + y_k, \\ 0 = -Ey_k + (A(r_k) - E)x_k + A_d(r_k)x_{k-d}, \end{array} \quad (13)$$

and construct a stochastic Lyapunov functional candidate as

$$\begin{cases} V(k, r_k) = \sum_{i=1}^{3} V_i(k, r_k), V_1(k, r_k) = x_k^TE^TX_{r_k}Ex_k, \\ V_2(k, r_k) = \sum_{l=k-d}^{k-1} x_l^TUx_l, \\ V_3(k, r_k) = \sum_{\theta=-d+1}^{0} \sum_{l=k-1+\theta}^{k-1} y_l^TE^TZEy_l, \end{cases} \quad (14)$$

where matrices $X_{r_k} > 0$, $U > 0, Z > 0$. Let the mode at time k be i, that is $r_k = i$. Recall that at time $k + 1$, the system may jump to any mode $r_{k+1} = j$. One can then obtain that

$$\begin{aligned} \Delta V_1(k) &= \mathbf{E}[V_1(k+1, r_{k+1})|r_k = i] - V_1(k, i) \\ &= \mathbf{E}[x_{k+1}^TE^TX_{r_{k+1}}Ex_{k+1}|r_k = i] \\ &\quad -x_k^TE^TX_iEx_k \\ &= \mathbf{E}[(x_k + y_k)^TE^TX_{r_{k+1}}E(x_k + y_k)|r_k = i] \\ &\quad -x_k^TE^TX_iEx_k \\ &= (x_k + y_k)^TE^T\bar{X}_iE(x_k + y_k) - x_k^TE^TX_iEx_k \end{aligned} \quad (15)$$

$$\begin{aligned} \Delta V_2(k) &= \mathbf{E}[V_2(k+1, r_{k+1})|r_k = i] - V_2(k, i) \\ &= \mathbf{E}[\sum_{l=k-d+1}^{k} x_l^TUx_l|r_k = i] - \sum_{l=k-d}^{k-1} x_l^TUx_l \\ &= x_k^TUx_k - x_{k-d}^TUx_{k-d}, \end{aligned} \quad (16)$$

$$\begin{aligned} \Delta V_3(k) &= \mathbf{E}[V_3(k+1, r_{k+1})|r_k = i] - V_3(k, i) \\ &= \mathbf{E}[\sum_{\theta=-d+1}^{0} \sum_{l=k+\theta}^{k} y_l^TE^TZEy_l|r_k = i] \\ &\quad - \sum_{\theta=-d+1}^{0} \sum_{l=k-1+\theta}^{k-1} y_l^TE^TZEy_l \\ &= dy_k^TE^TZEy_k - \sum_{l=k-d}^{k-1} y_l^TE^TZEy_l. \end{aligned} \quad (17)$$

From (15)-(17) and the second formula of (13), it is obtained that

$$\begin{aligned} \Delta V(k) &= \mathbf{E}[V(k+1, r_{k+1})|r_k = i] - V(k, i) \\ &= x_k^TA^T(i)\bar{X}_iA(i)x_k + 2x_k^TA^T(i)\bar{X}_iA_d(i)x_{k-d} \\ &\quad +x_{k-d}^TA_d^T(i)\bar{X}_iA_d(i)x_{k-d} - x_k^TE^TX_iEx_k \\ &\quad +d((A(i) - E)x_k + A_d(i)x_{k-d})^TZ \\ &\quad \cdot((A(i) - E)x_k + A_d(i)x_{k-d}) \\ &\quad +x_k^TUx_k - x_{k-d}^TUx_{k-d} - \sum_{l=k-d}^{k-1} y_l^TE^TZEy_l. \end{aligned} \quad (18)$$

From the first formula of (13), for any appropriate dimensions N_{i1}, N_{i2}, the following equation is true for $r_k = i$:

$$2(x_k^TN_{i1}+x_{k-d}^TN_{i2})(Ex_k-Ex_{k-d}-\sum_{l=k-d}^{k-1} Ey_l) = 0. \quad (19)$$

And from $RE = 0$, the following equation holds for any symmetric matrix S_i with appropriate dimensions and $r_k = i$:

$$\begin{aligned} 0 &= -\sum_{j=1}^{N} p_{ij}x_{k+1}^TE^TR^TS_jREx_{k+1} \\ &= -(A(i)x_k + A_d(i)x_{k-d})^TR^T\bar{S}_iR \\ &\quad \cdot(A(i)x_k + A_d(i)x_{k-d}). \end{aligned} \quad (20)$$

From Lemma 1, it is obtained that the following holds:

$$\begin{aligned} & -2(x_k^TN_{i1} + x_{k-d}^TN_{i2}) \sum_{l=k-d}^{k-1} Ey_l \\ = & -2 \sum_{l=k-d}^{k-1} \begin{bmatrix} x_k^T & x_{k-d}^T \end{bmatrix} \begin{bmatrix} N_{i1}^T & N_{i2}^T \end{bmatrix}^T Ey_l \\ \leq & \; d \begin{bmatrix} x_k \\ x_{k-d} \end{bmatrix}^T \begin{bmatrix} N_{i1} \\ N_{i2} \end{bmatrix} Z^{-1} \begin{bmatrix} N_{i1} \\ N_{i2} \end{bmatrix}^T \begin{bmatrix} x_k \\ x_{k-d} \end{bmatrix} \\ & + \sum_{l=k-d}^{k-1} y_l^TE^TZEy_l. \end{aligned} \quad (21)$$

Then, adding (19), (20) to (18), together with (21), it is obtained that

$$\Delta V(k) \leq \begin{bmatrix} x_k^T & x_{k-d}^T \end{bmatrix} \Lambda_i \begin{bmatrix} x_k^T & x_{k-d}^T \end{bmatrix}^T, \quad (22)$$

where

$$\Lambda_i = \begin{bmatrix} \Phi_{i11} & \Phi_{i12} \\ * & \Phi_{i22} \end{bmatrix} + d \begin{bmatrix} N_{i1} \\ N_{i2} \end{bmatrix} Z^{-1} \begin{bmatrix} N_{i1} \\ N_{i2} \end{bmatrix}^T. \quad (23)$$

Then inequality (7) is equivalent to $\Lambda_i < 0$, according to Schur complement. Let $\alpha_0 = \lambda_{\min}\{-\Lambda_i, i \in \mathcal{S}\}$, then $\alpha_0 > 0$. From (22), we obtain that for any $k \geq 0$

$$\mathbf{E}[V(k+1, r_{k+1})|r_k = i] \leq V(k, r_k) - \alpha_0 x_k^Tx_k. \quad (24)$$

Setting $k = 0$ and $k = 1$ in (24) yields

$$\mathbf{E}[V(1, r_1)|r_0] \leq V(0, r_0) - \alpha_0 x_0^Tx_0,$$

and

$$\mathbf{E}[V(2, r_2)|r_1] \leq V(1, r_1) - \alpha_0 x_1^Tx_1.$$

Then, we can get

$$\mathbf{E}[V(2, r_2)|r_0] \leq V(0, r_0) - \alpha_0 \sum_{l=0}^{1} \mathbf{E}[x_l^Tx_l|r_0].$$

Then, one can continue the iterative procedure (24) to obtain

$$\mathbf{E}[V(T+1, r_{T+1})|r_0] \le V(0, r_0) - \alpha_0 \sum_{l=0}^{T} \mathbf{E}[x_l^T x_l|r_0],$$

implying that

$$\sum_{l=0}^{\infty} \mathbf{E}[x_l^T x_l|r_0] \le \frac{1}{\alpha_0} V(0, r_0) < \infty.$$

This indicates that system (5) is stochastically stable. The proof is completed.

When $E = I$ in system (5), then $R = 0$, and we get in the case the following result for normal systems.

Corollary 1. System $x_{k+1} = A(r_k)x_k + A_d(r_k)x_{k-d}$ is stochastically stable, if for each mode $i \in \mathcal{S}$, there exist matrices $X_i > 0$, $Z > 0$, $U > 0$, N_{i1} and N_{i2} satisfying the following set of coupled LMIs:

$$\begin{bmatrix} \bar{\Phi}_{i11} & \bar{\Phi}_{i12} & dN_{i1} \\ * & \bar{\Phi}_{i22} & dN_{i2} \\ * & * & -dZ \end{bmatrix} < 0,$$

where

$$\begin{aligned} \bar{\Phi}_{i11} = {} & A^T(i)\bar{X}_i A(i) - X_i + U + N_{i1} + N_{i1}^T \\ & + d(A(i) - I)^T Z(A(i) - I), \\ \bar{\Phi}_{i12} = {} & A^T(i)\bar{X}_i A_d(i) - N_{i1} + N_{i2}^T \\ & + d(A(i) - I)^T Z A_d(i), \\ \bar{\Phi}_{i22} = {} & A_d^T(i)\bar{X}_i A_d(i) - U - N_{i2} \\ & - N_{i2}^T + dA_d^T(i) Z A_d(i). \end{aligned}$$

Remark 2. In (7), without loss of generality, the matrix S_i can be chosen as positive definite directly.

Although (7) is a set coupled LMIs, but it is difficult to use it to the design a robust state feedback controller for system (1). In order to design the robust state feedback stabilization controller for system (1) in the LMI setting, the following theorem is given.

Theorem 2. System (5) is regular, causal and stochastically stable, if for each mode $i \in \mathcal{S}$, and given scalars t_{i1}, t_{i2}, ϵ_1, ϵ_2, ϵ_3, there exist matrices $P_i > 0$, $S_i > 0$, $W > 0$, $\bar{U} > 0$, and L satisfying the following set of coupled LMIs:

$$\left[\begin{array}{ccccccc} \Psi_{i11} & \Psi_{i12} & dt_{i1}W & L^T A^T(i) & \epsilon_2 I_n & dL^T(A(i)-E)^T & \epsilon_1 L \\ * & \Psi_{i22} & dt_{i2}W & L^T A_d^T(i) & \epsilon_3 I_n & dL^T A_d^T(i) & 0 \\ * & * & -dW & 0 & 0 & 0 & 0 \\ * & * & * & \bar{P}_i + L + L^T & 0 & 0 & 0 \\ * & * & * & * & -\bar{S}_i & 0 & 0 \\ * & * & * & * & * & -dW & 0 \\ * & * & * & * & * & * & -P_i \end{array}\right] < 0, \quad (25)$$

where

$$\begin{aligned} \Psi_{i11} = {} & \bar{U} + \epsilon_1 L^T E^T + \epsilon_1 EL + t_{i1}EL \\ & + t_{i1}L^T E^T + \epsilon_2 L^T A^T(i) R^T + \epsilon_2 RA(i)L \\ \Psi_{i12} = {} & -t_{i1}EL + t_{i2}L^T E^T \\ & + \epsilon_3 L^T A^T(i) R^T + \epsilon_2 RA_d(i)L \\ \Psi_{i22} = {} & -\bar{U} - t_{i2}EL - t_{i2}L^T E^T \\ & + \epsilon_3 L^T A_d^T(i) R^T + \epsilon_3 RA_d(i)L. \end{aligned} \quad (26)$$

Proof. First, based on Lemma 1 and $X_i > 0$, $S_i > 0$, for any matrix L with appropriate dimensions, and scalars ϵ_1 ϵ_2 ϵ_3, the following inequalities holds

$$-\bar{X}_i^{-1} \le L^T \bar{X}_i L + L^T + L, \quad (27)$$

$$-L^T E^T X_i EL \le \epsilon_1 L^T E^T + \epsilon_1 EL + \epsilon_1^2 X_i^{-1}, \quad (28)$$

$$\begin{aligned} & -\begin{bmatrix} L^T A^T(i) \\ L^T A_d^T(i) \end{bmatrix} R^T \bar{S}_i R \begin{bmatrix} L^T A^T(i) \\ L^T A_d^T(i) \end{bmatrix}^T \\ \le {} & \begin{bmatrix} L^T A^T(i) \\ L^T A_d^T(i) \end{bmatrix} R^T \begin{bmatrix} \epsilon_2 I_n \\ \epsilon_3 I_n \end{bmatrix}^T \\ & + \begin{bmatrix} \epsilon_2 I_n \\ \epsilon_3 I_n \end{bmatrix} R \begin{bmatrix} L^T A^T(i) \\ L^T A_d^T(i) \end{bmatrix}^T \\ & + \begin{bmatrix} \epsilon_2 I_n \\ \epsilon_3 I_n \end{bmatrix} \bar{S}_i^{-1} \begin{bmatrix} \epsilon_2 I_n \\ \epsilon_3 I_n \end{bmatrix}^T . \end{aligned} \quad (29)$$

Since $X_i > 0$, applying Schur complement, (7) is equivalent to

$$\bar{\Phi}(i) = \begin{bmatrix} \bar{\Phi}_{i11} & \bar{\Phi}_{i12} & dN_{i1} & A^T(i) \\ * & \bar{\Phi}_{i22} & dN_{i2} & A_d^T(i) \\ * & * & -dZ & 0 \\ * & * & * & -\bar{X}_i^{-1} \end{bmatrix} < 0, \quad (30)$$

where

$$\begin{aligned} \bar{\Phi}_{i11} & = -A^T(i)R^T \bar{S}_i RA(i) - E^T X_i E + U + N_{i1}E \\ & \quad + E^T N_{i1}^T + d(A(i) - E)^T Z(A(i) - E), \\ \bar{\Phi}_{i12} & = -A^T(i)R^T \bar{S}_i RA_d(i) - N_{i1}E \\ & \quad + E^T N_{i2}^T + d(A(i) - E)^T Z A_d(i), \\ \bar{\Phi}_{i22} & = -A_d^T(i)R^T \bar{S}_i RA_d(i) - U - N_{i2}E \\ & \quad - E^T N_{i2}^T + dA_d^T(i) Z A_d(i). \end{aligned} \quad (31)$$

According to inequality (27), it is obtained that

$$\bar{\Phi}(i) \le \hat{\Phi}(i)$$

$$= \begin{bmatrix} \bar{\Phi}_{i11} & \bar{\Phi}_{i12} & dN_{i1} & A^T(i) \\ * & \bar{\Phi}_{i22} & dN_{i2} & A_d^T(i) \\ * & * & -dZ & 0 \\ * & * & * & L^T \bar{X}_i L + L^T + L \end{bmatrix}. \quad (32)$$

Obviously, if $\hat{\Phi}(i) < 0$, then $\bar{\Phi}(i) < 0$, and $\Phi(i) < 0$. From $\hat{\Phi}(i) < 0$, it follows that $L^T \bar{X}_i L + L^T + L < 0$, then L is nonsingular. Let $T = \text{diag}\{L, L, Z^{-1}, I_n\}$, and

$$\begin{aligned} & N_{i1} = t_{i1}L^{-T}, N_{i2} = t_{i2}L^{-T}, L^T U L = \bar{U}, \\ & L^T X_i L = P_i, W = Z^{-1}. \end{aligned} \quad (33)$$

Then from $T^T \hat{\Phi}(i) T < 0$, according to inequalities (28), (29), and applying Schur complement, it is obtained that (25) holds. The proof is completed.

In the following, we focus on the design of a robust state feedback controller in the form of (6) for system (1) such that the resulting closed-loop system is regular, causal and robust stochastically stable. The closed-loop system formed by system (1) and the state feedback (6) is

$$Ex_{k+1} = [A(k,r_k) + B(k,r_k)K(r_k)]x_k + A_d(k,r_k)x_{k-d}, \tag{34}$$

From Theorem 2, replacing $A(i)$, $A_d(i)$ in (25) with $A(k,i) + B(k,i)K(i)$ and $A_d(k,i)$, then the closed-loop system (34) is regular, causal and robust stochastically stable. Notice (3), it follows that

$$\Psi_K(i) + \Psi_{ia}\Delta(k,i)\Psi_{ib} + (\Psi_{ia}\Delta(k,i)\Psi_{ib})^T < 0, \tag{35}$$

where

$$\Psi_K(i) = \begin{bmatrix} \Psi_{i1K} & \Psi_{i2K} & dt_{i1}W & L^T(A(i)+B(i)K(i))^T & \epsilon_2 I_n & dL^T(A(i)+B(i)K(i)-E)^T & \epsilon_1 I_n \\ * & \Psi_{i22} & dt_{i2}W & L^T A_d^T(i) & \epsilon_3 I_n & dL^T A_d^T(i) & 0 \\ * & * & -dW & 0 & 0 & 0 & 0 \\ * & * & * & \bar{P}_i + L + L^T & 0 & 0 & 0 \\ * & * & * & * & -\bar{S}_i & 0 & 0 \\ * & * & * & * & * & -dW & 0 \\ * & * & * & * & * & * & -P_i \end{bmatrix} < 0,$$

$$\begin{aligned} \Psi_{i1k} &= \bar{U} + \epsilon_1 L^T E^T + \epsilon_1 EL + t_{i1}EL + t_{i1}L^T E^T \\ &\quad + \epsilon_2 L^T(A(i)+B(i)K(i))^T R^T \\ &\quad + \epsilon_2 R(A(i)+B(i)K(i))L, \\ \Psi_{i2K} &= -t_{i1}EL + t_{i2}L^T E^T \\ &\quad + \epsilon_3 L^T(A(i)+B(i)K(i))^T R^T + \epsilon_2 R A_d(i) L, \\ \Psi_{ia} &= \begin{bmatrix} \epsilon_2(RE_1(i))^T & \epsilon_3(RE_1(i))^T & 0 \\ E_1^T(i) & 0 & dE_1^T(i) & 0 \end{bmatrix}^T, \\ \Psi_{ib} &= \begin{bmatrix} (F_a(i)+F_b(i)K(i))L & F_d(i)L & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \end{aligned} \tag{36}$$

Based on Lemma 2, (35) holds if and only if there exists a scalar $\lambda_i > 0$ such that

$$\Psi_K(i) + \lambda_i \Psi_{ia}\Psi_{ia}^T + \lambda_i^{-1}\Psi_{ib}^T\Psi_{ib} < 0, \tag{37}$$

According to Schur complement, (37) is equivalent to

$$\begin{bmatrix} \Psi_K(i) & \lambda_i\Psi_{ia} & \Psi_{ib}^T \\ * & -\lambda_i I & 0 \\ * & * & -\lambda_i I \end{bmatrix} < 0. \tag{38}$$

Let $K(i)L = \bar{K}(i)$, it is obtained that

$$\begin{bmatrix} \Psi_{i1\bar{K}} & \Psi_{i2\bar{K}} & dt_{i1}W & (A(i)L+B(i)\bar{K}(i))^T & \epsilon_2 I_n & d(A(i)L+B(i)\bar{K}(i)-EL)^T & \epsilon_1 L & \epsilon_2\lambda_i RE_1(i) & (F_a(i)L+F_b(i)\bar{K}(i))^T \\ * & \Psi_{i22} & dt_{i2}W & L^T A_d^T(i) & \epsilon_3 I_n & dL^T A_d^T(i) & 0 & \epsilon_3\lambda_i RE_1(i) & (F_d(i)L)^T \\ * & * & -dW & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & \bar{P}_i + L + L^T & 0 & 0 & 0 & \lambda_i E_1(i) & 0 \\ * & * & * & * & -\bar{S}_i & 0 & 0 & 0 & 0 \\ * & * & * & * & * & -dW & 0 & d\lambda_i E_1(i) & 0 \\ * & * & * & * & * & * & -P_i & 0 & 0 \\ * & * & * & * & * & * & * & -\lambda_i I & 0 \\ * & * & * & * & * & * & * & * & -\lambda_i I \end{bmatrix} < 0, \tag{39}$$

$$\begin{aligned} \Psi_{i1\bar{K}} &= \bar{U} + \epsilon_1 L^T E^T + \epsilon_1 EL + t_{i1}EL \\ &\quad + t_{i1}L^T E^T + \epsilon_2(A(i)L+B(i)\bar{K}(i))^T R^T \\ &\quad + \epsilon_2 R(A(i)L+B(i)\bar{K}(i)), \\ \Psi_{i2\bar{K}} &= -t_{i1}EL + t_{i2}L^T E^T + \epsilon_2 R A_d(i) L \\ &\quad + \epsilon_3(A(i)L+B(i)\bar{K}(i))^T R^T, \end{aligned} \tag{40}$$

Based on the above discussion, we have

Theorem 3. Let t_{i1}, t_{i2}, ϵ_1, ϵ_2 and ϵ_3 be given scalars. There exists a robust state feedback control law for system (1) such that the resulting closed-loop system is regular, causal and robust stochastically stable, if for each mode $i \in \mathcal{S}$, there exist matrices $P_i > 0$, $S_i > 0$, $W > 0$, $\bar{U} > 0$, $\bar{K}(i)$, L, and scalar $\lambda_i > 0$ such that the set of the coupled LMIs (39) holds, the controller gain is given by $K(i) = \bar{K}(i)L^{-1}$.

Remark 3. The conditions (7), (25), (39) given in Theorem 1, Theorem 2 and Theorem 3, respectively are LMIs and delay-dependent. If the parameters t_{i1}, t_{i2}, ϵ_1, ϵ_2, ϵ_3 that were introduced in Theorem 2, Theorem 3 are not chosen first, then (25), (39) are not LMIs. So in Theorem 2, Theorem 3, t_{i1}, t_{i2}, ϵ_1, ϵ_2, ϵ_3 are chosen first, and the optimal values of the parameters can be found by the approach stated in [6, Remark 5]. A numerical solution to this problem can be obtained by using a numerical optimization algorithm, such as **fminsearch** in Optimization Toolbox.

IV. EXAMPLE

Consider the following uncertain time-delay discrete-time singular system

$$E = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix},$$

$$A(1) = \begin{bmatrix} 3 & -3 \\ 2 & 1 \end{bmatrix},\ A_d(1) = \begin{bmatrix} 0.1 & -0.1 \\ 0 & -0.2 \end{bmatrix},$$

$$B(1) = \begin{bmatrix} -2 \\ -3 \end{bmatrix}, E_1(1) = \begin{bmatrix} 0.002 \\ 0.001 \end{bmatrix}, F_b(1) = 0.001,$$

$$F_a(1) = \begin{bmatrix} 0.001 & 0 \end{bmatrix},\ F_d(1) = \begin{bmatrix} 0.001 & 0.001 \end{bmatrix},$$

$$A(2) = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix},\ A_d(2) = \begin{bmatrix} 0 & -0.2 \\ 0.1 & -0.1 \end{bmatrix},$$

$$B(2) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, E_1(2) = \begin{bmatrix} 0.001 \\ 0.002 \end{bmatrix}, F_b(2) = 0.001,$$

$$F_a(2) = \begin{bmatrix} 0.005 & 0.005 \end{bmatrix}, F_d(2) = \begin{bmatrix} 0 & -0.001 \end{bmatrix}.$$

The time-delay $d = 7$. The mode switching is governed by a Markov chain that has the following transition probability

$$p_{11} = 0.75,\ p_{12} = 0.25;\ p_{21} = 0.3,\ p_{22} = 0.7.$$

Let $R = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, $t_{11} = 0.05$, $t_{12} = 0.005$, $t_{21} = -0.004$, $t_{22} = 0.001$, $\epsilon_1 = 0.5$, $\epsilon_2 = -2$, $\epsilon_3 = -0.06$. Solving the LMIs (39), we get

$$P_1 = \begin{bmatrix} 15.1528 & 1.4169 \\ 1.4169 & 1.1651 \end{bmatrix}, P_2 = \begin{bmatrix} 15.5186 & 2.3950 \\ 2.3950 & 0.6785 \end{bmatrix},$$

$$W = \begin{bmatrix} 312.9124 & 37.4353 \\ 37.4353 & 153.8325 \end{bmatrix},$$

$$\bar{U} = \begin{bmatrix} 2.5979 & 0.2965 \\ 0.2965 & 0.6437 \end{bmatrix}$$

$$S_1 = \begin{bmatrix} 489.7182 & -1.6862 \\ -1.6862 & 440.6565 \end{bmatrix},$$

$$S_2 = \begin{bmatrix} 431.4255 & 11.5454 \\ 11.5454 & 436.9694 \end{bmatrix},$$

$$L = \begin{bmatrix} -20.3060 & -0.8535 \\ -6.0873 & -3.1401 \end{bmatrix},$$

$$\bar{K}(1) = \begin{bmatrix} -14.5944 & -4.7568 \end{bmatrix}, \lambda_1 = 422.6738,$$

$$\bar{K}(2) = \begin{bmatrix} -42.4461 & 4.2179 \end{bmatrix}, \lambda_2 = 423.5840.$$

Thus the gain matrices of a robust state feedback stabilization controller can be obtained as

$$K(1) = \begin{bmatrix} 0.2881 & 1.4365 \end{bmatrix},$$

$$K(2) = \begin{bmatrix} 2.7141 - 2.0809 \end{bmatrix}.$$

V. CONCLUSIONS

In this paper, the stochastic stability and robust stochastic stabilization via state feedback for time-delay discrete-time Markovian jump singular systems with parameter uncertainties are discussed. Based on stochastic Lyapunov functional, a delay-dependent linear matrix inequalities (LMIs) condition for the time-delay discrete Markovian jump singular systems to be regular, causal and stochastically stable is given. With this condition, the problem of robust stochastic stabilization are solved.

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