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Shortest paths on 3-D simple Lie groups with nonholonomic constraint

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Abstract— **In this paper we study the Carnot-Caratheodory metrics on $SU(2) \simeq S^3$, $SO(3)$ and $SL(2)$ induced by their Cartan decomposition and by the Killing form. Besides computing explicitly geodesics and conjugate loci, we compute the cut loci (globally) and we give the expression of the Carnot-Caratheodory distance as the inverse of an elementary function.**

For $SU(2)$ the cut locus is a maximal circle without one point. In all other cases the cut locus is a stratified set. To our knowledge, this is the first explicit computation of the whole cut locus in sub-Riemannian geometry, except for the trivial case of the Heisenberg group.

Keywords: left-invariant sub-Riemannian geometry, Carnot-Caratheodory distance, global structure of the cut locus

AMS subject classifications: 22E30, 49J15, 53C17

I. INTRODUCTION

In this paper we study the global structure of the cut locus (set of points reached optimally by more than one geodesic) for the simplest sub-Riemannian structures on three dimensional simple Lie groups (i.e. $SU(2)$, $SO(3)$ and $SL(2)$) namely, the left-invariant sub-Riemannian structure induced by their Cartan decomposition and by the Killing form.

Let G be a simple real Lie group of matrices with associated Lie algebra $\mathbf{L}$ and Killing form $\mathrm{Kil}(\cdot,\cdot)$. Let $\mathbf{L} = \mathbf{k} \oplus \mathbf{p}$ be its Cartan decomposition with the usual commutation relations $[\mathbf{k},\mathbf{k}] \subseteq \mathbf{k}, \quad [\mathbf{p},\mathbf{p}] \subseteq \mathbf{k}, \quad [\mathbf{k},\mathbf{p}] \subseteq \mathbf{p}$. If $\mathbf{L}$ is non compact we also require $\mathbf{k}$ to be the maximal compact subalgebra of $\mathbf{L}$. The most natural left-invariant sub-Riemannian structure that one can define on G is the one in which the distribution is generated by left translations of $\mathbf{p}$ and the sub-Riemannian metric $< \cdot,\cdot >$ at the identity is generated by a scalar multiple of the Killing form restricted to $\mathbf{p}$. The scalar must be chosen positive or negative in such a way that the scalar product is positive definite. We call G, endowed with such a sub-Riemannian structure, a $\mathbf{k} \oplus \mathbf{p}$ **sub-Riemannian manifold**.

$\mathbf{k} \oplus \mathbf{p}$ sub-Riemannian manifolds have very special features: there are no strict abnormal minimizers and the Hamiltonian system given by the Pontryagin Maximum Principle is integrable in terms of elementary functions (products of exponentials). More precisely, if we write the distribution at a point $g \in G$ as $\Delta(g) = g\mathbf{p}$, we have the following expression for geodesics parametrized by arclength, starting at time zero from g_0 ([2], [5]):

$$g(t) = g_0 e^{(A_k + A_p)t} e^{-A_k t}, \tag{1}$$

where $A_k \in \mathbf{k}$, $A_p \in \mathbf{p}$, and we have $< A_p, A_p >= 1$. Thanks to left-invariance, with no loss of generality we can always assume g_0 to be the identity and we will do so all along the paper.

In all three-dimensional cases $\mathbf{p}$ has dimension 2, while $\mathbf{k}$ has dimension 1. Writing $\mathbf{p} = \mathrm{span}\{p_1, p_2\}$ where $\{p_1, p_2\}$ is an orthonormal frame for the sub-Riemannian structure (i.e. $< p_i, p_j >= \delta_{ij}$) and $\mathbf{k} = \mathrm{span}\{k\}$, we can write $A_p = \cos(\theta)p_1 + \sin(\theta)p_2$ and $A_k = ck$ with $\theta \in \mathbb{R}/2\pi, \ c \in \mathbb{R}$. The map associating to the triple (θ, c, t) the final point of the corresponding geodesic starting from the identity, is called the *exponential map*:

$$\begin{array}{ccc} & \mathrm{Exp}: & \\ S^1 \times \mathbb{R} \times \mathbb{R}^+ & \rightarrow & G \\ (\theta, c, t) & \mapsto & \mathrm{Exp}(\theta, c, t) = e^{(A_k + A_p)t} e^{-A_k t}. \end{array}$$

For three dimensional $\mathbf{k}\oplus\mathbf{p}$ sub-Riemannian manifolds, the local structure of the sub-Riemannian spheres, cut loci and conjugate loci starting from the identity has been described by Agrachev (unpublished) and, due to cylindrical symmetry of the Killing form in the $\mathbf{p}$ subspace, it is very similar to the one of the Heisenberg group. Indeed, locally, the cut locus coincides with the first conjugate locus (i.e. the set where local optimality is lost) and it is made by two connected one-dimensional manifolds adjacent to the identity and transversal to the distribution, see Figure 1.

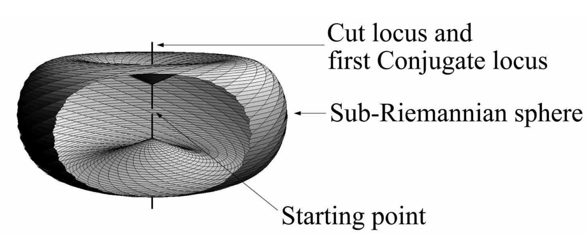

Fig. 1. Local structure of sub-Riemannian spheres, cut and conjugate loci for 3-dim $\mathbf{k} \oplus \mathbf{p}$ sub-Riemannian manifolds.

However the global structure of the cut locus was still unknown. Indeed, to our knowledge, no global structure of the cut locus is known in sub-Riemannian geometry apart from the one of the Heisenberg group.

The main result of our paper is the following (proofs can be found in [7]):

Theorem 1: Let K_{Id} be the cut locus starting from the identity. We have the following:

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- for $SU(2)$, $K_{\rm Id}$ is a maximal circle S^1 without one point (the identity).

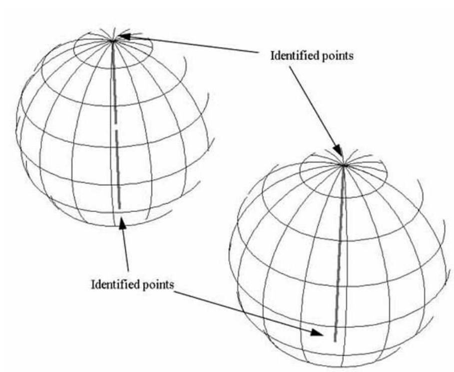

Fig. 2. The cut locus for the $\mathbf{k} \oplus \mathbf{p}$ sub-Riemannian manifold $SU(2)$.

- for $SO(3)$, $K_{\rm Id}$ is a stratified set made by two manifolds glued in one point. The first manifold is $\mathbb{RP}^2$, the second manifold is a maximal circle S^1 without one point (the identity).

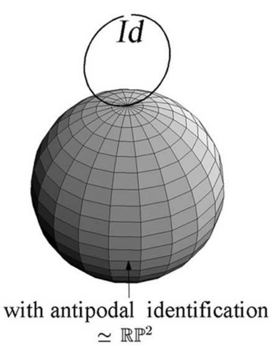

Fig. 3. The cut locus for the $\mathbf{k} \oplus \mathbf{p}$ sub-Riemannian manifold $SO(3)$.

- for $SL(2)$, $K_{\rm Id}$ is a stratified set made by two manifolds glued in one point. The first manifold is $\mathbb{R}^2$, the second manifold is a circle S^1 without one point (the identity).

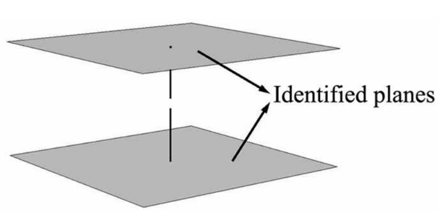

Fig. 4. The cut locus for the $\mathbf{k} \oplus \mathbf{p}$ sub-Riemannian manifold $SL(2)$.

For all cases the one dimensional strata contains the cut locus appearing in the local analysis.

Once the cut locus is computed, one can provide the expression of the sub-Riemannian distance from the identity. Theorem 3 below gives the sub-Riemannian distance for $SU(2)$. The proof, given in [7], can be adapted to get similar results in the cases of $SO(3)$ and $SL(2)$. This theorem and its analogs for $SO(3)$ and $SL(2)$ are useful to give estimates for the fundamental solutions of the hypoelliptic heat equation induced by the sub-Riemannian structure ([4], [6]). Moreover this Theorem can be seen as the answer, in the case of $SU(2)$, to the question (formulated in [9]) about the possibility of inverting the matrix equation (1), i.e., for every matrix $g \in SU(2)$, find a matrix $A = A_k + A_p$, with $< A_p, A_p >= 1$, solution to the equation $g = g_0 e^{(A_k+A_p)t} e^{-A_k t}$. If $\beta \neq 0$ then this equation has one and only one solution, otherwise it has more than one solution (indeed infinitely many, see Section III).

II. Basic Definitions

In this paper we are concerned with sub-Riemannian manifolds of dimension 3, defined by a pair of vector fields F_1 and F_2 such that for all $q \in M$, $Span\{F_1(q), F_2(q), [F_1(q), F_2(q)]\} = T_q M$, i.e. the so called 3-D **contact case**, for which there are no abnormal extremals. In this case the geodesics computed via the Pontryagin Maximum Principle (see for instance [2], [10]) are all and only the projections on M of a solution $(\lambda(t), q(t))$ for the Hamiltonian system on T^*M corresponding to:

$$H(\lambda, q) = \frac{1}{2}\left(< \lambda_1, F_1(q) >^2 + < \lambda_2, F_2(q) >^2\right)$$

with $q \in M$, $\lambda \in T_q^*M$ and satisfying $H(\lambda(0), q(0)) \neq 0$. We fix $H = \frac{1}{2}$ for having $q(.)$ parametrized by arclength.

Fix $q_0 \in M$. For every $\lambda_0 \in T_{q_0}^*M$ satisfying $H(\lambda_0, q_0) = \frac{1}{2}$ and every $t > 0$ define the *exponential map* $\mathrm{Exp}(\lambda_0, t)$ as the projection on M of the solution, evaluated at time t, of the Hamiltonian system associated with H, with initial condition $\lambda(0) = \lambda_0$ and $q(0) = q_0$. Notice that condition $H(\lambda_0, q_0) = \frac{1}{2}$ defines a hypercylinder $\Lambda_{q_0} \simeq S^1 \times \mathbb{R}$ in $T_{q_0}^*M$.

Definition 1: The **conjugate locus from** q_0 is the set C_{q_0} of critical values of the map

$$\mathrm{Exp}: \begin{array}{ccc} \Lambda_{q_0} \times \mathbb{R}^+ & \to & M \\ (\lambda_0, t) & \mapsto & \mathrm{Exp}(\lambda_0, t). \end{array}$$

For every $\bar\lambda_0 \in \Lambda_{q_0}$, let $t(\bar\lambda_0)$ be the n-th positive time, if it exists, for which the map $(\lambda_0, t) \mapsto \mathrm{Exp}(\lambda_0, t)$ is singular at $(\bar\lambda_0, t(\bar\lambda_0))$. The **n-th conjugate locus** from q_0 $C_{q_0}^n$ is the set $\{\mathrm{Exp}(\bar\lambda_0, t(\bar\lambda_0)) \mid t(\bar\lambda_0) \text{ exists}\}$.

The **cut locus** from q_0 is the set K_{q_0} of points reached optimally by more than one geodesic, i.e., the set

$$K_{q_0} = \Big\{q \in M \mid \exists\ \lambda_1, \lambda_2 \in \Lambda_{q_0},\ \lambda_1 \neq \lambda_2,\ t \in \mathbb{R}^+ \ \text{such that} \ \begin{array}{l} q = \mathrm{Exp}(\lambda_1, t) = \mathrm{Exp}(\lambda_2, t), \quad \text{and} \\ \mathrm{Exp}(\lambda_1, \cdot), \mathrm{Exp}(\lambda_2, \cdot) \text{ optimal in } [0, t] \end{array}\Big\}$$

<u>*Remark*</u> **1:** Let $(M, \Delta, \mathbf{g})$ be a sub-Riemannian manifold. Fix $q_0 \in M$ and assume: **(i)** each point of M is reached by an optimal geodesic starting from q_0; **(ii)** there are no abnormal

minimizers. The following facts are well known (a proof in the 3-D contact case can be found in [3]).

- the first conjugate locus $C^1_{q_0}$ is the set of points where the geodesics starting from q_0 lose local optimality;
- if $q(.)$ is a geodesic starting from q_0 and $\bar{t}$ is the first positive time such that $q(\bar{t}) \in K_{q_0} \cup C^1_{q_0}$, then $q(.)$ loses optimality in $\bar{t}$, i.e. it is optimal in $[0,\bar{t}]$ and not optimal in $[0,t]$ for any $t > \bar{t}$;
- if a geodesic $q(.)$ starting from q_0 loses optimality at $\bar{t} > 0$, then $q(\bar{t}) \in K_{q_0} \cup C^1_{q_0}$;

As a consequence, when the first conjugate locus is included in the cut locus (as in our cases), the cut locus is the set of points where the geodesics lose optimality.

Remark **2:** It is well known that, while in Riemannian geometry K_{q_0} is never adjacent to q_0, in sub-Riemannian geometry this is always the case. See [1].

A. $\mathbf{k} \oplus \mathbf{p}$ *sub-Riemannian manifolds*

Let $\mathbf{L}$ be a simple Lie algebra and $\mathrm{Kil}(X,Y) = Tr(ad_X \circ ad_Y)$ its Killing form. Recall that the Killing form defines a non-degenerate pseudo scalar product on $\mathbf{L}$. In the following we recall what we mean by a Cartan decomposition of $\mathbf{L}$.

Definition 2: A Cartan decomposition of a simple Lie algebra $\mathbf{L}$ is any decomposition of the form:

$$\mathbf{L} = \mathbf{k} \oplus \mathbf{p}, \text{ where } [\mathbf{k},\mathbf{k}] \subseteq \mathbf{k}, \quad [\mathbf{p},\mathbf{p}] \subseteq \mathbf{k}, \quad [\mathbf{k},\mathbf{p}] \subseteq \mathbf{p}. \quad (2)$$

Definition 3: Let G be a simple Lie group with Lie algebra $\mathbf{L}$. Let $\mathbf{L} = \mathbf{k} \oplus \mathbf{p}$ be a Cartan decomposition of $\mathbf{L}$. In the case in which G is noncompact assume that $\mathbf{k}$ is the maximal compact subalgebra of $\mathbf{L}$.

On G, consider the distribution $\Delta(g) = g\mathbf{p}$ endowed with the Riemannian metric $\mathbf{g}_g(v_1,v_2) =< g^{-1}v_1, g^{-1}v_2 >$ where $< \ , \ >:= \alpha \ \mathrm{Kil}\big|_{\mathbf{p}}(\ , \ )$ and $\alpha < 0$ (resp. $\alpha > 0$) if G is compact (resp. non compact).

In this case we say that $(G,\Delta,\mathbf{g})$ is a $\mathbf{k} \oplus \mathbf{p}$ sub-Riemannian manifold.

Let $\{X_j\}$ be an orthonormal frame for the subspace $\mathbf{p} \subset \mathbf{L}$, with respect to the metric given in Definition 3. Then the problem of finding the shortest curve steering the identity to a point $g_1 \in G$ can be written as the left-invariant optimal control problem with fixed time T

$$\dot{g} = g\left(\sum_j u_j X_j\right), \quad u_j \in L^\infty(0,T)$$

$$\int_0^T \sum_j u_j^2(t)\ dt \to \min, \quad g(0) = \mathrm{Id}, \quad g(T) = g_1.$$

For details, see [2]. This problem admits a solution, see for instance [8, Ch. 5].

For $\mathbf{k}\oplus\mathbf{p}$ sub-Riemannian manifolds, the Hamiltonian system given by the Pontryagin Maximum Principle is integrable and the explicit expression of geodesics starting from the identity and parameterized by arclength is

$$g(t) = e^{(A_k+A_p)t}e^{-A_k t}, \quad (3)$$

where $A_k \in \mathbf{k}$, $A_p \in \mathbf{p}$ and $< A_p, A_p >= 1$. See [2], [5].

III. $SU(2)$, $SO(3)$ and $SL(2)$, their geodesics and their cut loci

In this section we describe the groups $SU(2)$, $SO(3)$ and $SL(2)$ and we apply formula (3) in order to get the explicit expressions for geodesics. Besides this, we compute cut loci. Details and proofs can be found in [7].

A. *The* $\mathbf{k} \oplus \mathbf{p}$ *problem on* $SU(2)$

The Lie group $SU(2)$ is the group of unitary unimodular 2×2 complex matrices

$$SU(2) = \left\{ \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} \in \mathrm{Mat}(2,\mathbb{C}) \mid |\alpha|^2 + |\beta|^2 = 1 \right\}.$$

The Lie algebra of $SU(2)$ is the algebra of antihermitian traceless 2×2 complex matrices. A basis of $su(2)$ is $\{p_1, p_2, k\}$ where

$$p_1 = \tfrac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad p_2 = \tfrac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \quad k = \tfrac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}. \quad (4)$$

whose commutation relations are $[p_1,p_2] = k \quad [p_2,k] = p_1 \quad [k,p_1] = p_2$. Recall that for $su(2)$ we have $\mathrm{Kil}(X,Y) = 4\mathrm{Tr}(XY)$ and, in particular, $\mathrm{Kil}(p_i,p_j) = -2\delta_{ij}$. The choice of the subspaces $\mathbf{k} = \mathrm{span}\{k\} \quad \mathbf{p} = \mathrm{span}\{p_1,p_2\}$ provides a *Cartan decomposition* for $su(2)$. Moreover, $\{p_1,p_2\}$ is a orthonormal frame for the inner product $<\cdot,\cdot>= -\frac{1}{2}\mathrm{Kil}(\cdot,\cdot)$ restricted to $\mathbf{p}$. Defining $\Delta(g) = g\mathbf{p}$ and $\mathbf{g}_g(v_1,v_2) =< g^{-1}v_1, g^{-1}v_2 >$, we have that $(SU(2),\Delta,\mathbf{g})$ is a $\mathbf{k}\oplus\mathbf{p}$ sub-Riemannian manifold.

Remark **3:** Observe that all the $\mathbf{k}\oplus\mathbf{p}$ structures that one can define on $SU(2)$ are equivalent. For instance, one could set $\mathbf{k} = \mathrm{span}\{p_1\}$ and $\mathbf{p} = \mathrm{span}\{p_2,k\}$.

1) Expression of geodesics: We compute the explicit expression of geodesics using the formula (3). Consider an initial covector $\lambda = \lambda(\theta,c) = \cos(\theta)p_1 + \sin(\theta)p_2 + ck \in \Lambda_{\mathrm{Id}}$. The corresponding exponential map is $\mathrm{Exp}(\theta,c,t) := \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}$ with

$$\begin{aligned} \alpha &= \tfrac{c\sin(\frac{ct}{2})\sin(\sqrt{1+c^2}\frac{t}{2})}{\sqrt{1+c^2}} + \cos(\tfrac{ct}{2})\cos(\sqrt{1+c^2}\tfrac{t}{2}) + \\ &+ i\left(\tfrac{c\cos(\frac{ct}{2})\sin(\sqrt{1+c^2}\frac{t}{2})}{\sqrt{1+c^2}} - \sin(\tfrac{ct}{2})\cos(\sqrt{1+c^2}\tfrac{t}{2})\right), \\ \beta &= \tfrac{\sin(\sqrt{1+c^2}\frac{t}{2})}{\sqrt{1+c^2}}\left(\cos(\tfrac{ct}{2}+\theta) + i\sin(\tfrac{ct}{2}+\theta)\right). \end{aligned}$$

2) The cut locus and distance for $SU(2)$*:*

Theorem 2: The cut locus for the $\mathbf{k}\oplus\mathbf{p}$ problem on $SU(2)$ is $K_{\mathrm{Id}} = e^{\mathbf{k}} \setminus \mathrm{Id} = \left\{e^{ck} \mid c \in (0,4\pi)\right\}$.

The cut locus is topologically a circle S^1 without a point (the identity). We give a picture of it in Figure 2.

Theorem 3: Let $g = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} \in SU(2)$. Its sub-Riemannian distance from Id is

$$d(g,\mathrm{Id}) = \begin{cases} 2\sqrt{\arg(\alpha)\,(2\pi - \arg(\alpha))} & \text{if} \quad \beta = 0 \\ \psi(\alpha) & \text{if} \quad \beta \neq 0 \end{cases},$$

where $\arg(\alpha) \in [0,2\pi]$ and $\psi(\alpha) = t$ the unique solution of $\begin{cases} -\frac{ct}{2} + \arctan\left(\frac{c}{\sqrt{1+c^2}}\tan\left(\frac{\sqrt{1+c^2}t}{2}\right)\right) = \arg(\alpha) \\ \frac{\sin\left(\frac{\sqrt{1+c^2}t}{2}\right)}{\sqrt{1+c^2}} = \sqrt{1-|\alpha|^2} \\ t \in \left(0, \frac{2\pi}{\sqrt{1+c^2}}\right) \end{cases}$.

B. The $\mathbf{k}\oplus\mathbf{p}$ problem on $SO(3)$

The Lie group $SO(3)$ is the group of special orthogonal 3×3 real matrices

$$SO(3) = \left\{g \in \mathrm{Mat}(3,\mathbb{R}) \mid gg^T = \mathrm{Id}, \det(g) = 1\right\}.$$

The Lie algebra of $SO(3)$ is the algebra of skew-symmetric 3×3 real matrices. A basis of $so(3)$ is $\{p_1, p_2, k\}$ where

$$p_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \qquad p_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$
$$k = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

whose commutation relations are $[p_1,p_2] = k \quad [p_2,k] = p_1 \quad [k,p_1] = p_2$. For $so(3)$ we have $\mathrm{Kil}(X,Y) = \mathrm{Tr}(XY)$ so, in particular, $\mathrm{Kil}(p_i,p_j) = -2\delta_{ij}$. The choice of the subspaces $\mathbf{k} = \mathrm{span}\{k\} \quad \mathbf{p} = \mathrm{span}\{p_1,p_2\}$ provides a *Cartan decomposition* for $so(3)$. Moreover, $\{p_1,p_2\}$ is an orthonormal frame for the inner product $<\cdot,\cdot> = -\frac{1}{2}\mathrm{Kil}(\cdot,\cdot)$ restricted to $\mathbf{p}$. Defining $\Delta(g) = g\mathbf{p}$ and $\mathbf{g}_g(v_1,v_2) = < g^{-1}v_1, g^{-1}v_2 >$, we have that $(SO(3),\Delta,\mathbf{g})$ is a $\mathbf{k}\oplus\mathbf{p}$ sub-Riemannian manifold. As for $SU(2)$, all the $\mathbf{k}\oplus\mathbf{p}$ structures that one can define on $SO(3)$ are equivalent.

1) Expression of geodesics: Consider an initial covector $\lambda = \lambda(\theta,c) = \cos(\theta)p_1 + \sin(\theta)p_2 + ck \in \Lambda_{\mathrm{Id}}$: using formula (3), we have that the exponential map is

$$\mathrm{Exp}(\theta,c,t) = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

with

$$\begin{aligned}
a_{11} &= K_1\cos(ct) + K_2\cos(2\theta+ct) + K_3 c\sin(ct),\\
a_{12} &= K_1\sin(ct) + K_2\sin(2\theta+ct) - K_3 c\cos(ct),\\
a_{13} &= K_4\cos(\theta) + K_3\sin(\theta),\\
a_{21} &= -K_1\sin(ct) + K_2\sin(2\theta+ct) + K_3 c\cos(ct),\\
a_{22} &= K_1\cos(ct) - K_2\cos(2\theta+ct) + K_3 c\sin(ct),\\
a_{23} &= -K_3\cos(\theta) + K_4\sin(\theta),\\
a_{21} &= K_4\cos(\theta+ct) - K_3\sin(\theta+ct),\\
a_{22} &= K_3\cos(\theta+ct) + K_4\sin(\theta+ct),\\
a_{23} &= \frac{\cos\left(\sqrt{1+c^2}t\right) + c^2}{1+c^2},
\end{aligned}$$

$$\begin{aligned}
K_1 &= \frac{1 + \left(1+2c^2\right)\cos\left(\sqrt{1+c^2}t\right)}{2\left(1+c^2\right)},\\
K_2 &= \frac{1 - \cos\left(\sqrt{1+c^2}t\right)}{2\left(1+c^2\right)},\\
K_3 &= \frac{\sin\left(\sqrt{1+c^2}t\right)}{\sqrt{1+c^2}},\\
K_4 &= \frac{c\left(1-\cos\left(\sqrt{1+c^2}t\right)\right)}{1+c^2}.
\end{aligned}$$

2) The cut locus for $SO(3)$:

Theorem 4: The cut locus for the $\mathbf{k}\oplus\mathbf{p}$ problem on $SL(2)$ is $K_{\mathrm{Id}} = K_{\mathrm{Id}}^{sym} \cup K_{\mathrm{Id}}^{loc}$ with

$$\begin{aligned}
K_{\mathrm{Id}}^{sym} &= e^{2\pi k}e^{\mathbf{p}} = \left\{g \in SL(2) \mid g = g^T, \mathrm{Tr}g < 0\right\},\\
K_{\mathrm{Id}}^{loc} &= e^{\mathbf{k}} \setminus \mathrm{Id} =\\
&= \left\{\begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix} \mid \alpha \in R/2\pi, \alpha \neq 0\right\}.
\end{aligned}$$

The cut locus is a stratification: K_{Id}^{sym} is topologically $\mathbb{RP}^2$, while K_{Id}^{loc} is a circle S^1 without a point (the identity). We give a picture of it in Figure 3.

C. The $\mathbf{k}\oplus\mathbf{p}$ problem on $SL(2)$

The Lie group $SL(2)$ is the group of 2×2 real matrices with determinant 1

$$SL(2) = \{g \in \mathrm{Mat}(2,\mathbb{R}) \mid \det(g) = 1\}.$$

The Lie algebra of $SL(2)$ is the algebra of traceless 2×2 real matrices. A basis of $sl(2)$ is $\{p_1,p_2,k\}$ where

$$p_1 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad p_2 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
$$k = \frac{1}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

whose commutation relations are $[p_1,p_2] = -k \quad [p_2,k] = p_1 \quad [k,p_1] = p_2$. For $sl(2)$ we have $\mathrm{Kil}(X,Y) = 4\mathrm{Tr}(XY)$ and, in particular, $\mathrm{Kil}(p_i,p_j) = 2\delta_{ij}$. The choice of the subspaces $\mathbf{k} = \mathrm{span}\{k\} \quad \mathbf{p} = \mathrm{span}\{p_1,p_2\}$ provides a *Cartan decomposition* for $sl(2)$. For $sl(2)$ the Cartan decomposition is unique, since $\mathbf{k}$ must be the maximal compact subalgebra. Moreover, $\{p_1,p_2\}$ is a orthonormal frame for the inner product $<\cdot,\cdot> = \frac{1}{2}\mathrm{Kil}(\cdot,\cdot)$ restricted to $\mathbf{p}$. Defining $\Delta(g) = g\mathbf{p}$ and $\mathbf{g}_g(v_1,v_2) = < g^{-1}v_1, g^{-1}v_2 >$, we have that $(SL(2),\Delta,\mathbf{g})$ is a $\mathbf{k}\oplus\mathbf{p}$ sub-Riemannian manifold.

1) Expression of geodesics: Consider an initial covector $\lambda = \lambda(\theta,c) = \cos(\theta)p_1 + \sin(\theta)p_2 + ck \in \Lambda_{\mathrm{Id}}$: using formula (3), we have that the exponential map is

$$\mathrm{Exp}(\theta,c,t) = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

with

$$\begin{aligned}
a_{11} &= K_1 \cos\left(\frac{ct}{2}\right) + K_2 \left(\cos(\psi) + c\sin\left(\frac{ct}{2}\right)\right) \\
a_{12} &= K_1 \sin\left(\frac{ct}{2}\right) + K_2 \left(\sin(\psi) - c\cos\left(\frac{ct}{2}\right)\right) \\
a_{21} &= -K_1 \sin\left(\frac{ct}{2}\right) + K_2 \left(\sin(\psi) + c\cos\left(\frac{ct}{2}\right)\right) \\
a_{22} &= K_1 \cos\left(\frac{ct}{2}\right) + K_2 \left(-\cos(\psi) + c\sin\left(\frac{ct}{2}\right)\right) \\
\psi &= \theta + \frac{ct}{2} \\
K_1 &= \begin{cases} \mathrm{Cosh}\left(\sqrt{1-c^2}\,\frac{t}{2}\right) & c \in [-1,1] \\ \cos\left(\sqrt{c^2-1}\,\frac{t}{2}\right) & c \notin [-1,1] \end{cases}, \\
K_2 &= \begin{cases} \frac{\mathrm{Sinh}\left(\sqrt{1-c^2}\,\frac{t}{2}\right)}{\sqrt{1-c^2}} & c \in (-1,1) \\ \frac{t}{2} & c \in \{-1,1\} \\ \frac{\sin\left(\sqrt{c^2-1}\,\frac{t}{2}\right)}{\sqrt{c^2-1}} & c \notin [-1,1] \end{cases}.
\end{aligned}$$

2) The cut locus for $SL(2)$:

Theorem 5: The cut locus for the **k**⊕**p** problem on $SL(2)$ is $K_{\mathrm{Id}} = K_{\mathrm{Id}}^{sym} \cup K_{\mathrm{Id}}^{loc}$ with

$$\begin{aligned}
K_{\mathrm{Id}}^{sym} &= e^{2\pi k} e^{\mathbf{p}} = \left\{ g \in SL(2) \mid g = g^T, \mathrm{Tr} g < 0 \right\}, \\
K_{\mathrm{Id}}^{loc} &= e^{\mathbf{k}} \setminus \mathrm{Id} = \\
&= \left\{ \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix} \mid \alpha \in R/2\pi, \alpha \neq 0 \right\}.
\end{aligned}$$

The cut locus is a stratification: K_{Id}^{sym} is topologically a plane $\mathbb{R}^2$, while K_{Id}^{loc} is a circle S^1 without a point (the identity). We give a picture of it in Figure 4.

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