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Distributed Control of Vehicle Formations: a Decomposition Approach

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Abstract— **In this article we consider the problem of designing a controller for a formation of vehicles. We present a novel method that allows the synthesis of distributed controllers with $\mathcal{H}_\infty$ performance; the method is based on a modal decomposition technique and it makes use of LMIs (Linear Matrix Inequalities).**

One of the interesting results of this paper is that for certain relevant situations the size of the synthesis LMIs is independent of the size of the formation: this means that with a single, small LMI test it is possible to compute a controller with guaranteed disturbance rejection performance for all the formations of a kind. This method can then be considered as a significant generalization of the famous result presented by Fax and Murray in [10], where the authors showed a how it was possible to prove the stability of a formation with a single Nyquist diagram, irrespectively of the formation size.

I. INTRODUCTION

In recent times, the control theory community has put much effort into the development of methods for controlling vehicle formations. This interest is due to the variety of applications that have been made possible by modern technological advances, for example, in satellite formation flying [4], car platoons [21] or unmanned aerial vehicles [1], [11].

Vehicle formations can be considered as a set of identical agents which are basically dynamically decoupled, but share a common goal, i.e. keeping relative positions or fulfilling a common sensing task. This common goal introduces a coupling between the systems, making it necessary to consider the set of agents as a whole: if we assume the agent to be an l^{th} order model, and the number of agent is N, then the control problem is of order Nl, which can be difficult to manage for large N. Moreover, a centralized controller can be difficult to implement in practice, as it would require the presence of a central unit with knowledge of all the agents and that can command all of them. For these reasons, the literature presents different methods for either simplifying the computational complexity of the synthesis or for designing controllers in a *distributed* fashion, that means, with the agents managed by local controllers having a limited knowledge of the other agents. Different approaches to this problem have appeared in literature, from simple leader-follower architectures [22], to more complex algorithms based on consensus rules [8].

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In this paper we focus on the approach of [10], where it is assumed that every vehicle of the formation has identical dynamics and is controlled locally, and that every controller has only visibility of a limited set of the other vehicles. In this situation, the formation can be schematized as a *graph* [7] that mimics the information flow, and it is possible to prove a formation stability criterion which requires the evaluation of systems of the order of the agents, instead of the full order formation. After summarizing these results, we will show the main contribution of this paper, a kind of generalization from simple stability to $\mathcal{H}_\infty$ performance. This result is based on an LMI sufficient condition and it will allow the synthesis of distributed controllers for formations on the basis of disturbance attenuation criteria.

The paper is organized as follows. In Section II we introduce the notation and the notions of graph theory that are used in the article, while in Section III we briefly summarize the results of [10]. Section IV shows the novel approach that is proposed here and Section V shows how this approach can be used for $\mathcal{H}_\infty$ synthesis of a distributed controller. Finally, Section VI shows some special cases of particular interest, and Section VII contains an example of an application. The conclusions are in Section VIII.

II. PRELIMINARIES

We denote by $\mathbb{R}$ the field of real numbers and by $\mathbb{R}^{n\times m}$ the set of real $n \times m$ matrices. Let $\otimes$ indicate the Kronecker product, I_n the identity matrix of order n and let j be the imaginary unit. The notation $A \succ 0$ indicates that all the eigenvalues of the Hermitian matrix A are strictly positive, and the bullet $\bullet$ denotes a symbol that is either not relevant or clear from the context.

In this paper we will make use of some results of graph theory. For this reason, we will summarize here a few definitions and a theorem that will be fundamental in the following Sections. The interested reader can find out more in [7] and [5].

Definition 1: A *directed graph* $\mathcal{G}$ consists of a set of vertices $\mathcal{V}$ and a set of edges $\mathcal{A} \subset \mathcal{V}^2$, which can be interpreted as connections between vertices: for an edge (a, b), we call the vertex a the *tail* and b the *head*. We assume that each element of $\mathcal{A}$ is unique and that there are no self-loops, that means $(a, a) \notin \mathcal{A}\ \forall a \in \mathcal{V}$. A graph with the property that $(a, b) \in \mathcal{A} \Leftrightarrow (b, a) \in \mathcal{A}\ \forall a \in \mathcal{V}$ is called an *undirected graph*.

Graph theory becomes useful in control thanks to the properties of special kinds of matrices which are associated with graphs. For introducing these matrices, we assume that the N vertices of the graph $\mathcal{G}$ are enumerated, and each of them is denoted a_i.

Definition 2: The *normalized adjacency matrix* G of a graph is an $N \times N$ matrix defined by $G_{i,k} = 1/d_o(a_i)$ if $(a_i, a_k) \in \mathcal{A}$ and 0 otherwise; $d_o(a_i)$ is the *out-degree* of a_i, that is, the number of edges that feature a_i as tail (assume $d_o(a_i) \neq 0 \ \forall a_i$).

Definition 3: The *normalized Laplacian matrix*[1] $\mathcal{L}$ of a graph is defined as $\mathcal{L} = I_N - G$.

The property that we will use in this paper is stated in the following Theorem.

Theorem 4: The eigenvalues of the normalized Laplacian are located in a disk of radius 1 centered at $1 + 0j$ in the complex plane. In addition, for undirected graphs the eigenvalues of the normalized Laplacian are real, and thus they are located between 0 and 2.

In the next Section we will show a result that relates the stability properties of a formation to the eigenvalues of a normalized Laplacian matrix; this explains why it can be very important to have *a priori* information on the location of these eigenvalues.

III. FORMATION STABILITY

In this Section we report briefly the results of [10], in order to be able to build on them later. Let us consider a set of N identical linear systems (agents, vehicles, etc.), whose dynamics is modeled by the equation:

$$\dot{x}_i = Ax_i + Bu_i \tag{1}$$

where $x_i \in \mathbb{R}^l$ are the agents' states, $u_i \in \mathbb{R}^m$ their control inputs and $i = 1, \ldots, N$ is the index for the vehicles in the formation. From this it follows that the dynamics of the formation is described by the equation:

$$\dot{x} = (I_N \otimes A)x + (I_N \otimes B)u \tag{2}$$

where we have $x = [x_1^T \ x_2^T \ \ldots \ x_N^T]^T \in \mathbb{R}^{lN}$ and $u = [u_1^T \ u_2^T \ \ldots \ u_N^T]^T \in \mathbb{R}^{mN}$.

Let us now assume that each vehicle has a limited visibility with respect to the others; for this purpose we define the set $\mathcal{J}_i \subset [1, N] \backslash \{i\}$ of the vehicles that the i^{th} vehicle can sense. Then, each vehicle has the following measurements available for feedback control:

$$\begin{aligned} y_i &= C_a x_i \\ q_i &= \frac{1}{|\mathcal{J}_i|} C_b \sum_{j \in \mathcal{J}_i} (x_i - x_j) \end{aligned} \tag{3}$$

where $|\mathcal{J}_i|$ is the number of elements of the set $\mathcal{J}_i$ (assume all $|\mathcal{J}_i| \neq 0$: all agents can see at least one other agent). In this way, the global output function is equivalent to:

$$\begin{aligned} y &= (I_N \otimes C_a)x \\ q &= (\mathcal{L} \otimes C_b)x \end{aligned} \tag{4}$$

[1] In [10] this matrix is called just "Laplacian"; but, as stated there, in literature there are different definitions for it. In this article we have chosen to define the matrix used here as "normalized", in order to distinguish it from the other definition of the Laplacian matrix, that can be found e.g. in [13].

where as before $y = [y_1^T \ y_2^T \ \ldots \ y_N^T]^T \in \mathbb{R}^{r_y N}$ and $q = [q_1^T \ q_2^T \ \ldots \ q_N^T]^T \in \mathbb{R}^{r_q N}$; thanks to the definition of q in (3), it is possible to prove that $\mathcal{L}$ is indeed the normalized Laplacian of the graph that describes the information flow in the formation (i.e., an edge connects vertex i to vertex k iff agent k receives the output of agent i).

Let us now assume that each vehicle is locally controlled by identical local controllers K:

$$\begin{aligned} \dot{v}_i &= K_A v_i + K_{B_y} y_i + K_{B_q} q_i \\ u_i &= K_C v_i + K_{D_y} y_i + K_{D_q} q_i \end{aligned} \tag{5}$$

Then the following Theorem holds.

Theorem 5 (Formation stability): A local controller K as in (5) stabilizes the formation dynamics in (1), (3) if and only if it simultaneously stabilizes the following set of N "modal" subsystems:

$$\begin{aligned} \dot{\hat{x}}_i &= A\hat{x}_i + B\hat{u}_i \\ \hat{y}_i &= C_a \hat{x}_i \\ \hat{q}_i &= \lambda_i C_b \hat{x}_i \end{aligned} \tag{6}$$

where the λ_i are the eigenvalues of the matrix $\mathcal{L}$ of (4).

It has to be pointed out that λ_i can be complex, leading to complex-valued systems. The proof (see [10] for the details) is based on the fact that for any square matrix $\mathcal{L}$ there exists a Schur transformation [14] such as:

$$\mathcal{L} = T^{-1} U T$$

where T is unitary and U is upper diagonal, with the eigenvalues of $\mathcal{L}$ on the diagonal; both T and U can be complex-valued. Through this observation it is possible to show that the formation in closed loop is equivalent to a block upper diagonal system, and so its stability depends on the stability of the diagonal blocks, which are equivalent to the N systems of Theorem 5 in closed loop with the local controllers. This result is valid for any matrix $\mathcal{L}$, not only for Laplacians, but the use of Laplacians will allow having information on the λ_i without computing them.

It is also possible to derive a kind of Nyquist criterion for the formations in the case of single-input single-output agents; by comparing the agent equations and the modal subsystem equation (6), it is clear that if the transfer function from u_i to $C_b x_i$ of an agent is G, then the transfer functions of the modal subsystems (from $\hat{u}_i$ to $\hat{q}_i$) are $\lambda_i G$. Then, if we assume a feedback on q alone, if K is the transfer function of the controller, the formation is stable if and only if the point -1 is correctly encircled by the Nyquist diagram of all of the closed loop transfer functions $\lambda_i GK$ (see [10] for details.). This is equivalent to saying that the function GK must correctly encircle all the points $-\frac{1}{\lambda_i}$, so a single Nyquist diagram can be enough to grant the stability of the formation. Moreover, in certain situations, these points are restricted *a priori* to be in certain specific areas: for example, for normalized Laplacians of undirected formations, we will have $-\frac{1}{\lambda_i} \in \mathbb{R}$ and $-\infty \leq -\frac{1}{\lambda_i} \leqslant -\frac{1}{2}$ (Theorem 4). A Nyquist diagram that correctly encircles all this region will grant stability for all undirected formations; so with a single, simple test it is possible to prove a very general result. In Section VI we will show a kind of extension of this result to performance.

IV. THE DECOMPOSITION APPROACH

We take three steps forward with respect to what was shown in the previous Section. The first step is that we will not assume anymore that the formation is described by means of a Laplacian $\mathcal{L}$, which we now replace with a generic "pattern matrix" $\mathcal{P}$; this matrix is not necessarily a Laplacian, but of course it will still have to represent the pattern of communication between the agents, so we can imagine it as sparse matrix. Secondarily, we will allow a certain (limited) dynamic interaction between the subsystems, an interaction that follows the same pattern matrix, as it will be shown shortly. At last, we assume this $\mathcal{P}$ (or $\mathcal{L}$) to be diagonalizable. Then we can prove a variant of Theorem 5.

Theorem 6 (Decomposition): Consider an Nl-th order linear time invariant system described by the equations:

$$\begin{aligned} \dot{x} &= (I_N \otimes A_a + \mathcal{P} \otimes A_b)x + (I_N \otimes B_a + \mathcal{P} \otimes B_b)u \\ y &= (I_N \otimes C_a)x \\ q &= (\mathcal{P} \otimes C_b)x \end{aligned} \tag{7}$$

with $A_a, A_b \in \mathbb{R}^{l\times l}$, $B_a, B_b \in \mathbb{R}^{l\times m_u}$, $C_a \in \mathbb{R}^{r_y \times l}$ and $C_b \in \mathbb{R}^{r_q \times l}$. We assume that $\mathcal{P}$ is diagonalizable, with: $\mathcal{P} = Z^{-1}\Lambda Z$, where Λ is a diagonal matrix containing the eigenvalues λ_i of $\mathcal{P}$. Then:

1) the system in (7) is equivalent to the following set of N systems:

$$\begin{aligned} \dot{\hat{x}}_i &= (A_a + \lambda_i A_b)\hat{x}_i + (B_a + \lambda_i B_b)\hat{u}_i \\ \hat{y}_i &= C_a \hat{x}_i \\ \hat{q}_i &= \lambda_i C_b \hat{x}_i \end{aligned} \tag{8}$$

2) the controller (5) stabilizes the formation if and only if it simultaneously stabilizes all the systems in (8).

Proof: For 1), consider the following change of variables:

$$\begin{aligned} \hat{x} &= (Z \otimes I_l)x, & \hat{u} &= (Z \otimes I_{m_u})u, \\ \hat{y} &= (Z \otimes I_{r_y})y, & \hat{q} &= (Z \otimes I_{r_q})q \end{aligned} \tag{9}$$

which turns (7) into:

$$\begin{aligned} \dot{\hat{x}} &= (Z \otimes I_l)(I_N \otimes A_a + \mathcal{P} \otimes A_b)(Z \otimes I_l)^{-1}\hat{x} + \\ &\quad + (Z \otimes I_l)(I_N \otimes B_a + \mathcal{P} \otimes B_b)(Z \otimes I_{m_u})^{-1}\hat{u} \\ \hat{y} &= (Z \otimes I_{r_y})(I_N \otimes C_a)(Z \otimes I_l)^{-1}\hat{x} \\ \hat{q} &= (Z \otimes I_{r_q})(\mathcal{P} \otimes C_b)(Z \otimes I_l)^{-1}\hat{x} \end{aligned} \tag{10}$$

From the properties of Kronecker product [2] we have that:

$$\begin{aligned} &(Z \otimes I_l)(I_N \otimes A_a + \mathcal{P} \otimes A_b)(Z \otimes I_l)^{-1} = \\ &= (Z I_N Z^{-1}) \otimes (I_l A_a I_l) + (Z\mathcal{P}Z^{-1}) \otimes (I_l A_b I_l) = \\ &= I_N \otimes A_a + \Lambda \otimes A_b \end{aligned} \tag{11}$$

so (7) is equivalent to:

$$\begin{aligned} \dot{\hat{x}} &= (I_N \otimes A_a + \Lambda \otimes A_b)\hat{x} + (I_N \otimes B_a + \Lambda \otimes B_b)\hat{u} \\ \hat{y} &= (I_N \otimes C_a)\hat{x} \\ \hat{q} &= (\Lambda \otimes C_b)\hat{x} \end{aligned} \tag{12}$$

Since all of the matrices in this last expression are block diagonal, then the system is equivalent to the N independent systems of (8). Then 2) is a straightforward consequence of the first part of the Theorem. ■

Notice that Theorem 5 states an equivalence only for the stability of the systems: the modal systems in (6) are stable if and only if the original system is stable. This last Theorem instead states that the systems themselves are equivalent, through (9), in their dynamics from the input to the output. This means that we can evaluate also the performance of a system by looking at its decomposed version.

Before deriving the control synthesis method, we redefine the systems which are object of this paper according to the framework that is common in literature for evaluating disturbance rejection performance. We will define them in a very general way in order to include all the systems for which a decomposition can be achieved like in Theorem 6. We also move to a discrete time setting, for reasons that will be clear later on.

Definition 7: Let us consider the Nl-th order linear dynamical system described by:

$$\begin{cases} x(k+1) = \mathcal{A}x(k) + \mathcal{B}_w w(k) + \mathcal{B}_u u(k) \\ z(k) = \mathcal{C}_z x(k) + \mathcal{D}_{zw} w(k) + \mathcal{D}_{zu} u(k) \\ y(k) = \mathcal{C}_y x(k) + \mathcal{D}_{yw} w(k) \end{cases} \tag{13}$$

Where $u \in \mathbb{R}^{Nm_u}$ is the control input, $w \in \mathbb{R}^{Nm_w}$ is the disturbance, $y \in \mathbb{R}^{Nr_y}$ is the measured output and $z \in \mathbb{R}^{Nr_z}$ is the performance output. We call such systems "decomposable systems" iff $\mathcal{P}$ is diagonalizable ($\mathcal{P} = Z^{-1}\Lambda Z$) and

$$\begin{aligned} \mathcal{A} &= I_N \otimes A_a + \mathcal{P} \otimes A_b \\ \mathcal{B}_\bullet &= I_N \otimes B_{\bullet,a} + \mathcal{P} \otimes B_{\bullet,b} \\ \mathcal{C}_\bullet &= I_N \otimes C_{\bullet,a} + \mathcal{P} \otimes C_{\bullet,b} \\ \mathcal{D}_{\bullet\bullet} &= I_N \otimes D_{\bullet\bullet,a} + \mathcal{P} \otimes D_{\bullet\bullet,b} \end{aligned} \tag{14}$$

In the case that $\mathcal{P}$ is symmetric, then we call the system a "symmetric decomposable system"; then 1) Z is real and orthogonal ($Z^{-1} = Z^T$) and 2) Λ is real [15].

Remark 8: In the remainder of the paper we will consider only symmetric systems, as the two properties mentioned above greatly simplify the reasoning. Non symmetric systems are still manageable at the cost of some extra attentions (like dealing with complex-valued LMIs; see [18] for details). We generalize Theorem 6 for this kind of systems.

Theorem 9: A symmetric decomposable system of order Nl as described in Definition 7 is equivalent to N independent "modal" subsystems of order l. Each of these subsystems has only m_u inputs, m_d disturbances, r_z performance outputs and r_y measured outputs:

$$\begin{cases} \hat{x}_i(k+1) = \mathbf{A}_i \hat{x}_i(k) + \mathbf{B}_{w,i}\hat{w}_i(k) + \mathbf{B}_{u,i}\hat{u}_i(k) \\ \hat{z}_i(k) = \mathbf{C}_{z,i}\hat{x}_i(k) + \mathbf{D}_{zw,i}\hat{w}_i(k) + \mathbf{D}_{zu,i}\hat{u}_i(k) \\ \hat{y}_i(k) = \mathbf{C}_{y,i}\hat{x}_i(k) + \mathbf{D}_{yw,i}\hat{w}_i(k) \\ \text{for } i = 1, \dots, n \end{cases} \tag{15}$$

where for all the matrices in bold font it holds that:

$$\mathbf{M}_{\bullet,i} = M_{\bullet,a} + \lambda_i M_{\bullet,b} \tag{16}$$

where the λ_i are the eigenvalues of $\mathcal{P}$ and the matrices $M_{\bullet,a}$, $M_{\bullet,b}$ are defined as in (14). Conversely, it is true that all the sets of systems as in (15) for which the parameterization (16) holds are equivalent to a decomposable system.

Proof: The system is decomposed with the following change of variables:

$$x = (Z \otimes I_l)\hat{x}, \quad w = (Z \otimes I_{m_w})\hat{w}, \quad u = (Z \otimes I_{m_u})\hat{u}, \\ z = (Z \otimes I_{r_z})\hat{z}, \quad y = (Z \otimes I_{r_y})\hat{y} \tag{17}$$

For the details see [18]. ■

This last Theorem basically says that a decomposable system (that is, a system for which all the state space matrices $\mathcal{M}$ can be parameterized as $\mathcal{M} = I_N \otimes M_a + \mathcal{P} \otimes M_b$) is equivalent to a set of N independent systems with state matrices $\mathbf{M}_i = M_a + \lambda_i M_b$. Conversely, if a set of N independent systems has state matrices which are parameterized as $\mathbf{M}_i = M_a + \lambda_i M_b$, then they are equivalent to a decomposable system. This is of fundamental importance, because:

1) for a decomposable system, it is possible to simplify control problems by evaluating them in the frame of the N independent systems, which are of quite smaller order;
2) if the state space matrices of the controllers of the N independent systems are parameterized as in (16) (with the same $\mathcal{P}$), then the controller in its untransformed form will be a decomposable system with the same sparsity of the plant.

These considerations will be used in the next Section. Notice that the idea of the decomposition is already present in literature for other kinds of systems, for example for symmetrically interconnected systems [16] and for circulant systems [3], that in some cases overlap with the decomposable systems defined here.

Remark 10: From now on, we will always use the bold font to identify matrices which can be parameterized according to (16).

V. $\mathcal{H}_\infty$ CONTROLLER SYNTHESIS

For the class of decomposable systems, problems can be approached in the domain of the transformed variables, where the system is equivalent to a set of smaller independent modal subsystems. Once the solution has been obtained independently for each subsystem, one can retrieve the solution to the original problems through the inverse of (17).

For example, let us consider for a symmetric decomposable system as in (13) the problem of finding a stabilizing static state feedback:

$$u(k) = \mathcal{K}x(k) \tag{18}$$

which yields an $\mathcal{H}_\infty$ norm from w to z smaller than γ.

The approach to the solution would now be to find N $\mathcal{H}_\infty$ controllers for the system in its decomposed state as in (15) and then retrieve the controller in the non-decomposed form.

A. A note on the $\mathcal{H}_\infty$ norm

The first question that we have to solve is how the $\mathcal{H}_\infty$ norms of subsystems which make the system in the decomposed form relate to the $\mathcal{H}_\infty$ norm of the undecomposed system. This question is quickly answered through the following Lemma.

Lemma 11: Let T_{wz} be the transfer function of a discrete time symmetric decomposable system (Definition 7) from the disturbance w to the output z; let $\hat{T}_{\hat{w}\hat{z}}$ be the transfer function of the same system after transforming it with (17), from the new disturbance $\hat{w}$ to the new output $\hat{z}$, and let us call $\hat{T}_{\hat{w}_i\hat{z}_i}$ the transfer functions of each of the N modal subsystems into which the system can be decomposed, from $\hat{w}_i$ to $\hat{z}_i$. Then it holds:

$$||T_{wz}||_{\mathcal{H}_\infty} = \max_i ||\hat{T}_{\hat{w}_i\hat{z}_i}||_{\mathcal{H}_\infty} \tag{19}$$

With this Lemma we can then approach the problem of $\mathcal{H}_\infty$ synthesis in the following way: by constraining the $\mathcal{H}_\infty$ norm of all the modal subsystems to be smaller than a certain value γ, we then have the guarantee that the norm of the global, untransformed system will be smaller than γ as well.

B. LMI-based $\mathcal{H}_\infty$ control with state feedback

The basic LMI approach for solving the problem is to find a feasible solution to the following set of decoupled inequalities [12]:

$$\begin{bmatrix} X_i & \mathbf{A}_i X_i + \mathbf{B}_{u,i} L_i & \mathbf{B}_{w,i} & 0 \\ * & X_i & 0 & X_i \mathbf{C}_{z,i}^T + L_i^T \mathbf{D}_{zu,i}^T \\ * & * & I_{m_w} & \mathbf{D}_{zw,i}^T \\ * & * & * & \gamma^2 I_{r_z} \end{bmatrix} \succ 0 \\ \text{for } i = 1 \dots N \tag{20}$$

where $\mathbf{A}_i = A_a + \lambda_i A_b$, $\mathbf{B}_{u,i} = B_{u,a} + \lambda_i B_{u,b}$, etc. and the symbol $*$ is used to fill in the block symmetric matrix without repetitions of symbols. The decision variables are $X_i = X_i^T$ and L_i. These LMIs have a solution if and only if a controller that yields an $\mathcal{H}_\infty$ norm smaller than γ exists.

The static state gains can then be obtained as $K_i = L_i X_i^{-1}$; if we then construct a block diagonal matrix K by putting these K_i matrices in the diagonal blocks, the gain for the global, untransformed system will be:

$$\mathcal{K} = (Z \otimes I_{m_u}) K (Z^{-1} \otimes I_l) \tag{21}$$

In general this matrix $\mathcal{K}$ will have no sparsity, so the controller will be a full global controller, which will correspond to the solution of the $\mathcal{H}_\infty$ control problem for the global system.

C. Distributed $\mathcal{H}_\infty$ control with state feedback

Theorem 9 basically states that to a "bold" matrix as in (16) corresponds a matrix $\mathcal{M} = I_N \otimes M_a + \mathcal{P} \otimes M_b$ for the untransformed global systems. So the matrix $\mathcal{K}$ obtained through (21) would be sparse, that is, of the form $\mathcal{K} = I_N \otimes K_a + \mathcal{P} \otimes K_b$, if and only if the K_i matrices are parameterized as $K_i = K_a + \lambda_i K_b$. A controller of this kind could be implemented locally, as the computation of the control input for each agent would require only local information: the state of the agent itself and that of its "neighbors", the ones which are connected to it through the pattern matrix, as shown in Fig. 1.

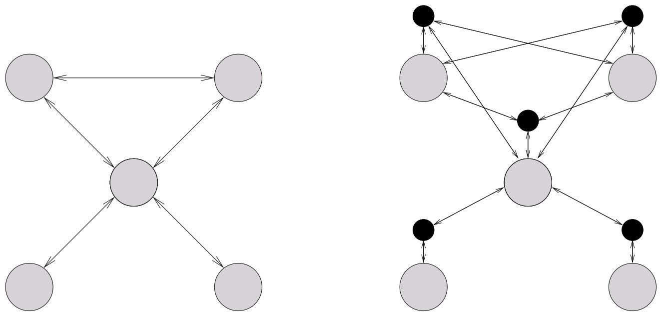

Fig. 1. On the left, a formation of 5 agents; the arrows indicate the interconnections among the agents as in the pattern matrix $\mathcal{P}$. These interactions could be dynamic (through the A_b matrix), or in the performance indices (through $C_{z,b}$), or of other kind. On the right, the smaller circles represent local controllers implementing a distributed controller; the controller follows the pattern too and thus it uses only local information.

This objective can be easily achieved by adding to the set of LMIs in (20) the following constraints:

$$\begin{array}{c} X_i = X \\ L_i = \mathbf{L}_i = L_a + \lambda_i L_b \end{array} \quad \text{for } i = 1 \dots N \tag{22}$$

where we write now L_i in bold to comply with the notation rule of Remark 10. This means that also K_i will be a "bold" matrix:

$$K_i = \mathbf{L}_i X^{-1} = L_a X^{-1} + \lambda_i L_b X^{-1} = \mathbf{K}_i \tag{23}$$

This "trick" is quite similar to the so-called *multiobjective optimization* [19], where in order to optimize different parameters of a system, it is necessary to equate the matrix associated to the Lyapunov function (the X_i matrices in this case) in different LMIs. This method is also called *Lyapunov shaping*. Of course the introduction of the constraints adds conservatism to the LMIs, so the method is not based anymore on an "if and only if" statement. We summarize this result in the following Theorem.

Theorem 12: Consider a discrete time symmetric decomposable system (Definition 7). A sufficient condition for the existence of a *sparse* static state feedback gain $\mathcal{K}$ as in (18) of the kind:

$$\mathcal{K} = I_N \otimes K_a + \mathcal{P} \otimes K_b \tag{24}$$

that yields a $||T_{wz}||_{\mathcal{H}_\infty} < \gamma$ is that the following set of LMIs has a feasible solution:

$$\begin{bmatrix} X & \mathbf{A}_i X + \mathbf{B}_{u,i}\mathbf{L}_i & \mathbf{B}_{w,i} & 0 \\ * & X & 0 & X\mathbf{C}_{z,i}^T + \mathbf{L}_i^T \mathbf{D}_{zu,i}^T \\ * & * & I_{m_w} & \mathbf{D}_{zw,i}^T \\ * & * & * & \gamma^2 I_{r_z} \end{bmatrix} \succ 0 \quad \text{for } i = 1 \dots N \tag{25}$$

where $X = X^T$ and $\mathbf{L}_i = L_a + \lambda_i L_b$ are the optimization variables; $K_a = L_a X^{-1}$, $K_b = L_b X^{-1}$.

The constraints can be quite conservative, considering that the number of LMIs may be large, and each of them must have the same Lyapunov matrix. But there is in the literature another approach to the problem that allows some relaxation. In [6] it is proven that the synthesis LMIs can be replaced with equivalent ones where an additional matrix is introduced, and this additional matrix can be equated in all the LMIs in the place of the Lyapunov matrix for multiobjective optimization. This method is called *G-shaping* from the symbol of this additional matrix (G), and it is available only for discrete time (this is why we use this assumption).

Then, according to [6], the set of LMIs in (20) can be replaced equivalently by:

$$\begin{bmatrix} X_i & \mathbf{A}_i G_i + \mathbf{B}_{u,i} L_i & \mathbf{B}_{w,i} & 0 \\ * & G_i + G_i^T - X_i & 0 & G_i^T \mathbf{C}_{z,i}^T + L_i^T \mathbf{D}_{zu,i}^T \\ * & * & I_{m_w} & \mathbf{D}_{zw,i}^T \\ * & * & * & \gamma^2 I_{r_z} \end{bmatrix} \succ 0 \quad \text{for } i = 1 \dots N \tag{26}$$

where the decision variables are $X_i = X_i^T$, G_i and L_i. The introduction of the constraints:

$$\begin{array}{c} G_i = G \\ L_i = \mathbf{L_i} = L_a + \lambda_i L_b \end{array} \quad \text{for } i = 1 \dots N \tag{27}$$

lets us arrive at the following Theorem, which is one of the main results of this paper.

Theorem 13: Consider a discrete time symmetric decomposable system (Definition 7). A sufficient condition for the existence of a *sparse* static state feedback controller $\mathcal{K}$ as in (18) of the kind: $\mathcal{K} = I_N \otimes K_a + \mathcal{P} \otimes K_b$ that yields a $||T_{wz}||_{\mathcal{H}_\infty} < \gamma$ is that the following set of LMIs has a feasible solution:

$$\begin{bmatrix} X_i & \mathbf{A}_i G + \mathbf{B}_{u,i}\mathbf{L}_i & \mathbf{B}_{w,i} & 0 \\ * & G + G^T - X_i & 0 & G^T \mathbf{C}_{z,i}^T + \mathbf{L}_i^T \mathbf{D}_{zu,i}^T \\ * & * & I_{m_w} & \mathbf{D}_{zw,i}^T \\ * & * & * & \gamma^2 I_{r_z} \end{bmatrix} \succ 0 \quad \text{for } i = 1 \dots N \tag{28}$$

where $X_i = X_i^T$, G and $\mathbf{L}_i = L_a + \lambda_i L_b$ are the optimization variables; $K_a = L_a G^{-1}$, $K_b = L_b G^{-1}$.

D. Distributed $\mathcal{H}_\infty$ control with dynamic output feedback

The same kind of approach can be used for the synthesis of distributed dynamic output feedback. Again, using a result from [6] for each single modal subsystem and introducing a parameterization we can get all the state space matrices of the controller as in (16). We report the result in the following Theorem.

Theorem 14: Consider a discrete time symmetric decomposable system (Definition 7), in one of the cases of Table I. A sufficient condition for the existence of a decomposable dynamic output feedback controller of the kind:

$$\begin{cases} x_c(k+1) = A_c x_c(k) + B_c y(k) \\ u(k) = C_c x_c(k) + D_c y(k) \end{cases} \tag{29}$$

that yields a $||T_{wz}||_{\mathcal{H}_\infty} < \gamma$ is that the set of LMI constraints in (30) (at the top of next page) has a feasible solution. The decision variables are X, Y, S and $P_i = P_i^T$, $H_i = H_i^T$, J_i, $\mathbf{L}_i = L_a + \lambda_i L_b$, $\mathbf{F}_i = F_a + \lambda_i F_b$, $\mathbf{Q}_i = Q_a + \lambda_i Q_b$, $\mathbf{R}_i = R_a + \lambda_i R_b$ for $i = 1, \dots, N$. Table I shows also additional constraints which might be needed. The state space matrices

$$\begin{bmatrix} P_i & J_i & \mathbf{A}_i X + \mathbf{B}_{u,i}\mathbf{L}_i & \mathbf{A}_i + \mathbf{B}_{u,i}\mathbf{R}_i\mathbf{C}_{y,i} & \mathbf{B}_{w,i} + \mathbf{B}_{u,i}\mathbf{R}_i\mathbf{D}_{yw,i} & 0 \\ * & H_i & \mathbf{Q}_i & Y\mathbf{A}_i + \mathbf{F}_i\mathbf{C}_{y,i} & Y\mathbf{B}_{w,i} + \mathbf{F}_i\mathbf{D}_{yw,i} & 0 \\ * & * & X + X^T - P_i & I_l + S^T - J_i & 0 & X^T\mathbf{C}_{z,i}^T + \mathbf{L}_i^T\mathbf{D}_{zu,i}^T \\ * & * & * & Y + Y^T - H_i & 0 & \mathbf{C}_{z,i}^T + \mathbf{C}_{y,i}^T\mathbf{R}_i^T\mathbf{D}_{zu,i}^T \\ * & * & * & * & I_{m_w} & \mathbf{D}_{zw,i}^T + \mathbf{D}_{yw,i}^T\mathbf{R}^T\mathbf{D}_{zu,i}^T \\ * & * & * & * & * & \gamma^2 I_{r_z} \end{bmatrix} \succ 0 \quad \text{for } i = 1, \ldots, N \tag{30}$$

of the controller (in the decomposed form) can be retrieved through the following relations:

$$\begin{cases} VU = S - YX \quad (\text{with } U, V \text{ non singular}) \\ \mathbf{D}_{c,i} = \mathbf{R}_i \\ \mathbf{C}_{c,i} = (\mathbf{L}_i - \mathbf{R}_i\mathbf{C}_{y,i}X)\, U^{-1} \\ \mathbf{B}_{c,i} = V^{-1}(\mathbf{F}_i - Y\mathbf{B}_{u,i}\mathbf{R}_i) \\ \mathbf{A}_{c,i} = V^{-1}(\mathbf{Q}_i - Y(\mathbf{A}_i + \mathbf{B}_{u,i}\mathbf{R}_i\mathbf{C}_{y,i})X + \\ \qquad -V\mathbf{B}_{c,i}\mathbf{C}_{y,i}X)\, U^{-1} - V^{-1}Y\mathbf{B}_{u,i}\mathbf{C}_{c,i} \end{cases} \tag{31}$$

TABLE I

CASES AND ADDITIONAL CONSTRAINTS FOR OUTPUT FEEDBACK.

	Case	Additional constraints
1	$C_{y,b} = 0, B_{u,b} = 0$	none
2	$B_{u,b} = 0$	$R_b = 0, F_{u,b} = 0$
3	$C_{y,b} = 0$	$R_b = 0$

In this last Theorem, (30) and (31) are adapted versions of the formulas shown in [6]. The additional constraints in Table I are necessary to make sure that the state space matrices found with (31) are always parameterized as in (16); this means that in (31) we must avoid products between bold matrices. For example, we have that:

$$\begin{gathered} \mathbf{C}_{c,i} = (\mathbf{L}_i - \mathbf{R}_i\mathbf{C}_{y,i}X)\, U^{-1} = \\ = (L_{c,a} + \lambda_i L_{c,b} - (R_a + \lambda_i R_b)(C_{y,a} + \lambda_i C_{y,b})X)\, U^{-1} \end{gathered} \tag{32}$$

So if we want $\mathbf{C}_{c,i}$ to be parameterized as $\mathbf{C}_{c,i} = C_{c,a} + \lambda_i C_{c,b}$, then we either need to have $\mathbf{C}_{y,i}$ constant ($C_{y,b} = 0$) or to set $\mathbf{R}_i$ as constant ($R_b = 0$).

VI. EXTENSIONS

In the previous Section we have shown a method for computing suboptimal $\mathcal{H}_\infty$ controllers for a formation of N systems, through the solution of a set on N cross-coupled LMIs. Actually there are some special cases where this computation can be further reduced, as we point out in the following Remark.

Remark 15: There are some cases when the set of N LMIs in (28) or in (30) are actually equivalent to just two LMIs. Considering (28), if $B_{u,b} = 0$ and $D_{zu,b} = 0$ (that is, if $\mathcal{B}_u$ and $\mathcal{D}_{zu}$ are block diagonal) then $\mathbf{B}_u = B_{u,b}$ and $\mathbf{D}_{zu} = D_{zu,a}$, that means, they are constant with respect to i. Thus there are no more products involving two "bold" matrices, making the matrices in the inequality all affine in λ_i. This allows expressing all the LMIs as a convex combination of the two which contain the extreme (maximum and minimum) values of λ_i. Since solving LMIs is a convex optimization problem [20], then the feasibility of the two inequalities with the extreme values of λ_i will guarantee the feasibility of the whole set. For (30), under the conditions of Table I, a similar reduction in the complexity can be done in the case of $B_{u,b} = 0$, $D_{zu,b} = 0$, $D_{yw,b} = 0$.

With these assumptions, the problem of finding a distributed controller is no longer depending on the size of the formation; whatever N is, the size of the LMIs and the number of decision variables involved are always the same.

There is one last observation to be made, that completes this article and makes a final connection with the work and the theory of graphs that was cited in the beginning.

Remark 16: Consider the situation of Remark 15. If $\mathcal{P}$ is a normalized Laplacian matrix of an undirected graph, then $0 \leqslant \lambda_i \leqslant 2$. Then the synthesis LMIs, either (28) or (30), can be used to find a controller with guaranteed boundaries on the $\mathcal{H}_\infty$ for all the possible undirected formations, if we assume a maximum λ_i of 2 and a minimum one of 0.

As a last consideration, the reader should also be aware that equivalents of Theorem 13 and Theorem 14 are available for the $\mathcal{H}_2$ case, and that the methods shown here apply to a wider class of systems than just vehicle formations (for example, agents with dynamic interactions). The interested reader can find out more in [18].

VII. EXAMPLE

As an example, we apply the methods of this article to a problem taken from [9]. Let us consider a swarm of satellites orbiting around a planet on a circular orbit; the small perturbations of the motion of each one of them with respect to the nominal circular trajectory are described by the Clohessy-Wiltshire equations [17]:

$$\begin{cases} \ddot{x}_1 = 3\omega_n^2 x_1 + 2\omega_n \dot{x}_2 + a_1 \\ \ddot{x}_2 = -2\omega_n \dot{x}_1 + a_2 \\ \ddot{x}_3 = -\omega_n^2 x_3 + a_3 \end{cases}$$

where x_1, x_2 and x_3 are respectively the displacements in the radial, tangential and out-of-plane direction with respect to an ideal body which is covering perfectly the circular orbit at an angular speed ω_n; a_1, a_2 and a_3 are the accelerations of the spacecraft due to either propulsion or external disturbances.

Let us now assume that N satellites are uniformly[2] distributed on the same circular orbit, and that we would like to design a controller that minimizes the error on their relative positions, with an $\mathcal{H}_\infty$ criterion. This means that if we consider the set of satellites as a single system, all the

[2] This assumption is not critical, the relative distance between the satellites just determines a "bias" term in the x_2 direction, that can be neglected in the synthesis procedure.

matrices will be block diagonal except $\mathcal{C}_z$. If we want each satellite to interact only with the one preceding and the one following it, then we can assume as output the following:

$$z_{x_h,i} = -\frac{1}{2}x_{h,i-1} + x_{h,i} - \frac{1}{2}x_{h,i+1} \quad \text{for } h = 1, 2, 3$$

where $x_{h,i}$ indicates the h^{th} coordinate of the i^{th} satellite (the index i has to be considered modulo N).

In this way, we can express $\mathcal{C}_z$ with the help of a symmetric normalized Laplacian matrix $\mathcal{L}$:

$$\mathcal{L} = \begin{bmatrix} 1 & -\frac{1}{2} & 0 & \cdots & 0 & -\frac{1}{2} \\ -\frac{1}{2} & 1 & -\frac{1}{2} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -\frac{1}{2} & 0 & 0 & \cdots & -\frac{1}{2} & 1 \end{bmatrix}$$

As a last addition, we consider a non-zero $\mathcal{D}_{zu}$ matrix in order to penalize the use of the actuators (the consumption of propellant).

We formulate the problem in discrete time; as we are in the case of Remark 16, then just two LMIs (one for $\lambda = 0$ and the other for $\lambda = 2$) are sufficient to solve the dynamic output feedback problem, for any number of satellites. Fig. 2 compares the performance of different controllers: 1) a completely decentralized one, which computes the control actions for each satellite only on the base of its (absolute) measures, 2) an optimal, centralized one and 3) the distributed controller computed with the method of this paper. As it could be expected, the centralized controller offers the best performance; but we can also see that the distributed controller is quite close to it with respect to the decentralized solution.

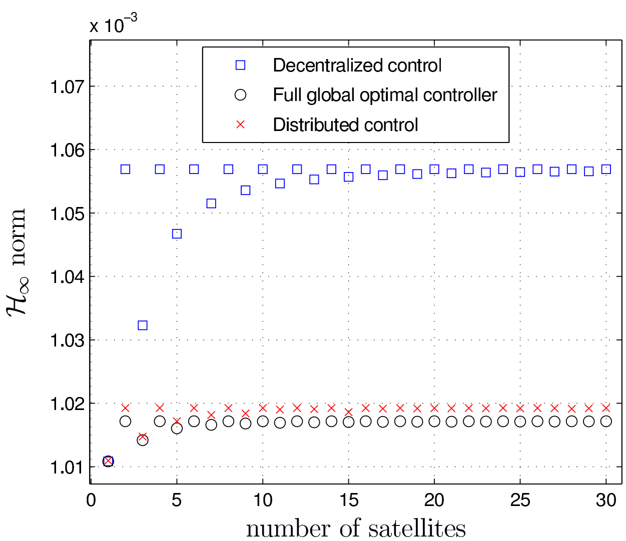

Fig. 2. $\mathcal{H}_\infty$ norm of the controlled formation with three different types of controllers.

VIII. CONCLUSIONS

In this article we have shown a simple method for designing $\mathcal{H}_\infty$ controllers for formations. The method can be considered as a spin-off of the ideas shown in [10]. One of the most interesing results of this reference is a variant of the Nyquist criterion that relates the stability of the formation to the number of encirclements of certain points in the Nyquist plot. As these points are related to the eigenvalues of a normalized Laplacian, then it is possible to define a region where they are restricted to be, and this makes it possible to find a controller that stabilizes all the possible formations of a kind with a single simple test. In fact, we can claim that Theorem 14 together with Remark 16 are a generalization of this method from simple stability to performance: as shown in the example, a single LMI test, which does not grow with the size of the formation, can be enough to guarantee disturbance rejection performance for all the formations of a kind.

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