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Minimal Dimension Internal Models for Lower Triangular Systems

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***Abstract*— Recursively designed internal models have been extensively applied in the output regulation problem for lower-triangular systems. In particular, one internal model component is constructed at each step to compensate for the steady state of one plant state (or input). This design method is simple with the potential expense of high dimension when some components are duplicate. This paper proposes a novel approach of designing a minimal dimension internal model by eliminating any possible duplication.**

I. Introduction and Problem Formulation

Consider the class of lower-triangular systems, with relative degree of r, as follows:

$$\begin{aligned} \dot{z} &= \psi(z, x_1, v, w) \\ \dot{x} &= \varphi(z, x, v, w) + Rx + bu \\ e &= x_1 - q(v, w) \end{aligned} \tag{1}$$

where $z \in \mathbb{R}^{n-r}$ and $x := [x_1, \cdots, x_r]^\top \in \mathbb{R}^r$ are the states, $u \in \mathbb{R}$ is the input, $e \in \mathbb{R}$ is the tracking error, $w \in \mathbb{R}^p$ represents the unknown (constant) parameters, and $v \in \mathbb{R}^q$ represents the reference trajectories and/or external disturbances. In (1), the notations are defined as follows:

$$\varphi(z, x, v, w) := \begin{bmatrix} \varphi_1(z, \vec{x}_1, v, w) \\ \vdots \\ \varphi_r(z, \vec{x}_r, v, w) \end{bmatrix}, \quad \vec{x}_i := \begin{bmatrix} x_1 \\ \vdots \\ x_i \end{bmatrix},$$

$$R = \begin{bmatrix} 0 & I_{(r-1)\times(r-1)} \\ 0 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 0_{(r-1)\times 1} \\ 1 \end{bmatrix}.$$

We assume that the functions ψ, φ, q are sufficiently smooth, $w \in \mathbb{W} \subset \mathbb{R}^p$ with $\mathbb{W}$ a compact set, and $v(t)$ is generated by an autonomous exosystem

$$\dot{v} = Sv, \ v(0) \in \mathbb{V}_0 \tag{2}$$

where $\mathbb{V}_0$ is a compact subset of $\mathbb{R}^q$. We also assume that the exosystem (2) is neutrally stable in the sense that all eigenvalues of S are simple with zero real parts. Clearly, under the neutral stability condition, we have $v(t) \in \mathbb{V}, t \geq 0$ for some compact subset $\mathbb{V}$ of $\mathbb{R}^q$ if $v(0) \in \mathbb{V}_0$. In fact, the exogenous signal $v(t)$ produced by the exosystem (2) represents finite combinations of constants and/or sinusoids with unknown amplitudes or phases, but known frequencies. Now, the problem studied in this paper can be precisely formulated as follows.

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Definition 1.1: *Global Robust Output Regulation Problem (GRORP):* For the system (1), to design a controller

$$u = k(x, \eta), \ \dot{\eta} = \lambda(x, \eta) \tag{3}$$

where $\eta \in \mathbb{R}^l$ is the compensator state with l to be specified later and the functions k and λ are continuously differentiable in their arguments, such that the states of the closed-loop system are bounded and $\lim_{t\to\infty} e(t) = 0$ for all initial state $[z(0), x(0), \eta(0)]^\top \in \mathbb{R}^{n+l}$ [1], all $w \in \mathbb{W}$, and all $v(t) \in \mathbb{V}$ generated by the exosystem (2). ∎

The output regulation problem for the class of linear systems has been thoroughly studied in the 1970s (see, e.g., [7], [8]). A salient outcome of this research is the internal model principle which enables the conversion of the output regulation problem into an eigenvalue placement problem for an augmented linear system. For the class of nonlinear systems, the same problem was first treated for the special case in which the exogenous signals are constant [9], [14], etc. The nonlinear output regulation problem with time varying exogenous signals was first studied in 1990 by Isidori and Byrnes without considering the parameter uncertainty [17]. A celebrated contribution of [17] is to link the solvability of the regulator equations to that of the output regulation problem. The robust version of the same problem was pursued by quite a few people in [2], [10], [20], etc. Various solvability conditions have been given which impose assumptions on the solution of regulator equations.

The aforementioned research on output regulation problem was restricted to local stability because its underpinning was Lyapunov linearization technique. With the introduction of non-local techniques, the research on output regulation problem has been experiencing a vigorous growth. The growth began from the stabilization problem, which is a special case of the output regulation problem. Specifically, in the last two decades, thanks to the introduction of differential geometry to the structural analysis of nonlinear systems, and the development of control synthesis methodology "backstepping", it becomes possible to study the non-local stabilization problem for a high-dimensional system by recursively studying a sets of low-dimensional subsystems connected in a certain way, e.g., lower-triangular form. In the simplest case where the system contains no uncertainty, the stabilization problem was solved using the backstepping technique in a series of papers [1], [19], [26]. In the more complex case where the system contains static uncertainty but not dynamic uncertainty, the stabilization problem was also extensively studied by many

[1]For column vectors $a_1, \cdots, a_n$, the column vector obtained by stacking them is denoted by $[a_1, \cdots, a_n]^\top := [a_1^\top \ \cdots \ a_n^\top]^\top$.

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researchers. Some of the representative results can be found in books [16], [21], [23] and so on. For the most general case where the system contains both static uncertainty and dynamic uncertainty, the stabilization problem was studied in [18] using the small gain theorem in the context of the input-to-state-stability (ISS). Recently, a Lyapunov approach is developed in [6] whose advantage is that an explicit Lyapunov function for the closed-loop system is constructed simultaneously.

The development in stabilization problem has facilitated the growth of the output regulation. Some authors have also addressed the semiglobal or global output regulation problem for nonlinear systems with special structures such as lower-triangular systems. The global robust output regulation of systems in output feedback form, a special class of lower-triangular systems, was studied in [5], [25], [27], etc. The semi-global robust output regulation problem for more general lower-triangular systems was studied in [15], [20], [22], etc. More recently, the global robust output regulation problem formulated in this paper was first studied in [13] and a series of accompanying papers. More results on output regulation can be found in some monographes including [2], [12], [24], etc.

As we discussed above, the basic manipulation for the stabilization or output regulation problem of lower-triangular systems is called "backstepping", i.e., recursively designing virtual controllers at each step. So, in the existing results of global robust output regulation problem, e.g, [13], the internal models are also recursively designed. Specifically, one internal model component is constructed compensating for the steady-state valve of one state x_i or input u at each recursive step. This recursive idea is simple and the manipulation is easy. However, a disadvantage of this method is the possible existence of duplicated components in the internal model when the steady-state values of more than one states (or input) share a common mode. In this situation, this internal model design method costs a high dimension. The main contribution of this paper is to bring a minimal dimension internal model by eliminating any possible duplication.

The remaining sections are organized as follows. The existing results on output regulation problem are revisited in Section II. Then, a minimal internal model is introduced in Section III with a numerical example. Finally, the paper is concluded in Section IV.

II. Revisit of the Existing Results

In this section, the existing output regulation theory, mainly developed in [13], is revisited and represented for better citation. A necessary condition for the output regulation problem is that the steady-state values of the plant input and state are well defined. Specifically, the so-called output regulator equations admit a well defined solutions. To this end, we first list the following assumption.

Assumption 2.1: There exists a sufficiently smooth function $\mathbf{z}(v,w)$, such that,

$$\frac{\partial \mathbf{z}(v,w)}{\partial v} Sv = \psi(\mathbf{z}(v,w), q(v,w), v, w), \tag{4}$$

for all $v \in \mathbb{V}$, $w \in \mathbb{W}$, ∎

Remark 2.1: Due to the special lower-triangular structure, Assumption 2.1 guarantees that the regulator equations associated with the system (1), i.e., (4) and

$$\begin{aligned} \frac{\partial \mathbf{x}(v,w)}{\partial v} Sv &= \varphi(\mathbf{z}(v,w), \mathbf{x}(v,w), v, w) \\ &\quad + R\mathbf{x}(v,w) + b\mathbf{u}(v,w) \\ 0 &= \mathbf{x}_1(v,w) - q(v,w), \end{aligned}$$

admit a solution given by $[\mathbf{z}(v,w), \mathbf{x}(v,w), \mathbf{u}(v,w))]^\intercal$ corresponding to the steady-state values of $[z, x, u]^\intercal$. ∎

Clearly, when $u = \mathbf{u}(v,w)$, an invariant manifold is defined by

$$\{[z, x, v]^\intercal \mid z = \mathbf{z}(v,w), x = \mathbf{x}(v,w)\}.$$

Now, let

$$\mathcal{z} := z - \mathbf{z}(v,w),\ \mathcal{x} := x - \mathbf{x}(v,w),\ \mathcal{u} := u - \mathbf{u}(v,w) \tag{5}$$

be the displacement of the state and input away from the invariant manifold. Then, by noting $e = \mathcal{x}_1$, a stricter definition of the global robust output regulation is given below (see [4]).

Definition 2.1: *Global Robust Output Regulation Problem 2 (GRORP2):* The GRORP with $\lim_{t\to\infty} e(t) = 0$ enhanced to $\lim_{t\to\infty}[\mathcal{z}(t), \mathcal{x}(t), \mathcal{u}(t)]^\intercal = 0$. ∎

Since the steady-state state and input depend on the uncertainties v and w, they can't be used in feedback. Otherwise, the problem becomes a trivial stabilization problem of designing a controller $\mathcal{u}$ using $\mathcal{x}$ in feedback through the direct cancellation (5). Therefore, we expect to construct an observer (or called an internal model) for these steady-state values, which are denoted by

$$\hbar(v,w) := [\ \mathbf{x}_2(v,w) \quad \cdots \quad \mathbf{x}_r(v,w) \quad \mathbf{u}(v,w)\ ]^\intercal.$$

We assume the vector $\hbar(v,w)$ can be reproduced by the dynamics given below .

Assumption 2.2: There exists a sufficiently smooth function $\tau(v,w) : \mathbb{R}^{q+p} \mapsto \mathbb{R}^l$ for an integer l, vanishing at the origin, such that, for all trajectories $v(t)$ of the exosystem,

$$\begin{aligned} \frac{d\tau(v,w)}{dt} &= \Phi\tau(v,w) \\ \hbar(v,w) &= \Psi\tau(v,w). \end{aligned} \tag{6}$$

And the pair (Ψ, Φ) is observable. ∎

In literature [13], the dynamics (6) is called a steady-state generator. It is used to produce the steady-state state and steady-state input of a nonlinear system when these steady-state values are perturbed away from zero by exogenous disturbances or required to be away from zero to track some reference signals. Moreover, to explicitly construct an observer associated with the steady-state generator, a certain observability condition should be imposed on it as shown in Assumption 2.2 that (Ψ, Φ) is observable.

The existence of a linear steady-state generator (6), or the satisfaction of Assumption 2.2, has been well investigated in literature. In particular, the equivalence of the immersion, polynomial, and trigonometric polynomial conditions on $\hbar(v,w)$ was studied in [11]. All of these conditions lead to the existence of (6). For example, if $\hbar_i(v,w)$ is a polynomial function of v for $i=1,\ldots,r$, then, we have

$$\tau_i(v,w) = [\hbar_i(v,w), L_{Sv}\hbar_i(v,w), \cdots, L_{Sv}^{l_i-1}\hbar_i(v,w)]^\intercal$$

for an integer l_i, such that

$$\begin{aligned} \frac{d\tau_i(v,w)}{dt} &= \Phi_i\tau_i(v,w) \\ \hbar_i(v,w) &= \Psi_i\tau_i(v,w). \end{aligned}$$

As a result, we can construct (6) by stacking the vectors and matrices as follows:

$$\tau(v,w) = [\tau_1(v,w), \cdots, \tau_r(v,w)]^\intercal,$$
$$\Phi = \begin{bmatrix} \Phi_1 & 0 & 0 \\ \vdots & \ddots & \vdots \\ 0 & 0 & \Phi_r \end{bmatrix}, \ \Psi = \begin{bmatrix} \Psi_1 & 0 & 0 \\ \vdots & \ddots & \vdots \\ 0 & 0 & \Psi_r \end{bmatrix}. \tag{7}$$

In literature, all steady-state generators and hence internal models are designed using the above procedure (7) for lower-triangular systems. As a result, their dimension is $l = l_1 + \cdots + l_r$. However, a disadvantage of this method is that the internal model dimension is not always optimized. There may exist an integer $l < l_1 + \cdots + l_r$ as shown in the following example.

Example 2.1: Consider

$$S = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \ \hbar(v) = \begin{bmatrix} v_1 \\ v_2 + v_3 \end{bmatrix}.$$

On one hand, we have

$$\tau_1(v) = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}, \ \tau_2(v) = \begin{bmatrix} v_2 + v_3 \\ -v_1 \\ -v_2 \end{bmatrix}$$

and

$$\Phi_1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \ \Psi_1 = \begin{bmatrix} 1 & 0 \end{bmatrix}$$
$$\Phi_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}, \ \Psi_2 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}.$$

As a result, (7) suggests a five dimension steady-state generator. On the other hand, we note Assumption 2.2 simply holds for

$$\tau(v) = v, \ \Phi = S, \ \Psi = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

whose dimension is three. This simple example reveals that the extra two dimensions are caused by the duplicated components $[v_1, v_2]^\intercal$ in $\tau_1(v)$ and $\tau_2(v)$. ∎

For a nonlinear system, there may be a variety of internal model candidates associated with steady-state generators. In this paper, we let $\tau(v,w)$ be the function vector with minimal dimension satisfying Assumption 2.2. In general, such a steady-state generator (6) doesn't have the special structure of (7) which is required in the existing results (see, e.g., [3], [13]). Nevertheless, an interesting feature of this paper is to show how this minimal dimension steady-state generator (6) leads to a minimal dimension internal model for lower-triangular systems. In other words, we will construct an internal model without relying on the special structure of (7).

III. A Minimal Dimension Internal Model

To facilitate the construction of a minimal dimension internal model for lower-triangular systems, the steady-state generator (6) is first rearranged to have a certain lower-triangular structure. This rearrangement is stated in the following lemma.

Lemma 3.1: *For an observable pair (Ψ,Φ), there exists a nonsingular matrix T, such that the matrices $C := \Psi T^{-1}$ and $A := T\Phi T^{-1}$, denoted by $(\Psi,\Phi) \sim (C,A)$ [2], satisfy the following property:*

$$C := \begin{bmatrix} C_1 & 0 & 0 & \cdots & 0 \\ C_{21} & C_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ C_{r1} & C_{r2} & \cdots & C_{r(r-1)} & C_r \end{bmatrix}$$
$$A := \begin{bmatrix} A_1 & 0 & 0 & \cdots & 0 \\ A_{21} & A_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ A_{r1} & A_{r2} & \cdots & A_{r(r-1)} & A_r \end{bmatrix} \tag{8}$$

where $C_i \in R^{1\times l_i}$ and $A_i \in \mathbb{R}^{l_i\times l_i}$ for an integer $l_i \geq 0$ satisfying $\sum_{i=1}^r l_i = l$ and the other matrices have the appropriate dimensions. The pair (C_i, A_i) is observable for $i = 1,\cdots,r$. ∎

Proof: The proof can be given by using the canonical decomposition for r times. First, since (Ψ,Φ) is observable, the canonical decomposition shows

$$(\Psi,\Phi) \sim \left(\begin{bmatrix} C_1 & 0 \\ \bar{\mathcal{C}}_2 & \mathcal{C}_2 \end{bmatrix}, \begin{bmatrix} A_1 & 0 \\ \bar{\mathcal{A}}_2 & \mathcal{A}_2 \end{bmatrix} \right)$$

where the pairs (C_1, A_1) and $(\mathcal{C}_2, \mathcal{A}_2)$ are observable. Now, for $i = 2,\cdots,r-1$, if $(\mathcal{C}_i, \mathcal{A}_i)$ is observable, then, again, the canonical decomposition shows

$$(\mathcal{C}_i, \mathcal{A}_i) \sim \left(\begin{bmatrix} C_i & 0 \\ \bar{\mathcal{C}}_{i+1} & \mathcal{C}_{i+1} \end{bmatrix}, \begin{bmatrix} A_i & 0 \\ \bar{\mathcal{A}}_{i+1} & \mathcal{A}_{i+1} \end{bmatrix} \right)$$

where the pairs (C_i, A_i) and $(\mathcal{C}_{i+1}, \mathcal{A}_{i+1})$ are observable. By using the mathematical induction, and letting $C_r = \mathcal{C}_r$ and $A_r = \mathcal{A}_r$, we can define the matrices C and A as in (8) satisfying $(\Psi,\Phi) \sim (C,A)$. ∎

By using the nonsingular matrix T given in Lemma 3.1, we define

$$\theta(v,w) := T\tau(v,w).$$

[2]For matrices $A, B \in \mathbb{R}^{n\times n}$ and $C, D \in \mathbb{R}^{m\times n}$, $(C,A) \sim (D,B)$ is defined as $D = CT^{-1}$ and $B = TAT^{-1}$ for a nonsingular matrix T.

On one hand, the steady-state vector $\hbar(v,w)$ can be generated by

$$\begin{aligned}\frac{d\theta(v,w)}{dt} &= A\theta(v,w)\\ \hbar(v,w) &= C\theta(v,w).\end{aligned}$$

On the other hand, the uncertain term $\hbar(v,w)$ is contained in the measurement output

$$y = Rx + bu$$

where x is the state and u is the controller output through

$$y = R\mathcal{x} + b\mathcal{u} + \hbar(v,w).$$

Here, we note

$$Rx + bu = [x_2, \cdots, x_r, u]^{\mathsf{T}}.$$

So, this measurement output y can be produced by the following system, viewing θ being a state, i.e., $\theta(t) = \theta(v(t), w)$, for the neatness of presentation,

$$\begin{aligned}\dot{\theta} &= A\theta\\ y &= R\mathcal{x} + b\mathcal{u} + C\theta.\end{aligned} \tag{9}$$

Since θ represents the steady-state values, the problem becomes trivial if θ is measurable. In our situation, θ is not a part of measurement state, which motives us to build an observer for θ. A possible one, motivated by the Luenberger observer, is given as follows:

$$\begin{aligned}\dot{\eta} &= A\eta + L(y - \hat{y})\\ \hat{y} &= C\eta\end{aligned} \tag{10}$$

where

$$L := \text{block diag}\{L_1, \ldots, L_r\}, \quad L_i \in \mathbb{R}^{r_i \times 1}$$

is chosen such that $M := A - LC$ is Hurwitz. In particular, the matrices $M_i = A_i - L_iC_i$ are Hurwitz for $i = 1, \cdots, r$.

Remark 3.1: The observer (10) is also called an internal model candidate. It has a property that, at the steady space, i.e., $\mathcal{x} = 0$, $\mathcal{u} = 0$, and $\eta = \theta$, its dynamics reduce to (9). In other words, at the steady space, the trajectories of the internal model candidate are governed by the steady-state generator (9). Moreover, if the state η of the internal model candidate is driven to asymptotically approach θ by an appropriately designed controller u, then the internal model candidate is further called an internal model. The concept of internal model (candidate) used here is closely related to what introduced in [13] . ∎

The internal model candidate (10) does reduce the dimension cost against the existing methods in dealing with lower-triangular systems. Next, we should further show that this minimal dimension internal model candidate works effectively to make the resulted stabilization problem solvable. To this end, we perform the coordinate transformation:

$$(\eta, v, z, x, u) \mapsto (\zeta, v, \mathcal{z}, \xi, \omega):$$
$$\begin{cases}\zeta = \eta - \theta(v,w)\\ \mathcal{z} = z - \mathbf{z}(v,w)\\ \xi = x - \mathbf{x}(v,w) - QC\zeta = \mathcal{x} - QC\zeta\\ \omega = u - \mathbf{u}(v,w) - b^{\mathsf{T}}C\zeta = \mathcal{u} - b^{\mathsf{T}}C\zeta\end{cases},$$
$$Q := \begin{bmatrix} 0 & 0\\ I_{(r-1)\times(r-1)} & 0\end{bmatrix}.$$

Then, noting $Rx + bu = R\xi + b\omega + C\eta$, we have

$$\begin{aligned}\dot{\zeta} &= M\zeta + L(R\xi + b\omega + C\zeta)\\ \dot{\mathcal{z}} &= \bar{\psi}(\mathcal{z}, \xi_1, v, w)\\ \dot{\xi} &= \bar{\varphi}(\mathcal{z}, \xi, \zeta, v, w) + R\xi + b\omega + C\zeta\end{aligned} \tag{11}$$

where

$$\begin{aligned}\bar{\psi}(\mathcal{z}, \xi_1, v, w) &:= \psi(\mathcal{z} + \mathbf{z}(v,w), \xi_1 + \mathbf{x}_1(v,w), v, w)\\ &\quad -\psi(\mathbf{z}(v,w), \mathbf{x}_1(v,w), v, w)\\ \bar{\varphi}(\mathcal{z}, \xi, \zeta, v, w) &:= \varphi(\mathcal{z} + \mathbf{z}(v,w), \xi + QC\zeta + \mathbf{x}(v,w),\\ &\quad v, w) - \varphi(\mathbf{z}(v,w), \mathbf{x}(v,w), v, w)\\ &\quad -QC\{M\zeta + L(R\xi + b\omega + C\zeta)\}.\end{aligned}$$

In particular, the state $\xi = x - QC\eta - \bar{b}(x_1 - e)$ is measurable and the input $\omega = u - b^{\mathsf{T}}C\eta$ is implementable where $\bar{b} = [1\ \ 0_{1\times(r-1)}]^{\mathsf{T}}$. Therefore, we can design a controller ω using feedback ξ for the system (11). The following Proposition shows this controller can be converted to the one solving the GRORP of the original system.

Proposition 3.1: *Consider the system (1) and (2) under Assumptions 2.1 and 2.2, there exists an internal model candidate (10), such that the augmented system (11) has the property that $[\zeta, \mathcal{z}, \xi]^{\mathsf{T}} = 0$ is the equilibrium point for the undriven system with $\omega = 0$ for all $w \in \mathbb{W}$ and $v \in \mathbb{V}$. Moreover, if there exists a controller*

$$\omega = \kappa(\xi), \tag{12}$$

with sufficiently smooth function κ satisfying $\kappa(0) = 0$, such that the equilibrium point is globally asymptotically stable for all $v \in \mathbb{V}$ and $w \in \mathbb{W}$. Then, the GRORP2 for the original system (1) and (2) is solved by a corresponding controller

$$\begin{aligned}u &= \kappa(x - QC\eta - \bar{b}(x_1 - e)) + b^{\mathsf{T}}C\eta\\ \dot{\eta} &= A\eta + L(Rx + bu - C\eta).\end{aligned} \tag{13}$$

∎

Proof: The proof is straightforward and thus omitted here. ∎

By Proposition 3.1, it suffices to solve the global robust stabilization problem of (11) viewing (v, w) as external uncertainties. To make the stabilization tractable, we introduce another set of coordinate transformation:

$$(\zeta, \mathcal{z}, \xi, \omega) \mapsto (\vartheta, \mathcal{z}, \xi, \omega) : \vartheta = \zeta - L\xi\ .$$

As a result, we have

$$\begin{aligned}\dot{\vartheta} &= M\vartheta + ML\xi - L\bar{\varphi}(z, \xi, \vartheta + L\xi, v, w)\\ \dot{z} &= \bar{\psi}(z, \xi_1, v, w)\\ \dot{\xi} &= \bar{\varphi}(z, \xi, \vartheta + L\xi, v, w) + R\xi + b\omega + C\zeta. \quad (14)\end{aligned}$$

Clearly, the solvability of the global stabilization problem for the system (11) is nothing but that for the system (14). So, what left is to look into the system (14) to give the solvability condition of the global stabilization problem. By noting that the matrices M, L, Q, C and $QCLR$ have the (block) lower-triangular structures and $QCLb = 0$, we can put the system (14) in the form of

$$\begin{aligned}\dot{\vartheta}_i &= M_i\vartheta + \gamma_i(z, \vec{\xi}_i, \vec{\vartheta}_{i-1}, v, w)\\ \dot{z} &= \bar{\psi}(z, \xi_1, v, w)\\ \dot{\xi}_i &= \phi_i(z, \vec{\xi}_i, \vec{\vartheta}_i, v, w) + \xi_{i+1} \text{ with } \xi_{r+1} := \omega\\ & \quad i = 1, \cdots, r \qquad (15)\end{aligned}$$

where $z_i \in \mathbb{R}^{l_i}$, $\xi_i \in \mathbb{R}$, $\vec{\vartheta}_i := [\vartheta_1, \cdots, \vartheta_i]^\intercal$, $\vec{\xi}_i := [\xi_1, \cdots, \xi_i]^\intercal$, and γ_i and ϕ_i are sufficiently smooth functions varnishing at their origins $[z, \vartheta, \xi]^\intercal = 0$.

The stabilization problem of (15) has been well studied in literature, e.g., [6], [13], [18], under different assumptions. For instance, the result cited in Proposition 3.2 is given under the following assumption.

Assumption 3.1: The system $z = \bar{\psi}(z, \xi_1, v, w)$ is robustly input-to-state stable with z as state and ξ_1 as input, and has a known continuously differentiable gain function. ∎

Proposition 3.2: *Consider the system (15) with $\vartheta_i \in \mathbb{R}^{l_i}$, $z \in \mathbb{R}^{n-r}$, $\xi_i \in \mathbb{R}$, $v \in \mathbb{V}$, and $w \in \mathbb{W}$. If M_i is Hurwitz, the functions γ_i and ϕ_i are sufficiently smooth satisfying $\gamma_i(0, 0, 0, v, w) = 0$, and $\phi_i(0, 0, 0, v, w) = 0$, then, under Assumption 3.1, there exists a sufficiently smooth controller $\omega = \kappa(\xi)$ such that the equilibrium point of the closed-loop system is globally asymptotically stable.* ∎

Now, by combining Propositions 3.1 and 3.2, it is ready to give the main result as follows.

Theorem 3.1: *Consider a lower-triangular system (1) and (2) under Assumptions 2.1, 2.2 and 3.1. Then, there exists a sufficiently function κ, such that the GRORP2 is solved by the controller (13).* ∎

Finally, this section is closed by a numerical example.

Example 3.1: Consider a nonlinear system

$$\begin{aligned}\dot{z} &= -z + e\\ \dot{x}_1 &= x_2\\ \dot{x}_2 &= w_1 z + w_2 z x_2^2 + u\\ e &= x_1 - v_1 \qquad (16)\end{aligned}$$

and an exosystem

$$\dot{v}_1 = v_2,\ \dot{v}_2 = -v_1,$$

where the uncertainties $w_1, w_2, v_1, v_2 \in [-1\ 1]$.

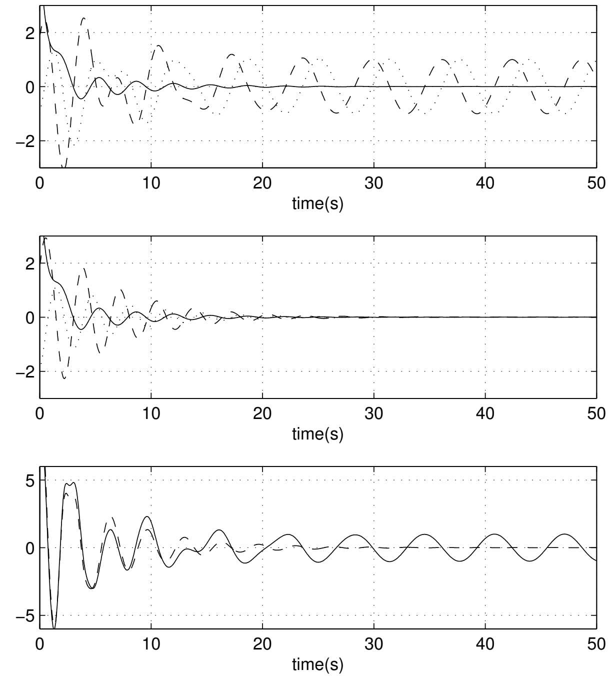

Fig. 1. The controller (18) works to drive the states $[z, x, u]^\intercal$ to the steady state values $[\mathbf{z}(v, w), \mathbf{x}(v, w), \mathbf{u}(v, w)]^\intercal$. Top graph: the trajectories of z (solid), x_1 (dotted), and x_2 (dashed). Middle graph: the tracking error profiles of $\mathcal{Z}$ (solid), $\mathcal{X}_1$ (dotted), and $\mathcal{X}_2$ (dashed). Bottom graph: the control profiles of u (solid) and $\mathcal{U}$ (dashed). (Simulation parameters: $z(0) = 5, x_1(0) = -1, x_2(0) = 2, v(0) = [1, 0]^\intercal, w_1 = 0.8, w_2 = -0.5$.)

First, we note that Assumption 2.1 is satisfied with

$$\mathbf{z}(v, w) = 0, \mathbf{x}_1(v, w) = v_1, \mathbf{x}_2(v, w) = v_2, \mathbf{u}(v, w) = -v_1.$$

Clearly, the GRORP2 for this system has been solved by a recursively designed internal model of dimension four in [13]. However, we see that a two dimension internal model is suggested in this paper by eliminating the duplicated mode between $\mathbf{x}_2(v, w)$ and $\mathbf{u}(v, w)$.

In particular, we can verify that Assumption 2.2 holds for $\tau(v) = [v_2, -v_1]^\intercal$ and

$$\Phi = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \Psi = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Let $A = \Phi$ and $C = \Psi$, and pick a matrix

$$L = \begin{bmatrix} 3 & 0 \\ 1 & 0 \end{bmatrix} \text{ such that } M = A - LC = \begin{bmatrix} -3 & 1 \\ -2 & 0 \end{bmatrix}$$

is Hurwitz. Then, the internal model can be given by (10). Obviously, Assumption 3.1 is satisfied. Now, by using the algorithm in [13], we can design a partial state feedback stabilizer

$$\omega = -30(\xi_1 + \xi_2) - 15(\xi_1 + \xi_2)^3 \qquad (17)$$

for the system (14). As a result, the overall controller for the GRORP2 can be given as follows,

$$\begin{aligned} u &= -30(e + x_2 - \eta_1) - 15(e + x_2 - \eta_1)^3 + \eta_2 \\ \dot{\eta} &= \begin{bmatrix} -3 & 1 \\ -2 & 0 \end{bmatrix} \eta + \begin{bmatrix} 3 \\ 1 \end{bmatrix} x_2. \end{aligned} \quad (18)$$

The profiles of states and tracking errors with the controller (18) are shown in Fig. 1. We can see that both the states and inputs approach their steady-state values, i.e., $\lim_{t\to\infty}[z(t), x(t), u(t)]^\tau = 0$.

∎

IV. Conclusion

The recursively designed internal models for lower-triangular systems have the disadvantage of duplication which costs a high dimension. A minimal dimension internal model has been developed in this paper to eliminate any possible duplication. For the convenience of presentation, only linear internal models are considered in this paper. These linear internal models can be easily extended to nonlinear ones by using the technique introduced in [13].

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