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Robust State Estimation Using Error Sensitivity Penalizing

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Abstract— **This paper deals with robust state estimation when parametric uncertainties nonlinearly affect a plant state-space model. A new framework is suggested on the basis of simultaneous minimization of nominal estimation errors and the sensitivities of estimation errors to model uncertainties. Under the condition that plant parameters are differentiable with respect to modelling errors, an analytic solution is derived for the optimal estimator which can be recursively realized. The computational complexity of the derived filter is comparable to that of the Kalman filter. Numerical simulations show that the obtained filter may have smaller estimation variance than other methods.**

Key Words—-**recursive estimation, regularized least-squares, robustness, state estimation, structured parametric uncertainty.**

I. INTRODUCTION

State estimation is one of the most important research fields in signal processing and industrial automation. The major reasons behind long and extensive attentions to this problem appear to lie upon the fact that there usually exist plant variables that can not be directly measured, and some widely adopted optimal control strategies, such as $\mathcal{H}_2$ and $\mathcal{H}_\infty$ control, can be divided into optimal state estimation and optimal state feedback [3], [5], [6], [7]. As modelling errors are unavoidable in any plant dynamics description, it is essential that performances of a state estimator do not change appreciably when the actual values of plant parameters deviate to some reasonable extents from their nominal ones. An estimator with this property is usually called a robust estimator/filter [1], [4], [8], [9], [10], [13].

Until now, many methods have been proposed for designing a robust state estimator. Among them, the $\mathcal{H}_\infty$ approach, the set-valued estimation approach, and the guaranteed cost paradigm are regarded as the most widely adopted ones. These approaches enrich the arsenal for state estimator analysis and synthesis. It is not an easy task, however, to recursively realize a $\mathcal{H}_\infty$ approach based robust filter, as certain existence conditions should be verified at every estimation step. Similar problems arise when the set-valued estimation approach or the guaranteed cost paradigm is adopted [12], [14], [16]. Note that recursiveness is an important property of filters, especially when a plant under investigation has time varying dynamics and on-line estimation is required [2], [6], [15].

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To overcome this difficulty, a regularized least-squares based framework is suggested in [14] for robust filter design. An attractive characteristic of this method is that the derived filter shares the same form of the well known and widely applied Kalman filter. This property makes it formally very easy to recursively realize the corresponding computations, and various well developed recursive schemes for the Kalman filter can be directly adopted. In the realization of this filter, however, it is necessary to optimize a cost function at every estimation step. Although it has been proved that this cost function has a unique minimum, no analytic expression is available for its optimum. On the other hand, based on extensive numerical simulation studies, some empirical values have been suggested for the optimal filter design parameter, but this approximation is not always valid. Other insufficiencies of this method include that plant parameters are required to depend linearly on a uncertainty block, and when there are structural constraints on the parametric uncertainty block, only upper bounds are available for the influences of modelling errors over filter performances and conservatism will generally be introduced into estimator design.

In robust filter design, another paradigm is based on penalizing the sensitivity of estimation errors to parameter variations [9], [11]. This paradigm has been adopted to filter design for single-input single-output systems using transfer function representation and spectral factorization. Numerical simulations show that when the penalizing factor is appropriately selected, which is usually not a very difficult task, the obtained filter may outperform that designed using the $\mathcal{H}_\infty$ approach. The derived results, however, are only applicable to linear time invariant systems.

In this paper, we investigate robust state estimation for multi-input multi-output systems. Modelling errors are permitted to affect plant parameters in a relatively arbitrary way, and the plant under investigation is allowed to have time varying dynamics. On the basis of a relation between the Kalman filter and regularized least-squares, as well as sensitivity penalizing on estimation errors to parameter variations, an analytic expression has been derived for the optimal estimator, provided that plant parameters are differentiable with respect to modelling errors. This estimator can be recursively implemented and has a comparable computational complexity with the Kalman filter. Numerical simulations show that this estimator may perform better than that of [14].

The rest of this paper is as follows. In the next section, a plant state-space model is given and some related previous conclusions are introduced. The robust state estimator is derived in section III, while some numerical simulation results are reported in Section IV. Finally, Section V concludes this paper.

II. Plant Dynamics Description and Preliminary Results

Assume that the dynamics of a plant can be described by the following state-space representation

$$x_{i+1} = A_i(\varepsilon_i)x_i + B_i(\varepsilon_i)u_i, \qquad i \geq 0 \tag{1}$$
$$y_i = C_i(\varepsilon_i)x_i + v_i \tag{2}$$

Here, x_0, u_i and v_i are uncorrelated zero-mean random variables with a covariance matrix

$$\mathcal{E}\left(\begin{bmatrix} x_0 \\ u_i \\ v_i \end{bmatrix}\begin{bmatrix} x_0 \\ u_i \\ v_i \end{bmatrix}^T\right) = \begin{bmatrix} \Pi_0 & & \\ & Q_i\delta_{ij} & \\ & & R_i\delta_{ij} \end{bmatrix}$$

in which $\mathcal{E}$ represents the operation of mathematical expectation, while Π_0, Q_i and R_i are known positive definite matrices. Moreover, δ_{ij} is the Kronecker delta function which equals to 1 when $i = j$ and to zero whenever $i \neq j$. Furthermore, ε_i stands for parametric modelling errors at the i-th sampled instant which consists of real valued scalar uncertainties $\varepsilon_{i,k}$, $k = 1, 2, \cdots, L$. It is assumed that all the entries of matrices $A_i(\varepsilon_i)$, $B_i(\varepsilon_i)$ and $C_i(\varepsilon_i)$ are known differentiable functions of ε_i, while $\varepsilon_{i,k}$, $k = 1, 2, \cdots, L$, are independent of each other. In addition to these, it is also assumed that $\varepsilon_{i,k}$ has been normalized to be contractive, that is, $|\varepsilon_{i,k}| \leq 1$.

When $\varepsilon_i \equiv 0$, $i = 0, 1, \cdots$, and x_0, u_i and v_i have normal distributions, it is well known that under the criterion of mean-squares, the Kalman filter is the optimal state estimator. Moreover, let $\hat{x}_{i|k}$ denote the optimal estimate of x_i based on the observations $y_j|_{j=0}^{k}$, and $P_{i|k}$ the estimation error covariance matrix $\mathcal{E}\left\{(x_i - \hat{x}_{i|k})(x_i - \hat{x}_{i|k})^T\right\}$. Then, when $\hat{x}_{i|i}$, $P_{i|i}$ and $y_j|_{j=0}^{i+1}$ are available, the optimal $\hat{x}_{i+1|i+1}$ can be obtained through solving the following minimization problem [2], [5], [14].

$$\hat{x}_{i+1|i+1} = A_i(0)\hat{x}_{i|i+1} + B_i(0)\hat{u}_{i|i+1} \tag{3}$$
$$\begin{bmatrix} \hat{x}_{i|i+1} \\ \hat{u}_{i|i+1} \end{bmatrix} = arg \min_{x_i,\, u_i} \left\{ ||x_i - \hat{x}_{i|i}||^2_{P_{i|i}^{-1}} + ||u_i||^2_{Q_i^{-1}} + ||y_{i+1} - C_{i+1}(0)x_{i+1}||^2_{R_{i+1}^{-1}} \right\} \tag{4}$$

Here, $||\xi||_W$ stands for the weighted Euclidean norm of vector ξ, that is, $||\xi||_W = \sqrt{\xi^T W \xi}$. In the following discussions, $||\xi||_W$ is usually abbreviated as $||\xi||$ when W is an identity matrix, in order to simplify notations.

Define matrices $H_i(\varepsilon_i, \varepsilon_{i+1})$, Φ_i and Ψ_i; vectors α_i and $\beta_i(\varepsilon_i, \varepsilon_{i+1})$, respectively as follows,

$$H_i(\varepsilon_i, \varepsilon_{i+1}) = C_{i+1}(\varepsilon_{i+1})[A_i(\varepsilon_i) \ \ B_i(\varepsilon_i)], \quad \Psi_i = R_{i+1}^{-1}$$
$$\Phi_i = \begin{bmatrix} P_{i|i}^{-1} & \\ & Q_i^{-1} \end{bmatrix}, \quad \alpha_i = \begin{bmatrix} x_i - \hat{x}_{i|i} \\ u_i \end{bmatrix}$$
$$\beta_i(\varepsilon_i, \varepsilon_{i+1}) = y_{i+1} - C_{i+1}(\varepsilon_{i+1})A_i(\varepsilon_i)\hat{x}_{i|i}$$

then, obviously, the cost function in the optimization problem of Equation (4) can be re-expressed by the following form with $\varepsilon_i = \varepsilon_{i+1} = 0$

$$||\alpha_i^T||^2_{\Phi_i} + ||H_i(\varepsilon_i, \varepsilon_{i+1})\alpha_i - \beta_i(\varepsilon_i, \varepsilon_{i+1}))||^2_{\Psi_i} \tag{5}$$

This means that the Kalman filter design can be converted to a regularized least-squares problem for which an analytic solution can be easily obtained through elementary algebraic operations [2], [5], [14].

III. Robust State Estimator Design

While the Kalman filter is optimal under a physically significant criterion and has been extensively applied to various engineering and financial problems, such as satellite orbit estimation, fading communication channel estimation, etc., it is also well known that its performances may be appreciably deteriorated by model uncertainties [5], [7], [12]. As modelling errors are generally unavoidable in actual filter designs, it is evidently essential that a physically meaningful filter should be robust against plant parameter deviations and/or inaccuracies. To achieve this purpose, it is suggested in [14] to minimize the cost function of Equation (5) with respect to modelling errors $(\varepsilon_{i,k}, \varepsilon_{i+1,k})|_{k=1}^{L}$ that maximize this cost function, that is, to minimize the cost function in the worst case. While this paradigm is widely adopted and physically attractive, in the derivation of the optimal estimator, Lagrange multipliers are introduced for which there are still not analytic expressions for the desirable filter. Although some empirical values are suggested for them, only the one uncertainty block case has been successfully dealt with.

For notational simplicity, denote $C_{i+1}(\varepsilon_{i+1})[A_i(\varepsilon_i) \ \ B_i(\varepsilon_i)]\alpha_i - (y_{i+1} - C_{i+1}(\varepsilon_{i+1})A_i(\varepsilon_i)\hat{x}_{i|i})$ by $e_i(\varepsilon_i, \varepsilon_{i+1})$. Then, when $A_i(\varepsilon_i)$, $B_i(\varepsilon_i)$ and $C_i(\varepsilon_i)$ are differentiable with respect to ε_i, it is obvious that for every $k = 1, 2, \cdots, L$, we have

$$\frac{\partial e_i(\varepsilon_i, \varepsilon_{i+1})}{\partial \varepsilon_{i,k}} = C_{i+1}(\varepsilon_{i+1})\left[\frac{\partial A_i(\varepsilon_i)}{\partial \varepsilon_{i,k}} \ \ \frac{\partial B_i(\varepsilon_i)}{\partial \varepsilon_{i,k}}\right]\alpha_i + C_{i+1}(\varepsilon_{i+1})\frac{\partial A_i(\varepsilon_i)}{\partial \varepsilon_{i,k}}\hat{x}_{i|i} \tag{6}$$
$$\frac{\partial e_i(\varepsilon_i, \varepsilon_{i+1})}{\partial \varepsilon_{i+1,k}} = \frac{\partial C_{i+1}(\varepsilon_{i+1})}{\partial \varepsilon_{i+1,k}}[A_i(\varepsilon_i) \ \ B_i(\varepsilon_i)]\alpha_i + \frac{\partial C_{i+1}(\varepsilon_{i+1})}{\partial \varepsilon_{i+1,k}}A_i(\varepsilon_i)\hat{x}_{i|i} \tag{7}$$

To reduce the sensitivity of the Kalman filter's performances to modelling errors, it is suggested in this paper to minimize the following cost function $J(\alpha_i)$ at every sampled time

$$J(\alpha_i) = \gamma\left[||\alpha_i||^2_{\Phi_i} + ||H_i(0,0)\alpha_i - \beta_i(0,0)||^2_{\Psi_i}\right] + (1-\gamma) \times \sum_{k=1}^{L}\left(\left|\left|\frac{\partial e_i(\varepsilon_i, \varepsilon_{i+1})}{\partial \varepsilon_{i,k}}\right|\right|^2 + \left|\left|\frac{\partial e_i(\varepsilon_i, \varepsilon_{i+1})}{\partial \varepsilon_{i+1,k}}\right|\right|^2\right)\Bigg|_{\substack{\varepsilon_i = 0 \\ \varepsilon_{i+1} = 0}} \tag{8}$$

and calculate an estimate of x_{i+1} using the formula of Equation (3). Here, γ is a non-negative scalar constant taking value in the interval $(0,\ 1]$. Clearly, in this cost function, a penalty has been added on the magnitude of the sensitivity of the matching error $H_i(\varepsilon_i, \varepsilon_{i+1})\alpha_i - \beta_i(\varepsilon_i, \varepsilon_{i+1})$ with respect to modelling errors $(\varepsilon_{i,k}, \varepsilon_{i+1,k})|_{k=0}^{L}$ at the plant nominal parameter values. With a little abuse of notation, this matching error is also called estimation error in this paper.

The design parameter γ in the above cost function takes a balance between the importance of nominal estimation performances and that of estimation performance degradation due to modelling errors. The bigger this parameter is, the more important the nominal estimation performances. In the extreme case, that is, when $\gamma = 1$ and/or $\frac{\partial A_i(\varepsilon_i)}{\partial \varepsilon_{i,k}} \equiv 0$, $\frac{\partial B_i(\varepsilon_i)}{\partial \varepsilon_{i,k}} \equiv 0$ and $\frac{\partial C_{i+1}(\varepsilon_{i+1})}{\partial \varepsilon_{i+1,k}} \equiv 0$, the above cost function is proportional to that of Equation (5) at $\varepsilon_i = \varepsilon_{i+1} = 0$ and both of them lead to the same optimal α_i. This implies that when there are no modelling errors and/or when the robustness of the estimator is not very important, the obtained estimator through minimizing the above cost function $J(\alpha_i)$ collapses to the well known Kalman filter.

Define matrices S_i and T_i respectively as

$$S_i = \begin{bmatrix} S_{i,1}(0,0) \\ S_{i,2}(0,0) \\ \vdots \\ S_{i,L}(0,0) \end{bmatrix}, \quad T_i = \begin{bmatrix} T_{i,1}(0,0) \\ T_{i,2}(0,0) \\ \vdots \\ T_{i,L}(0,0) \end{bmatrix}$$

in which for every $k = 1, 2, \cdots, L$,

$$S_{i,k}(\varepsilon_i, \varepsilon_{i+1}) = \begin{bmatrix} \frac{\partial C_{i+1}(\varepsilon_{i+1})}{\partial \varepsilon_{i+1,k}} A_i(\varepsilon_i) \\ C_{i+1}(\varepsilon_{i+1}) \frac{\partial A_i(\varepsilon_i)}{\partial \varepsilon_{i,k}} \end{bmatrix}$$

$$T_{i,k}(\varepsilon_i, \varepsilon_{i+1}) = \begin{bmatrix} \frac{\partial C_{i+1}(\varepsilon_{i+1})}{\partial \varepsilon_{i+1,k}} B_i(\varepsilon_i) \\ C_{i+1}(\varepsilon_{i+1}) \frac{\partial B_i(\varepsilon_i)}{\partial \varepsilon_{i,k}} \end{bmatrix}$$

Then, it can be directly proved that

$$\sum_{k=1}^{L} \left(\left|\left| \frac{\partial e_i(\varepsilon_i, \varepsilon_{i+1})}{\partial \varepsilon_{i,k}} \right|\right|^2 + \left|\left| \frac{\partial e_i(\varepsilon_i, \varepsilon_{i+1})}{\partial \varepsilon_{i+1,k}} \right|\right|^2 \right) \Bigg|_{\substack{\varepsilon_i = 0 \\ \varepsilon_{i+1} = 0}}$$
$$= ([S_i\ T_i]\alpha_i + S_i\hat{x}_{i|i})^T([S_i\ T_i]\alpha_i + S_i\hat{x}_{i|i}) \tag{9}$$

Substituting this relation into the cost function $J(\alpha_i)$, direct algebraic operations show that

$$\frac{\partial J(\alpha_i)}{\partial \alpha_i}\Bigg|_{\substack{\varepsilon_i = 0 \\ \varepsilon_{i+1} = 0}}$$
$$= 2\gamma \Bigg\{ \left(\Phi_i + H_i^T(0)\Psi_i H_i(0)) + \frac{1-\gamma}{\gamma}[S_i\ T_i]^T\ [S_i\ T_i] \right) \alpha_i - H_i^T(0)\Psi_i\beta_i(0) + \frac{1-\gamma}{\gamma}[S_i\ T_i]^T S_i\hat{x}_{i|i} \Bigg\} \tag{10}$$

From the definition of matrices Φ_i and Ψ_i, it is obvious that Φ_i is positive definite, while $H_i^T(0)\Psi_i H_i(0)$ is at least positive semi-definite. This means that when $0 < \gamma \leq 1$, the cost function $J(\alpha_i)$ is a strictly convex function and therefore has a unique minimum. Therefore, $J(\alpha_i)$ achieves its global optimal value at $\frac{\partial J(\alpha_i)}{\partial \alpha_i} = 0$. That is, the optimal α_i, denote it by α_{iopt}, is uniquely determined by

$$\left(\Phi_i + H_i^T(0)\Psi_i H_i(0)) + \frac{1-\gamma}{\gamma}[S_i\ T_i]^T\ [S_i\ T_i] \right) \alpha_{iopt}$$
$$= H_i^T(0)\Psi_i\beta_i(0) - \frac{1-\gamma}{\gamma}[S_i\ T_i]^T S_i\hat{x}_{i|i} \tag{11}$$

On the basis of this optimum characterization and the following relation,

$$\begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} = \begin{bmatrix} I & 0 \\ G_{21}G_{11}^{-1} & I \end{bmatrix} \times \begin{bmatrix} G_{11} & 0 \\ 0 & G_{22} - G_{21}G_{11}^{-1}G_{12} \end{bmatrix} \begin{bmatrix} I & G_{11}^{-1}G_{12} \\ 0 & I \end{bmatrix} \tag{12}$$

algebraic operations show that the optimal estimate of the plant state x_i can be computed using the following recursive procedure, which is very similar to that of the time and measurement-update form of the Kalman filter.

- Initialization. Designate $P_{0|0}$ and $\hat{x}_{0|0}$ respectively as $P_{0|0} = (\Pi_0^{-1} + C_0^T(0)R_0^{-1}C_0(0))^{-1}$ and $\hat{x}_{0|0} = P_{0|0}C_0^T(0)R_0^{-1}y_0$.
- Parameter modification. Define matrices $\hat{A}_i(0)$, $\hat{B}_i(0)$, $\hat{P}_{i|i}$ and $\hat{Q}_i$ respectively as follows

$$\hat{A}_i(0) = \left[A_i(0) - \frac{1-\gamma}{\gamma}\hat{B}_i(0)\hat{Q}_i T_i^T S_i \right] \left[I - \frac{1-\gamma}{\gamma}\hat{P}_{i|i}S_i^T S_i \right]$$
$$\hat{B}_i(0) = B_i(0) - \frac{1-\gamma}{\gamma}A_i(0)\hat{P}_{i|i}S_i^T S_i$$
$$\hat{P}_{i|i} = \left(P_{i|i}^{-1} + \frac{1-\gamma}{\gamma}S_i^T S_i \right)^{-1}$$
$$\hat{Q}_i = \left(Q_i^{-1} + \frac{1-\gamma}{\gamma}T_i^T \left[I + \frac{1-\gamma}{\gamma}S_i P_{i|i} S_i^T \right] T_i \right)^{-1}$$

- Plant state estimate updating. Calculate $\hat{x}_{i+1|i+1}$ and $P_{i+1|i+1}$ respectively as follows.

$$\hat{x}_{i+1|i+1} = \hat{A}_i(0)\hat{x}_{i|i} + P_{i+1|i+1}C_{i+1}^T(0)R_{i+1}^{-1} \times (y_{i+1} - C_{i+1}(0)\hat{A}_i(0)\hat{x}_{i|i})$$
$$P_{i+1|i} = A_i(0)\hat{P}_{i|i}A_i^T(0) + \hat{B}_i(0)\hat{Q}_i\hat{B}_i^T(0)$$
$$R_{e,i+1} = R_{i+1} + C_{i+1}(0)P_{i+1|i}C_{i+1}^T(0)$$
$$P_{i+1|i+1} = P_{i+1|i} - P_{i+1|i}C_{i+1}^T(0)R_{e,i+1}^{-1}C_{i+1}(0)P_{i+1|i}$$

The derivations are direct but tedious, and are omitted here due to space considerations. On the other hand, the estimate for the plant initial state x_0 is obtained through minimizing the following cost function

$$x_0^T\Pi_0^{-1}x_0 + (y_0 - C_0(0)x_0)^T R_0^{-1}(y_0 - C_0(0)x_0) \tag{13}$$

In addition to the above time and measurement-update form, it is also not difficult to derive a prediction form and an information form for the above robust state estimator. All of them take almost completely the same computation procedure as the corresponding Kalman filter. The only difference between the Kalman filter and the above robust state estimator is that, rather than directly using the nominal plant parameters and the minimal covariance matrix of the estimation error for the nominal model, in the recursive computation of the robust estimator, both plant parameters and estimation error covariance matrix are modified in every estimation step in order to take into account of estimation performance degradations due to modelling errors. The magnitude of modifications is mainly determined by the strength of the modelling error influences and the relative weighting factor γ which reflects a designer's balance between nominal estimation performances and performances deteriorations due to model inaccuracies.

Unlike the modifications of [14], there is no necessity to perform any on-line optimization in the above estimation procedure. Moreover, the restrictions are fairly weak on the influences of modelling errors on plant parameters. This is also different from that of [14], in which it is required that matrix $C_i(\varepsilon_i)$ is not affected by modelling errors, and matrices $A_i(\varepsilon_i)$ and $B_i(\varepsilon_i)$ depend linearly on a norm bounded uncertainty matrix.

The above results can be easily modified to the case in which modelling errors are time invariant. In this situation, the only required modification is to replace $S_{i,k}(\varepsilon_i, \varepsilon_{i+1})$ and $T_{i,k}(\varepsilon_i, \varepsilon_{i+1})$ in the above estimation algorithm respectively by $S_{i,k}(\varepsilon)$ and $T_{i,k}(\varepsilon)$ which are defined as follows

$$S_{i,k}(\varepsilon) = \frac{\partial C_{i+1}(\varepsilon)}{\partial \varepsilon_{\cdot k}} A_i(\varepsilon) + C_{i+1}(\varepsilon)\frac{\partial A_i(\varepsilon)}{\partial \varepsilon_{\cdot k}}$$
$$T_{i,k}(\varepsilon) = \frac{\partial C_{i+1}(\varepsilon)}{\partial \varepsilon_{\cdot k}} B_i(\varepsilon) + C_{i+1}(\varepsilon)\frac{\partial B_i(\varepsilon)}{\partial \varepsilon_{\cdot k}}$$

Here, ε represents modelling errors of the plant state-space model and $\varepsilon_{\cdot k}$ its k-th element.

IV. Numerical Simulations

In this section, we compare the performances of the derived robust state estimator with those of the Kalman filter and the robust state estimator of [14]. In these simulations, it is assumed that modelling errors are time-invariant, and every uncertainty parameter belongs to the interval $[-1,\ 1]$. 10^3 time-domain input-output pairs are generated for plant state estimation, in which all the plant initial states are set to zero, while disturbances u_i and v_i are produced according to normal distributions.

To compute the ensemble-average estimation error variance at every sampled instant, 5×10^2 simulations are performed for each set of numerical experiment settings. The size of the ensemble-average is approximated by the averaged value of the square of the Euclidean distance between the actual plant state and its estimate.

To investigate the influences of the design parameter γ on filter performances, 40 equally distributed samples are taken from the interval $[5.0000\times 10^{-3},\ 1.0000]$ in every numerical study. However, to make curves easy to be recognized, only part of the simulation results are included in the following figures. Specifically, the size of estimation error variances is provided for some typical values of the design parameter γ. Moreover, to reflect the fact that there are quite a few γs that make the suggested robust state estimator perform better than that of [14], the size of estimation error variances is also given for all the sampled γs at some typical instants. On the other hand, Kalman filters with both the nominal and the actual plant parameter values are realized in each collection of numerical simulations. The main purpose of this inclusion is to clarify performance deteriorations due to modelling errors. Furthermore, the design parameter λ is selected as $1.5\lambda_l$ for the filter of [14] which is recommended by the author.

A. Example 1

This example is adopted from [14], in which it is assumed that

$$A_i(\varepsilon) = \begin{bmatrix} 0.9802 & 0.0196 \\ 0.0000 & 0.9802 \end{bmatrix} + \begin{bmatrix} 0.0198 \\ 0.0000 \end{bmatrix} \times \varepsilon \times [0.0000\ \ 5.0000]$$
$$B_i(\varepsilon) = \begin{bmatrix} 1.0000 & \\ & 1.0000 \end{bmatrix}, \quad x_0 = \begin{bmatrix} 0.0000 \\ 0.0000 \end{bmatrix}$$
$$C_i(\varepsilon) = [1.0000\ \ -1.0000], \quad R_i = 1.0000$$
$$Q_i = \begin{bmatrix} 1.9608 & 0.0195 \\ 0.0195 & 1.9605 \end{bmatrix}, \quad \Pi_0 = \begin{bmatrix} 1.0000 & \\ & 1.0000 \end{bmatrix}$$

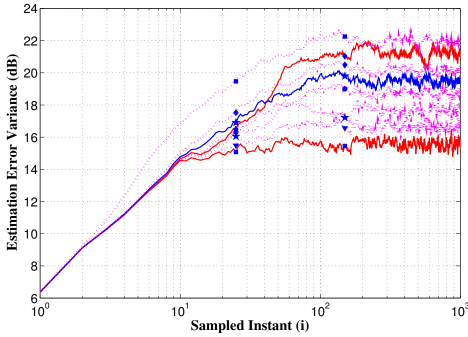
In the first set of simulations, the modelling error ε is fixed to be -0.8508. Figure 1 † shows the variations of estimation error variances with respect to time samples and the filter design parameter γ. An interesting observation from this figure is that when γ belongs to the interval $[0.4000,\ 0.9800]$, the robust state estimator of this paper outperforms both the filter suggested in [14] and the Kalman filter with nominal parameter values. Figure 1b also shows that at the sampled instants $i = 5 \times 10^2$ and $i = 10^3$, if γ takes the optimal value, which is approximately 0.8500, then, there is only about 1dB difference between the performances of the Kalman filter with actual parameter values and the robust state estimator derived in this paper. In this case, more than 2.5000dB performance improvement is obtained compared with the filter of [14], and more than 4.0000dB performance improvement is obtained compared with the Kalman filter based on nominal parameter values.

In another set of simulations, the modelling error ε is produced randomly and independently in each simulation according to a normal distribution with truncations. The mean and the standard variance of the normal distribution are set respectively to 0.0000 and 0.3333. In case that a generated ε has a magnitude greater than 1, it will be got rid of and another ε will be produced until a ε with magnitude not greater than 1 is obtained. The corresponding simulation results are given in Figure 2. Once again, it is observed that there exists a large interval of γ which leads to a robust estimator with better performances than the existent filters.

B. Example 2

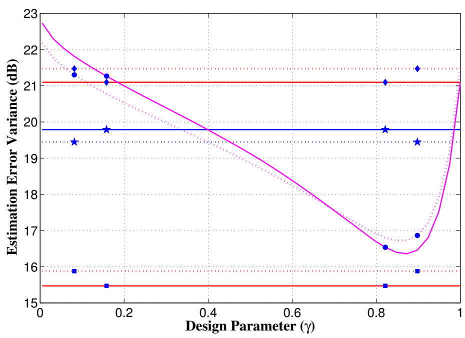
This example is modified from the previous one. Here, we assume that there are two uncertainties, ε_1 and ε_2, in a plant

†The same line styles and markers are used in this and the following 3 figures to represent the curves obtained from the same numerical simulation settings.



(a) Estimation Error Variance with a Fixed γ.

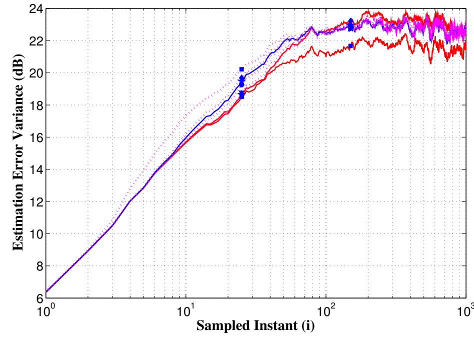
—□—: Kalman filter with actual parameters; —◇—: Kalman filter with nominal parameters; —⋆—: Filter of [14]; $\cdots\square\cdots$: New filter with $\gamma = 3.0513\times10^{-2}$; $\cdots\diamond\cdots$: New filter with $\gamma = 2.8564\times10^{-1}$; $\cdots\bullet\cdots$: New filter with $\gamma = 5.1526\times10^{-1}$; $\cdots\triangledown\cdots$: New filter with $\gamma = 7.9590\times10^{-1}$; $\cdots\star\cdots$: New filter with $\gamma = 9.4897\times10^{-1}$.



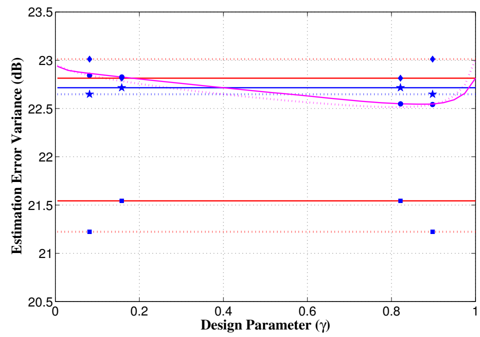
(b) Estimation Error Variance at a Fixed Instant.

$\cdots$: $i = 5\times10^2$; ——: $i = 10^3$. □: Kalman filter with actual parameters; ◇: Kalman filter with nominal parameters; $\star$: Filter of [14]; $\bullet$: Filter of this paper.

Fig. 1. Estimation Performances with a Fixed Modelling Error (Ex.1).



(a) Estimation Error Variance with a Fixed γ.



(b) Estimation Error Variance at a Fixed Instant.

Fig. 2. Estimation Performances with Random Modelling Errors (Ex.1).

state-space model. Specifically, it is assumed that

$$A_i(\varepsilon)=\begin{bmatrix}0.9802 & 0.0196\\ 0.0000 & 0.9802\end{bmatrix}+\begin{bmatrix}0.0198 & 0.0200\\ 0.0000 & 0.0396\end{bmatrix}\times \begin{bmatrix}\varepsilon_1 & \\ & \varepsilon_2\end{bmatrix}\begin{bmatrix}0.4375 & 5.0000\\ 0.0000 & 0.1094\end{bmatrix}$$

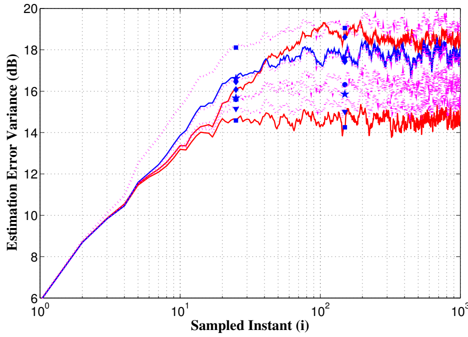
$$B_i(\varepsilon)=\begin{bmatrix}1.0000 & \\ & 1.0000\end{bmatrix}+\begin{bmatrix}0.0198 & 0.0200\\ 0.0000 & 0.0396\end{bmatrix}\times \begin{bmatrix}\varepsilon_1 & \\ & \varepsilon_2\end{bmatrix}\begin{bmatrix}1.7500 & 0.2188\\ 0.4375 & 0.8750\end{bmatrix}$$

All the other experimental settings keep unchanged. The simulation results are provided by Figure 3 when ε_1 and ε_2 are respectively fixed to be -0.8508 and -0.9432. From this figure, it is observed that there are still many γs that lead to a robust state estimator with better performances than both the filter suggested in [14] and the Kalman filter with nominal parameter values, and there is also a γ which leads to a robust estimator with closer performances than any other γ to those of the Kalman filter with the actual parameter values. Specifically, at the sampled instants $i = 5\times10^2$ and $i = 10^3$, the desirable interval for γ is approximately $[0.2500,\ 0.9800]$. If γ takes the optimal value, which is approximately 0.8300, the difference between the performances of the Kalman filter with actual parameter values and the robust state estimator derived in this paper is not greater than 1.0000dB. In this case, more than 2.5000dB performance improvement is obtained compared with both the filter of [14] and the Kalman filter based on nominal parameter values.

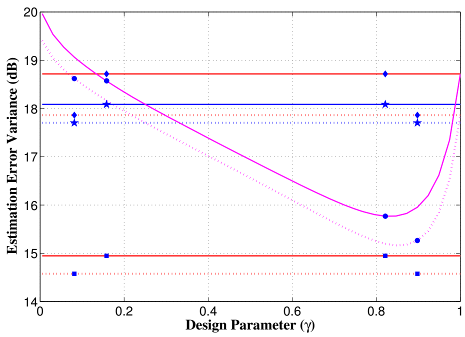
When both ε_1 and ε_2 are randomly and independently generated using the same method of Example 1 and a normal distribution of mean 0.0000 and standard variance 0.3333, the corresponding simulation results are shown in Figure 4. These results are consistent with those of Example 1.

From Figures 1-4, it is obvious that in plant state estimator designs, significant robustness improvements can be achieved through introducing an appropriate penalty on the sensitivity of estimation errors to modelling uncertainties. The optimal design parameter γ may lead to a robust estimator with performances close to those of the Kalman filter based on actual plant parameter values. Moreover, performances of the robust state estimator are continuous functions of this design parameter, and there are quite a lot of choices for it that make the corresponding estimator outperform the available recursive state estimators. These properties are attractive in actual filter design as there is still not a method to find the optimal filter design parameter. Furthermore, the stronger the modelling errors are on estimation performances, the smaller the desirable γ, which is consistent with physical intuitions.

Simulations have also been performed for which modelling

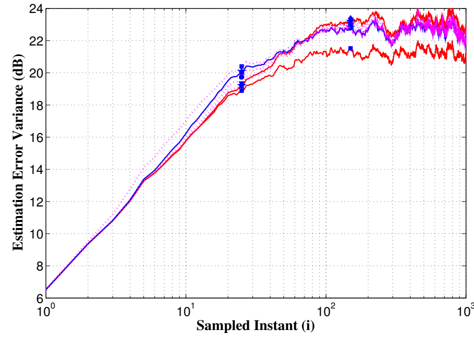


(a) Estimation Error Variance with a Fixed γ.

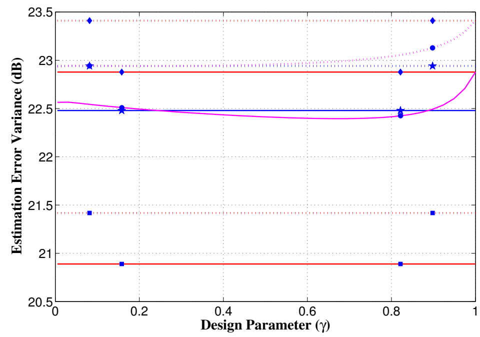


(b) Estimation Error Variance at a Fixed Instant.

Fig. 3. Estimation Performances with a Fixed Modelling Error (Ex.2).



(a) Estimation Error Variance with a Fixed γ.



(b) Estimation Error Variance at a Fixed Instant.

Fig. 4. Estimation Performances with Random Modelling Errors (Ex.2).

uncertainties are generated randomly according to the uniform distribution over $[-1,\ 1]$ which are suggested in [14]. The results are consistent with those of the aforementioned normal distribution with truncations.

V. Concluding Remarks

In this paper, a robust state estimator has been derived on the basis of penalizing the sensitivity of estimation errors with respect to plant model uncertainties. The obtained estimator takes completely the same form as that of the well known Kalman filter and can be easily implemented in a recursive manner. Moreover, its computational burden is also comparable to that of the Kalman filter. Attractive properties of this filter include that there are only very weak restrictions on the ways in which uncertainties affect plant parameters, and model uncertainties can take an arbitrary structure. Numerical simulations show that quite a lot of design parameters exist that lead to a robust estimator outperforming those based on worst case estimation error minimization. Moreover, estimation performances depend continuously on this design parameter. These properties make it relatively easy to find a penalizing factor that provides a robust state estimator with better performances.

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