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P·SPR·D Control for Affine Nonlinear System and Robot Manipulators

—Stability analysis based on K-Y-P Property and LaSalle's Invariance Principle—

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Abstract— **This paper is concerned with P·SPR·D control of affine nonlinear system and robot manipulators which are passive systems. P·SPR·D control consists of proportional(P) action + strict positive real(SPR) action + derivative(D) action. Such control can asymptotically stabilize the affine nonlinear system being of multi input and multi output. Stability analysis of the P·SPR·D control is made, based on the passivity theory and LaSalle's invariance principle. The L_2-gain disturbance attenuation problem is also investigated. Further a set-point servo problem (set-point tracking control) for the robot manipulator is also solved by the P·SPR·D control. The effectiveness of the proposed method is demonstrated by the simulation results for a two-link manipulator.**

I. Introduction

This paper investigates a PID-like control scheme for affine nonlinear system and robot manipulators. In regard to stabilizing control of affine nonlinear system there exist many studies as passivity theory[4, 5, 10, 11], exact linearization[6], back stepping method[7, 11], passivity based design of cascaded system[12], nonlinear H^∞ control[10] etc. But PID control has not been used so much except for the Lagrangian systems like robot manipulators.

We study stability analysis of P·SPR·D control imitating PID control for the affine nonlinear systems, based on the passivity theory and LaSalle's invariance principle[8]. (SPR is a short for strict positive real.) When the P·SPR·D controller is applied to a plant possessing the Kalman-Yakubovich-Popov (K-Y-P) property[4,5,11], we can prove that the closed-loop system becomes asymptotically stable by the P·SPR·D control, applying the passivity theory and LaSalle's theorem. The P·SPR·D control for Lagrangean system is seen in Ref.[13]. Passivity of PID controllers is investigated in Ref.[9] to some extent.

Section 2 makes a study of regulation problem for the affine nonlinear system in general. Section 3 is devoted to a set-point servo problem of robot manipulator system. In regard to robot manipulators, there exist many papers including Arimoto et al.[1,2] which applied PID control. So the feature of our paper is to apply the P·SPR·D control instead of the PID control. SPR stabilization of mechanical systems is discussed in the book[9] also. Section 4 investigates L_2 disturbance attenuation problem (γ-dissipativity) under the existince of disturbances. The L_2 disturbance attenuation by PID control, in case of robot manipulators, was analyzed in Ref.[3] as H^∞ design problem. Applying the P·SPR·D control, however, it is easy to solve the problem. Simulation results is presented in Section 5 to demonstrate the effectiveness of the P·SPR·D control.

II. P·SPR·D Control of Affine Nonlinear System

Let us consider an affine nonlinear system

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) + G(\boldsymbol{x})\boldsymbol{u} \tag{1}$$

$$\boldsymbol{y} = \boldsymbol{h}(\boldsymbol{x}) \tag{2}$$

where $\boldsymbol{x} \in R^n, \boldsymbol{u} \in R^m, \boldsymbol{y} \in R^m$ are the state vector, the control input and the measurable output. We assume that system (1),(2) is stabilizable.

Consider PID control for the regulation problem

$$\boldsymbol{u} = -K_P\boldsymbol{y} - K_I\int_0^t \boldsymbol{y}dt - K_D\dot{\boldsymbol{y}} \tag{3}$$

where $K_P \in R^{m\times m}, K_I \in R^{m\times m}, K_D \in R^{m\times m}$ are gain matrices corresponding to proportional, integral and derivative actions, respectively.

Introducing here a new state equation (an integlator)

$$\dot{\boldsymbol{\xi}} = -\boldsymbol{y} \tag{4}$$

the PID control (3) is expressed as

$$\boldsymbol{u} = -K_P\boldsymbol{y} + K_I\boldsymbol{\xi} - K_D\dot{\boldsymbol{y}} \tag{5}$$

Below we propose P·SPR·D control for asymptotical stabilization of affine nonlinear system, applying the passivity theory and LaSalle's invariance principle[8].

First the following is well known[6,11].

[**Theorem 1**] Assume that system (1),(2) is passive and zero state detectable[1] . Then the output feedback control

$$\boldsymbol{u} = -K_P\boldsymbol{y}$$

asymptotically stabilizes an equilibrium point $\boldsymbol{x}_e = \boldsymbol{0}$, where $K_P \in R^{m\times m}$ is a positive definite matrix.

Now consider the cascaded system of subsystem Σ_p and subsystem Σ_c :

$$\Sigma_c : \dot{\boldsymbol{\xi}} = D\boldsymbol{\xi} - \boldsymbol{y}, \quad D < 0 \tag{6}$$

$$\Sigma_p : \dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) + G(\boldsymbol{x})\boldsymbol{u} \tag{7}$$

$$\boldsymbol{y} = \boldsymbol{h}(\boldsymbol{x}) \tag{8}$$

where Σ_p represents the controlled object (1),(2). We consider here the strict positive real (SPR) system (6) instead of the integrator (4). Then the following theorem holds.

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[1] Nonlinear system (1),(2) is *zero state detectable* , if $\boldsymbol{x}(t) \to \boldsymbol{0}$ as $t \to \infty$ when $\boldsymbol{u}(t) = \boldsymbol{0}, \boldsymbol{y}(t) = \boldsymbol{0} \ \ \forall t \geq 0$

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[**Theorem 2**] Suppose that the cascaded system (6)∼(8) of subsystem Σ_p and subsystem Σ_c satisfies :

Assumption (a) Subsystem Σ_p is passive.

Assumption (b) Subsystem Σ_c is asymptotically stable as $\boldsymbol{y} = \boldsymbol{0}$, that is, there exists a positive definite function $U(\boldsymbol{\xi}) = \frac{1}{2}\boldsymbol{\xi}^T K_S \boldsymbol{\xi} > 0$ such that

$$\dot{U}(\boldsymbol{\xi}) = \boldsymbol{\xi}^T K_S D \boldsymbol{\xi} < 0 \tag{9}$$

Then if the system Σ_p is zero state detectable with respect to the output $\boldsymbol{y}$, the P·SPR·D control

$$\boldsymbol{u} = -K_P \boldsymbol{y} + K_S \boldsymbol{\xi} - K_D \dot{\boldsymbol{y}} \tag{10}$$

asymptotically stabilizes the closed-loop system of cascaded system of Σ_p and Σ_c at the equilibrium point $(\boldsymbol{x}_e, \boldsymbol{\xi}_e) = (\boldsymbol{0}, \boldsymbol{0})$, provided that K_P, K_S are positive definite matrices and K_D is semi-positive definite one.

(**Proof**) From Assumption (a), letting the semi-positive definite storage function of Σ_p as $W(\boldsymbol{x}) \geq 0$, $W(\boldsymbol{0}) = 0$, the so-called K-Y-P property

$$W_{\boldsymbol{x}}(\boldsymbol{x})\boldsymbol{f}(\boldsymbol{x}) \leq 0 \tag{11}$$

$$W_{\boldsymbol{x}}(\boldsymbol{x})G(\boldsymbol{x}) = \boldsymbol{y}^T \tag{12}$$

holds[4,5,10]. For the overall system consider a Lyapunov function candidate (semi-positive definite function)

$$\begin{aligned} V(\boldsymbol{x}, \boldsymbol{\xi}) &= W(\boldsymbol{x}) + U(\boldsymbol{\xi}) + \frac{1}{2}\boldsymbol{y}^T K_D \boldsymbol{y} \\ &= W(\boldsymbol{x}) + \frac{1}{2}\boldsymbol{\xi}^T K_S \boldsymbol{\xi} + \frac{1}{2}\boldsymbol{y}^T K_D \boldsymbol{y} \geq 0 \end{aligned} \tag{13}$$

and take a time derivative of $V(\boldsymbol{x}, \boldsymbol{\xi})$ along (6),(7),(10) and use (9),(11),(12) to get

$$\begin{aligned} &\dot{V}(\boldsymbol{x}, \boldsymbol{\xi}) \\ &= W_{\boldsymbol{x}}(\boldsymbol{x})\dot{\boldsymbol{x}} + \boldsymbol{\xi}^T K_S \dot{\boldsymbol{\xi}} + \boldsymbol{y}^T K_D \dot{\boldsymbol{y}} \\ &= W_{\boldsymbol{x}}(\boldsymbol{x})\{\boldsymbol{f}(\boldsymbol{x}) + G(\boldsymbol{x})\boldsymbol{u}\} + \boldsymbol{\xi}^T K_S (D\boldsymbol{\xi} - \boldsymbol{y}) + \boldsymbol{y}^T K_D \dot{\boldsymbol{y}} \\ &= W_{\boldsymbol{x}}(\boldsymbol{x})\boldsymbol{f}(\boldsymbol{x}) + W_{\boldsymbol{x}}(\boldsymbol{x})G(\boldsymbol{x})(-K_P\boldsymbol{y} + K_S\boldsymbol{\xi} - K_D\dot{\boldsymbol{y}}) \\ &\quad + \boldsymbol{\xi}^T K_S (D\boldsymbol{\xi} - \boldsymbol{y}) + \boldsymbol{y}^T K_D \dot{\boldsymbol{y}} \\ &= W_{\boldsymbol{x}}(\boldsymbol{x})\boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{y}^T(-K_P\boldsymbol{y} + K_S\boldsymbol{\xi} - K_D\dot{\boldsymbol{y}}) \\ &\quad + \boldsymbol{\xi}^T K_S D \boldsymbol{\xi} - \boldsymbol{\xi}^T K_S \boldsymbol{y} + \boldsymbol{y}^T K_D \dot{\boldsymbol{y}} \\ &\leq -\boldsymbol{y}^T K_P \boldsymbol{y} + \boldsymbol{\xi}^T K_S D \boldsymbol{\xi} \leq 0 \end{aligned} \tag{14}$$

Here $\dot{V}(\boldsymbol{x}, \boldsymbol{\xi})$ is semi-negative definite. Accordingly, Lyapunov's stability theorem cannot be applied, as $V(\boldsymbol{x}, \boldsymbol{\xi})$ is semi-positive definite and $\dot{V}(\boldsymbol{x}, \boldsymbol{\xi})$ is semi-negative definite. So we apply LaSalle's invariance principle[8] to prove that the overall system is asymptotically stable at the equilibrium $(\boldsymbol{x}, \boldsymbol{\xi}) = (\boldsymbol{0}, \boldsymbol{0})$.

Now let $\Omega_c = \{(\boldsymbol{x}, \boldsymbol{\xi}) | V(\boldsymbol{x}, \boldsymbol{\xi}) \leq c\}$ and suppose that Ω_c is bounded and $\dot{V}(\boldsymbol{x}, \boldsymbol{\xi}) \leq 0$ in Ω_c (c is a positive number such that $\dot{V}(\boldsymbol{x}, \boldsymbol{\xi}) \leq 0$). Here define Ω_E as a set of all points of Ω_c satisfying $\dot{V}(\boldsymbol{x}, \boldsymbol{\xi}) = 0$ and put

$$\Omega_E = \left\{(\boldsymbol{x}, \boldsymbol{\xi}) \mid \dot{V}(\boldsymbol{x}, \boldsymbol{\xi}) = 0,\ (\boldsymbol{x}, \boldsymbol{\xi}) \in \Omega_c\right\}$$

Since $K_P > 0$, $K_S D < 0$ from the condition of the theorem, $\dot{V}(\boldsymbol{x}, \boldsymbol{\xi}) = 0$ holds from (14) only when $\boldsymbol{\xi} = \boldsymbol{0},\ \boldsymbol{y} = \boldsymbol{0}$, that is,

$$\Omega_E = \{(\boldsymbol{x}, \boldsymbol{\xi}) \mid \boldsymbol{\xi} = \boldsymbol{0},\ \boldsymbol{y} = \boldsymbol{0},\ (\boldsymbol{x}, \boldsymbol{\xi}) \in \Omega_c\}$$

But, when $\boldsymbol{\xi} = \boldsymbol{0},\ \boldsymbol{y} = \boldsymbol{0}$, one has $\boldsymbol{u} = \boldsymbol{0}$ from (10). Thus it follows that

$$\Omega_E = \{(\boldsymbol{x}, \boldsymbol{\xi}) \mid \boldsymbol{\xi} = \boldsymbol{0},\ \dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}),\ \boldsymbol{y} = \boldsymbol{0},\ (\boldsymbol{x}, \boldsymbol{\xi}) \in \Omega_c\} \tag{15}$$

The subsystem Σ_p is zero state detectable from the condition of the theorem. Therefore, $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}),\ \boldsymbol{y} = \boldsymbol{h}(\boldsymbol{x}) = \boldsymbol{0}$ implies that $\boldsymbol{x}(t) \to \boldsymbol{0}$ as $t \to \infty$ in Ω_E by the definition of zero state detectability. Consequently, $(\boldsymbol{x}, \boldsymbol{\xi})$ satisfying $\dot{V}(\boldsymbol{x}, \boldsymbol{\xi}) = 0$ consists of only a point $(\boldsymbol{x}, \boldsymbol{\xi}) = (\boldsymbol{0}, \boldsymbol{0})$. Namely, letting Ω_M be the largest invariance set in Ω_E, Ω_M consists of only the equilibrium point $(\boldsymbol{x}_e, \boldsymbol{\xi}_e) = (\boldsymbol{0}, \boldsymbol{0})$. Thus, by LaSalle's invariance principle, all trajectories in Ω_c converge to Ω_M as $t \to \infty$, that is, converge to the equilibrium $(\boldsymbol{x}_e, \boldsymbol{\xi}_e) = (\boldsymbol{0}, \boldsymbol{0})$. Q.E.D

We call the PID type control (10) with (6) **P·SPR·D control**.

As is well known from Theorem 1, if the system is passive and zero state detectable, one can stabilize it by $\boldsymbol{u} = -K_P\boldsymbol{y}$. Hence the reason why we use the P·SPR·D control (10) is to improve control performance. It is noticed that there is a lot of freedom in regard to the best choice of parameter matrices $K_P,\ K_S,\ K_D$.

By the way, static state feedback control law may be obtained by the passivity based design[11,12] of the cascaded system also. Generally speaking, however, the control law using a storage function is complex. Besides, an advantage of the P·SPR·D control is of output feedback of simple structure.

III. P·SPR·D Control of Robot Manipulators

In this section we consider an application to a set-point problem of robot manipulators. An equation of motion of the manipulator with n degree of freedom can be obtained by the Euler-Lagrange formulation . Let $\boldsymbol{q}$ be the position of each link of manipulator, $\boldsymbol{\tau}$ the input torque, $\frac{1}{2}\dot{\boldsymbol{q}}^T M(\boldsymbol{q})\dot{\boldsymbol{q}}$ the kinetic energy, $U(\boldsymbol{q})$ the potential energy. The system then can be represented as

$$M(\boldsymbol{q})\ddot{\boldsymbol{q}} + \frac{1}{2}\dot{M}(\boldsymbol{q})\dot{\boldsymbol{q}} + S(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + \boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{\tau} \tag{16}$$

where $M(\boldsymbol{q})$ denotes the inertia matrix which is positive definite and bounded, $\boldsymbol{g}(\boldsymbol{q}) \triangleq U_q(\boldsymbol{q})^T$ is the gradient of the gravity potential energy and $S(\boldsymbol{q}, \dot{\boldsymbol{q}})$ denotes

$$S(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}} = \frac{1}{2}\left\{\dot{M}(\boldsymbol{q})\dot{\boldsymbol{q}} - \left[\frac{\partial}{\partial \boldsymbol{q}}\boldsymbol{q}^T M(\boldsymbol{q})\dot{\boldsymbol{q}}\right]^T\right\}$$

which is a skew-symmetric matrix. Letting $\boldsymbol{x}_1 = \boldsymbol{q} \in R^n$, $\boldsymbol{x}_2 = \dot{\boldsymbol{q}} \in R^n$, $\boldsymbol{x} = (\boldsymbol{x}_1, \boldsymbol{x}_2)^T$, and denoting the output by $\boldsymbol{y} = \boldsymbol{x}_2 \in R^n$, and the control input by $\boldsymbol{\tau} \in R^n$, the state space representation of (16) becomes as follows.

$$\dot{\boldsymbol{x}}_1 = \boldsymbol{x}_2 \tag{17a}$$

$$\begin{aligned}\dot{\boldsymbol{x}}_2 &= -M(\boldsymbol{x}_1)^{-1}\left\{\frac{1}{2}\dot{M}(\boldsymbol{x}_1)\boldsymbol{x}_2 + S(\boldsymbol{x}_1,\boldsymbol{x}_2)\boldsymbol{x}_2 + \boldsymbol{g}(\boldsymbol{x}_1)\right\} \\ &\quad + M(\boldsymbol{x}_1)^{-1}\tau \\ &\triangleq \boldsymbol{f}_2(\boldsymbol{x}_1,\boldsymbol{x}_2) + G_2(\boldsymbol{x}_1)\boldsymbol{\tau}\end{aligned} \tag{17b}$$

$$\boldsymbol{y} = \boldsymbol{x}_2 \tag{18}$$

Now taking a storage function equal to the kinetic energy + the potential energy

$$W(\boldsymbol{x}) = \frac{1}{2}\boldsymbol{x}_2^T M(\boldsymbol{x}_1)\boldsymbol{x}_2 + U(\boldsymbol{x}_1) - U(\boldsymbol{x}_1^*) \tag{19}$$

we calculate its time derivative with use of skew-symmetricity of $S(\boldsymbol{x}_1,\boldsymbol{x}_2)$ to obtain

$$\begin{aligned}\dot{W}(\boldsymbol{x}) &= \frac{\partial}{\partial \boldsymbol{x}_1}\left\{\frac{1}{2}\boldsymbol{x}_2^T M(\boldsymbol{x}_1)\boldsymbol{x}_2\right\}\dot{\boldsymbol{x}}_1 \\ &\quad + \frac{\partial}{\partial \boldsymbol{x}_2}\left\{\frac{1}{2}\boldsymbol{x}_2^T M(\boldsymbol{x}_1)\boldsymbol{x}_2\right\}\dot{\boldsymbol{x}}_2 + \frac{\partial U(\boldsymbol{x}_1)}{\partial \boldsymbol{x}_1}\dot{\boldsymbol{x}}_1 \\ &= \frac{1}{2}\boldsymbol{x}_2^T \dot{M}(\boldsymbol{x}_1)\boldsymbol{x}_2 + \boldsymbol{x}_2^T\{-\frac{1}{2}\dot{M}(\boldsymbol{x}_1)\boldsymbol{x}_2 - S(\boldsymbol{x}_1,\boldsymbol{x}_2)\boldsymbol{x}_2 \\ &\quad -\boldsymbol{g}(\boldsymbol{x}_1) + \boldsymbol{\tau}\} + \boldsymbol{g}(\boldsymbol{x}_1)^T\boldsymbol{x}_2 \le \boldsymbol{y}^T\boldsymbol{\tau}\end{aligned} \tag{20}$$

Therefore, the robot manipulator is passive with respect to the input $\boldsymbol{\tau}$ and the output $\boldsymbol{y} = \boldsymbol{x}_2$. Thus, the so-called K-Y-P property holds :

$$W_{\boldsymbol{x}_1}(\boldsymbol{x})\boldsymbol{x}_2 + W_{\boldsymbol{x}_2}(\boldsymbol{x})\boldsymbol{f}_2(\boldsymbol{x}_1,\boldsymbol{x}_2) \le 0 \tag{21a}$$

$$W_{\boldsymbol{x}_2}(\boldsymbol{x})G_2(\boldsymbol{x}_1) = \boldsymbol{y}^T \tag{21b}$$

Next let us consider a set-point servo problem (a set-point tracking control) with the desired set-point $(\boldsymbol{x}_1^*, \boldsymbol{0})$. For that we consider the following system which consists of the robot manipulator (17),(18) and the strict positive real element (23).

$$\dot{\boldsymbol{x}}_1 = \boldsymbol{x}_2 \tag{22a}$$

$$\begin{aligned}\dot{\boldsymbol{x}}_2 &= -M(\boldsymbol{x}_1)^{-1}\left\{\frac{1}{2}\dot{M}(\boldsymbol{x}_1)\boldsymbol{x}_2 + S(\boldsymbol{x}_1,\boldsymbol{x}_2)\boldsymbol{x}_2 + \boldsymbol{g}(\boldsymbol{x}_1)\right\} \\ &\quad + M(\boldsymbol{x}_1)^{-1}\boldsymbol{\tau} \\ &\triangleq \boldsymbol{f}_2(\boldsymbol{x}_1,\boldsymbol{x}_2) + G_2(\boldsymbol{x}_1)\boldsymbol{\tau}\end{aligned} \tag{22b}$$

$$\dot{\boldsymbol{\xi}} = D\boldsymbol{\xi} + (\boldsymbol{x}_1^* - \boldsymbol{x}_1) - \boldsymbol{x}_2, \quad D < 0 \tag{23}$$

$$\boldsymbol{y} = \boldsymbol{x}_2 \tag{24}$$

And set up a feedback compensator (P·SPR·D control with respect to $\boldsymbol{x}_1$) :

$$\boldsymbol{\tau} = K_P(\boldsymbol{x}_1^* - \boldsymbol{x}_1) + K_S\boldsymbol{\xi} - K_D\boldsymbol{x}_2 + \boldsymbol{g}(\boldsymbol{x}_1^*) \tag{25}$$

where $K_P,\ K_S,\ K_D$ are all positive definite diagonal matrices. Here $\boldsymbol{g}(\boldsymbol{x}_1^*)$, gravity force compensation at the desired value $\boldsymbol{x}_1^*$, corresponds to the so-called manual reset quantity of PID controller.

[Theorem 3] The closed-loop system (22)∼(25) of the robot manipulator with the P·SPR·D control is asymptotically stable at the equilibrium $(\boldsymbol{x}_1^*, \boldsymbol{0}, \boldsymbol{0})$, provided that positive definite diagonal matrices $K_P,\ K_S,\ K_D$ and negative definite D are appropriately chosen.

(Proof) At the equilibrium of system (22),(23),(25) hold the following relations.

$$\begin{aligned}\boldsymbol{0} &= \boldsymbol{x}_{2e} \\ \boldsymbol{0} &= -\boldsymbol{g}(\boldsymbol{x}_{1e}) + \boldsymbol{\tau}_e \\ \boldsymbol{0} &= D\boldsymbol{\xi}_e + (\boldsymbol{x}_1^* - \boldsymbol{x}_{1e})\end{aligned}$$

Thus it follows that $(\boldsymbol{x}_{1e} = \boldsymbol{x}_1^*,\ \boldsymbol{x}_{2e} = \boldsymbol{0},\ \boldsymbol{\xi}_e = \boldsymbol{0})$ is an equilibrium point, provided that $\boldsymbol{\tau}_e = \boldsymbol{g}(\boldsymbol{x}_1^*)$.

Now let us consider a Lyapunov function candidate

$$\begin{aligned}V(\boldsymbol{x},\boldsymbol{\xi}) &= W(\boldsymbol{x}) + \boldsymbol{g}(\boldsymbol{x}_1^*)^T(\boldsymbol{x}_1^* - \boldsymbol{x}_1) \\ &+ \frac{1}{2}\left[(\boldsymbol{x}_1^* - \boldsymbol{x}_1)^T\ \boldsymbol{\xi}^T\right]\begin{bmatrix} K_P - \overline{K} & \overline{K} \\ \overline{K}^T & K_S - \overline{K}\end{bmatrix}\begin{bmatrix}(\boldsymbol{x}_1^* - \boldsymbol{x}_1) \\ \boldsymbol{\xi}\end{bmatrix}\end{aligned} \tag{26}$$

where $K_P - \overline{K} > 0,\ K_S - \overline{K} > 0$ and $\begin{bmatrix} K_P - \overline{K} & \overline{K} \\ \overline{K}^T & K_S - \overline{K}\end{bmatrix}$ is a positive definite matrix. The first term in the right-hand side of (26) is a semi-positive definite function. Since the second term plus the third one is a quadrtic function of $\begin{bmatrix}(\boldsymbol{x}_1^* - \boldsymbol{x}_1) \\ \boldsymbol{\xi}\end{bmatrix}$ whose quadratic term is with the positive definite matrix, it has the minimum. Accordingly, $V(\boldsymbol{x},\boldsymbol{\xi})$ is a function bounded below.

Next calculate its time derivative along (22),(23),(25) with use of the K-Y-P property (21) to get

$$\begin{aligned}&\dot{V}(\boldsymbol{x},\boldsymbol{\xi}) \\ &= W_{\boldsymbol{x}_1}(\boldsymbol{x})\boldsymbol{x}_2 + W_{\boldsymbol{x}_2}(\boldsymbol{x})\boldsymbol{f}_2(\boldsymbol{x}_1,\boldsymbol{x}_2) \\ &+ W_{\boldsymbol{x}_2}(\boldsymbol{x})G_2(\boldsymbol{x}_1)\boldsymbol{\tau} - \boldsymbol{g}(\boldsymbol{x}_1^*)^T\boldsymbol{x}_2 \\ &+ \left[(\boldsymbol{x}_1^* - \boldsymbol{x}_1)^T\ \boldsymbol{\xi}^T\right]\begin{bmatrix} K_P - \overline{K} & \overline{K} \\ \overline{K}^T & K_S - \overline{K}\end{bmatrix}\begin{bmatrix}-\dot{\boldsymbol{x}}_1 \\ \dot{\boldsymbol{\xi}}\end{bmatrix} \\ &\le \boldsymbol{y}^T\boldsymbol{\tau} - \boldsymbol{g}(\boldsymbol{x}_1^*)^T\boldsymbol{x}_2 + \\ &\left[(\boldsymbol{x}_1^* - \boldsymbol{x}_1)^T\ \boldsymbol{\xi}^T\right]\begin{bmatrix} K_P - \overline{K} & \overline{K} \\ \overline{K}^T & K_S - \overline{K}\end{bmatrix} \\ &\qquad \times \begin{bmatrix}-\boldsymbol{x}_2 \\ D\boldsymbol{\xi} + (\boldsymbol{x}_1^* - \boldsymbol{x}_1) - \boldsymbol{x}_2\end{bmatrix} \\ &= \boldsymbol{x}_2^T(K_P(\boldsymbol{x}_1^* - \boldsymbol{x}_1) + K_S\boldsymbol{\xi} - K_D\boldsymbol{x}_2 + \boldsymbol{g}(\boldsymbol{x}_1^*)) - \boldsymbol{g}(\boldsymbol{x}_1^*)^T\boldsymbol{x}_2 \\ &+ \left[(\boldsymbol{x}_1^* - \boldsymbol{x}_1)^T\ \boldsymbol{\xi}^T\right] \times \\ &\begin{bmatrix} -(K_P - \overline{K})\boldsymbol{x}_2 + \overline{K}D\boldsymbol{\xi} + \overline{K}(\boldsymbol{x}_1^* - \boldsymbol{x}_1) - \overline{K}\boldsymbol{x}_2 \\ -\overline{K}^T\boldsymbol{x}_2 + (K_S - \overline{K})D\boldsymbol{\xi} + (K_S - \overline{K})(\boldsymbol{x}_1^* - \boldsymbol{x}_1) \\ -(K_S - \overline{K})\boldsymbol{x}_2\end{bmatrix} \\ &= \boldsymbol{x}_2^T(K_P(\boldsymbol{x}_1^* - \boldsymbol{x}_1) + K_S\boldsymbol{\xi} - K_D\boldsymbol{x}_2) \\ &+ \left[(\boldsymbol{x}_1^* - \boldsymbol{x}_1)^T\ \boldsymbol{\xi}^T\right]\begin{bmatrix}\overline{K} & \overline{K}D \\ (K_S - \overline{K}) & (K_S - \overline{K})D\end{bmatrix}\begin{bmatrix}(\boldsymbol{x}_1^* - \boldsymbol{x}_1) \\ \boldsymbol{\xi}\end{bmatrix} \\ &- (\boldsymbol{x}_1^* - \boldsymbol{x}_1)^T K_P\boldsymbol{x}_2 - \boldsymbol{\xi}^T K_S\boldsymbol{x}_2 \\ &= -\boldsymbol{x}_2^T K_D\boldsymbol{x}_2 + \left[(\boldsymbol{x}_1^* - \boldsymbol{x}_1)^T\ \boldsymbol{\xi}^T\right]\begin{bmatrix}\overline{K} & \overline{K}D \\ (K_S - \overline{K}) & (K_S - \overline{K})D\end{bmatrix} \\ &\times \begin{bmatrix}(\boldsymbol{x}_1^* - \boldsymbol{x}_1) \\ \boldsymbol{\xi}\end{bmatrix}\end{aligned} \tag{27}$$

Here we try to make

$$\begin{bmatrix} \overline{K} & \overline{K}D \\ (K_S-\overline{K}) & (K_S-\overline{K})D \end{bmatrix}$$

be negative definite. For that purpose, set $\overline{K}<0,\ K_S-\overline{K}=(\overline{K}D)^T$ and $D<-I$ such that we have $K_S=(I+D)\overline{K}>0$. Then the above matrix becomes

$$\begin{bmatrix} \overline{K} & \overline{K}D \\ (\overline{K}D)^T & \overline{K}D^2 \end{bmatrix}$$

Since the (1,1) element and the (2,2) element are $\overline{K}<0,\ \overline{K}D^2<0$, respectively, we can choose $\overline{K}<0$ and $D<0$ such that the above matrix becomes negative definite.

Consequently, $\dot{V}(\boldsymbol{x},\boldsymbol{\xi})$ becomes semi-negative definite, and it follows that the P·SPR·D control is stable in the sense of Lyapunov, but it is unknown if asymptotically stable. So we apply LaSalle's invariance principle.

Let $\Omega_c=\{(\boldsymbol{x},\boldsymbol{\xi})\mid V(\boldsymbol{x},\boldsymbol{\xi})\le c\}$ and suppose Ω_c is bounded and $\dot{V}(\boldsymbol{x},\boldsymbol{\xi})\le 0$ in Ω_c (c is a positive number such that $\dot{V}(\boldsymbol{x},\boldsymbol{\xi})\le 0$). Here define Ω_E as a set of all points of Ω_c satisfying $\dot{V}(\boldsymbol{x},\boldsymbol{\xi})=0$ and put

$$\Omega_E=\{(\boldsymbol{x},\boldsymbol{\xi})\mid \dot{V}(\boldsymbol{x},\boldsymbol{\xi})=0,\ (\boldsymbol{x},\boldsymbol{\xi})\in\Omega_c\} \tag{28}$$

From (27) $(\boldsymbol{x},\boldsymbol{\xi})$ satisfying $\dot{V}(\boldsymbol{x},\boldsymbol{\xi})=0$ is given as $\boldsymbol{x}_2=\boldsymbol{0},\ \boldsymbol{x}_1^*-\boldsymbol{x}_1=\boldsymbol{0},\ \boldsymbol{\xi}=\boldsymbol{0}$. So we have

$$\Omega_E=\{(\boldsymbol{x},\boldsymbol{\xi})|\boldsymbol{x}_1=\boldsymbol{x}_1^*,\boldsymbol{x}_2=\boldsymbol{0},\boldsymbol{\xi}=\boldsymbol{0},\ (\boldsymbol{x},\boldsymbol{\xi})\in\Omega_c\} \tag{29}$$

Accordingly, we know from (22),(23),(25) that $(\boldsymbol{x},\boldsymbol{\xi})$ in Ω_E consists of only the equilibrium point $(\boldsymbol{x}_{1e},\boldsymbol{x}_{2e},\boldsymbol{\xi}_e)=(\boldsymbol{x}_1^*,\boldsymbol{0},\boldsymbol{0})$ with $\boldsymbol{\tau}_e=\boldsymbol{g}(\boldsymbol{x}_1^*)$. Thus the largest invariance set Ω_M in Ω_E consists of the equilibrium point $(\boldsymbol{x}_{1e},\boldsymbol{x}_{2e},\boldsymbol{\xi}_e)=(\boldsymbol{x}_1^*,\boldsymbol{0},\boldsymbol{0})$. Therefore, by LaSalle's invariance principle all trajectories in Ω_c converges to Ω_M as $t\to\infty$. Thus $(\boldsymbol{x}_1^*,\boldsymbol{0},\boldsymbol{0})$ is aymptotically stable. Q.E.D

[Remark 1] Since the robot manipulator is not zero state detectable, one cannot apply Theorem 2 to attain asymptotical stabilization to the origin. In order to stabilize the origin $(\boldsymbol{x}_1,\boldsymbol{x}_2)=(\boldsymbol{0},\boldsymbol{0})$, one must apply Theorem 3 letting $\boldsymbol{x}_1^*=\boldsymbol{0}$.

Local asymptotical stability of PID control for robot manipulators was first proved by Arimoto[1, 2]. For comparison with P·SPR·D control, its proof based on the K-Y-P property is given in Appendix.

IV. L_2 Gain disturbance attenuation Problem

In this section we study L_2 disturbance attenuation problem under the existence of disturbance $\boldsymbol{w}$. Consider the following cascaded system.

$$\Sigma_c:\dot{\boldsymbol{\xi}}=D\boldsymbol{\xi}-\boldsymbol{y} \tag{30}$$

$$\Sigma_p:\dot{\boldsymbol{x}}=\boldsymbol{f}(\boldsymbol{x})+G(\boldsymbol{x})\boldsymbol{u}+J(\boldsymbol{x})\boldsymbol{w} \tag{31}$$

$$\boldsymbol{y}=\boldsymbol{h}(\boldsymbol{x}) \tag{32}$$

where $\boldsymbol{w}\in R^l$ is the disturbance vector.

L_2 disturbance attenuation problem is defined to obtain the P·SPR·D control such that the closed-loop system satisfies the following condition under the given disturbance attenuation level $\gamma>0$.

P1. When $\boldsymbol{w}=\boldsymbol{0}$, the closed-loop system is asymptotically stable at the equilibrium $(\boldsymbol{x},\boldsymbol{\xi})=(\boldsymbol{0},\boldsymbol{0})$.

p2. When $\boldsymbol{x}(\boldsymbol{0})=\boldsymbol{0}$, the following inequality holds for arbitrarily given $T>0$.

$$\int_0^T\|\boldsymbol{y}(t)\|^2dt\le\gamma^2\int_0^T\|\boldsymbol{w}(t)\|^2dt$$

It is noticed that P2 is equivallent to having L_2 gain below γ when $\boldsymbol{x}(\boldsymbol{0})=\boldsymbol{0}$, that is, $\|\boldsymbol{y}\|_2\le\gamma^2\|\boldsymbol{w}\|_2$. It implies that for all $\boldsymbol{w}\in L_2[0,T]$ and for the supply rate $s(\boldsymbol{y},\boldsymbol{w})=\frac{1}{2}\{\gamma^2\boldsymbol{w}^T\boldsymbol{w}-\boldsymbol{y}^T\boldsymbol{y}\}$, the following γ-dissipation inequality holds[10].

$$\dot{V}(\boldsymbol{x},\boldsymbol{\xi})\le\frac{1}{2}\{\gamma^2\boldsymbol{w}^T\boldsymbol{w}-\boldsymbol{y}^T\boldsymbol{y}\} \tag{33}$$

The following theorem solves the L_2 diturbance attenuation problem.

[Theorem 4] Suppose the cascaded system (30),(31),(32) satisfies Assumptions (a) and (b) in Theorem 2. Further $W(\boldsymbol{x})$ and $J(\boldsymbol{x})$ satisfy the matching condition

$$W_{\boldsymbol{x}}(\boldsymbol{x})J(\boldsymbol{x})=\boldsymbol{y}^TM(\boldsymbol{x})^T \tag{34}$$

where $M(\boldsymbol{x})\in R^{l\times m}$ denotes the function matrix and $M(\boldsymbol{x})^TM(\boldsymbol{x})=I_m$. In addition assume $K_P\ge\frac{1}{2}(1-\frac{1}{\gamma^2})I_m$. Then by the P·SPR·D control (10) the closed-loop system satisfies P2, that is, it possesses L_2 gain less than γ (i.e., γ-dissipation inequality holds.)

Furthermore, if subsystem Σ_p is zero state detectable with respect to the output $\boldsymbol{y}$, then by the P·SPR·D control (10) the closed-loop system satisfies P1 so that $(\boldsymbol{x},\boldsymbol{\xi})=(\boldsymbol{0},\boldsymbol{0})$ is asymptotically stable.

(**Proof**) To prove that the γ-dissipation inequality holds, make the following calculation for a storage function (13)(semi-positive definite function).

$$\begin{aligned}
&\dot{V}(\boldsymbol{x},\boldsymbol{\xi})+\frac{1}{2}\{\boldsymbol{y}^T\boldsymbol{y}-\gamma^2\boldsymbol{w}^T\boldsymbol{w}\}\\
&=\dot{W}(\boldsymbol{x})+\dot{U}(\boldsymbol{\xi})+\boldsymbol{y}^TK_D\dot{\boldsymbol{y}}+\frac{1}{2}\{\boldsymbol{y}^T\boldsymbol{y}-\gamma^2\boldsymbol{w}^T\boldsymbol{w}\}\\
&=W_{\boldsymbol{x}}(\boldsymbol{x})\{\boldsymbol{f}(\boldsymbol{x})+G(\boldsymbol{x})\boldsymbol{u}+J(\boldsymbol{x})\boldsymbol{w}\}\\
&\quad+\boldsymbol{\xi}^TK_S(D\boldsymbol{\xi}-\boldsymbol{y})+\boldsymbol{y}^TK_D\dot{\boldsymbol{y}}+\frac{1}{2}\{\boldsymbol{y}^T\boldsymbol{y}-\gamma^2\boldsymbol{w}^T\boldsymbol{w}\}\\
&=W_{\boldsymbol{x}}(\boldsymbol{x})\boldsymbol{f}(\boldsymbol{x})+W_{\boldsymbol{x}}(\boldsymbol{x})G(\boldsymbol{x})(-K_P\boldsymbol{y}+K_S\boldsymbol{\xi}-K_D\dot{\boldsymbol{y}})\\
&\quad+W_{\boldsymbol{x}}(\boldsymbol{x})J(\boldsymbol{x})\boldsymbol{w}+\boldsymbol{\xi}^TK_SD\boldsymbol{\xi}-\boldsymbol{\xi}^TK_S^T\boldsymbol{y}+\boldsymbol{y}^TK_D\dot{\boldsymbol{y}}\\
&\quad+\frac{1}{2}\{\boldsymbol{y}^T\boldsymbol{y}-\gamma^2\boldsymbol{w}^T\boldsymbol{w}\}
\end{aligned}$$

Here using Assumptions (a), (b) and the matching condition (34),

$$\begin{aligned}
&\le\boldsymbol{y}^T\{-K_P\boldsymbol{y}+K_S\boldsymbol{\xi}-K_D\dot{\boldsymbol{y}}\}+\boldsymbol{y}^TM(\boldsymbol{x})^T\boldsymbol{w}-\boldsymbol{\xi}^TK_S\boldsymbol{y}\\
&\quad+\boldsymbol{y}^TK_D\dot{\boldsymbol{y}}+\frac{1}{2}\{\boldsymbol{y}^T\boldsymbol{y}-\gamma^2\boldsymbol{w}^T\boldsymbol{w}\}\\
&=-\boldsymbol{y}^TK_P\boldsymbol{y}+\boldsymbol{y}^TM(\boldsymbol{x})^T\boldsymbol{w}-\frac{1}{2}\left\{\frac{1}{\gamma}\boldsymbol{y}^TM(\boldsymbol{x})^T-\gamma\boldsymbol{w}^T\right\}\\
&\quad\left\{\frac{1}{\gamma}M(\boldsymbol{x})\boldsymbol{y}-\gamma\boldsymbol{w}\right\}+\frac{1}{2}\boldsymbol{y}^T\boldsymbol{y}+\frac{1}{2}\frac{1}{\gamma^2}\boldsymbol{y}^TM(\boldsymbol{x})^TM(\boldsymbol{x})\boldsymbol{y}\\
&\quad-\frac{1}{2}\boldsymbol{w}^TM(\boldsymbol{x})\boldsymbol{y}-\frac{1}{2}\boldsymbol{y}^TM(\boldsymbol{x})^T\boldsymbol{w}
\end{aligned}$$

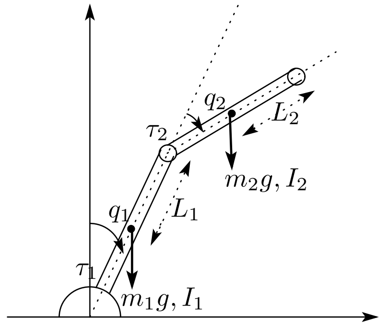

Fig.1. 2-Link Manipulator

$$= -\boldsymbol{y}^T\{K_P - \frac{1}{2}(1+\frac{1}{\gamma^2})I_m\}\boldsymbol{y} - \frac{1}{2}\left\{\frac{1}{\gamma}M(\boldsymbol{x})\boldsymbol{y} - \gamma\boldsymbol{w}\right\}^T \left\{\frac{1}{\gamma}M(\boldsymbol{x})\boldsymbol{y} - \gamma\boldsymbol{w}\right\} \quad (35)$$

Here using $K_P \geq \frac{1}{2}(1+\frac{1}{\gamma^2})I_m$

$$\leq -\frac{1}{2}\left\{\frac{1}{\gamma}M(\boldsymbol{x})\boldsymbol{y} - \gamma\boldsymbol{w}\right\}^T \left\{\frac{1}{\gamma}M(\boldsymbol{x})\boldsymbol{y} - \gamma\boldsymbol{w}\right\} \leq 0 \quad (36)$$

Consequently, γ-dissipation inequality (33) holds, and so it follows that we have L_2 gain below γ.

When $\boldsymbol{w} = \boldsymbol{0}$, P1 has been already concluded by Theorem 2. Q.E.D

V. SIMULATION

Let us apply the P·SPR·D control of robot manipulator studied in Section 3 to a 2-link manipulator depicted in Fig.1. Here generalized coordinates q_1, q_2 are relative joint angles, and $x_{11} \triangleq q_1$ denotes perpendicular angle (angle from vertical line) of link 1 and $x_{12} \triangleq q_2$ relative angle of link 2 from link 1, τ_1 *and* τ_2 denote torque of each link acting clockwise. L_1, L_2, m_1, m_2, I_1, I_2 denote the length, the mass and the inertia moment of each link, respectively.

A numerical example of 2-link manipulator is given as follows.

$$\begin{bmatrix} \dot{x}_{11} \\ \dot{x}_{12} \\ \dot{x}_{21} \\ \dot{x}_{22} \end{bmatrix} = \begin{bmatrix} x_{21} \\ x_{22} \\ f_{21}(\boldsymbol{x}_1, \boldsymbol{x}_2) + G_{211}(\boldsymbol{x}_1)\tau_1 + G_{212}(\boldsymbol{x}_1)\tau_2 \\ f_{22}(\boldsymbol{x}_1, \boldsymbol{x}_2) + G_{221}(\boldsymbol{x}_1)\tau_1 + G_{222}(\boldsymbol{x}_1)\tau_2 \end{bmatrix}$$

where

$$f_{21}(\boldsymbol{x}_1, \boldsymbol{x}_2) \triangleq \frac{-1}{\det\ M}[1.05\{(-6x_{21}x_{22} - 3x_{22}^2)\sin x_{12} + 5x_{21} - 117.6\sin x_{11} - 14.7\sin(x_{11}+x_{12})\} - (1+3\cos x_{12})(3x_{21}^2 \sin x_{12} + 5x_{22} - 14.7\sin(x_{11}+x_{12}))]$$

$$f_{22}(\boldsymbol{x}_1, \boldsymbol{x}_2) \triangleq \frac{-1}{\det\ M}[(-1-3\cos x_{12})\{(-6x_{21}x_{22} - 3x_{22}^2)\sin x_{12} + 5x_{21} - 117.6\sin x_{11} - 14.7\sin(x_{11}+x_{12})\} + (21.2+6\cos x_{12})(3x_{21}^2\sin x_{12} + 5x_{22} - 14.7\sin(x_{11}+x_{12}))]$$

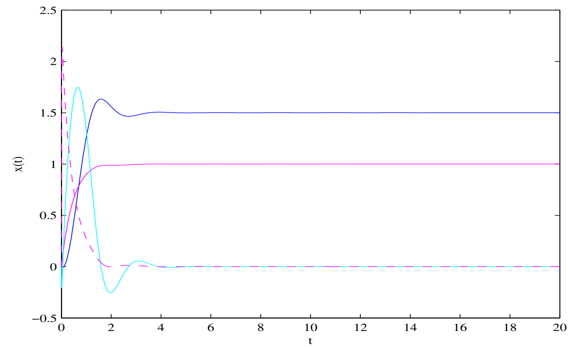

Fig.2. P·SPR·D Control

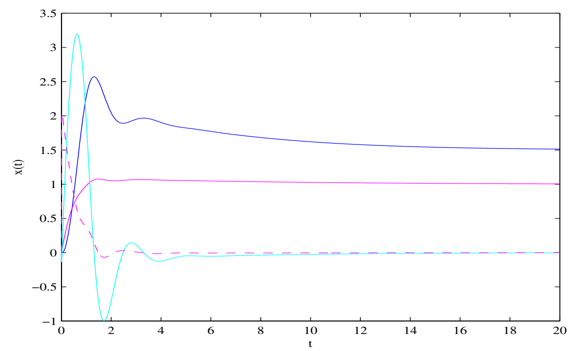

Fig.3. Ordinary PID Control

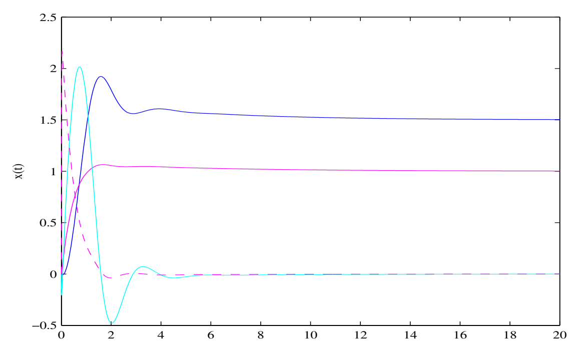

Fig.4. Ordinary PID+m0 Control

$$G_{211}(\boldsymbol{x}_1) \triangleq \frac{1.05}{\det\ M}, \quad G_{212}(\boldsymbol{x}_1) \triangleq \frac{1}{\det\ M}(1 - 3cosx_{12}),$$

$$G_{221} \triangleq \frac{1}{\det\ M}(-1 - 3cosx_{12}),$$

$$G_{222} \triangleq \frac{1}{\det\ M}(21.2 + 6cosx_{12})$$

where $\det M = 21.26 + 0.3\cos x_{12} - 9(\cos x_{12})^2$.

Further, $\boldsymbol{g}(\boldsymbol{x}_1)$ is also given as

$$\begin{bmatrix} g_1(\boldsymbol{x}_1) \\ g_2(\boldsymbol{x}_1) \end{bmatrix} = \begin{bmatrix} -117.6\sin x_{11} - 14.7\sin(x_{11}+x_{12}) \\ -14.7\sin(x_{11}+x_{12}) \end{bmatrix}$$

Applying Theorem 3, let us solve a set-point servo problem with $\boldsymbol{x}_1^* = (1.5,\ 1)^T$. We set the SPR element as (23) and take an initial state as $(\boldsymbol{x}_1(0), \boldsymbol{x}_2(0)) =$

$(\mathbf{0},\mathbf{0})$. The simulation results is shown in Fig.2, when $D = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$, $K_S = \begin{bmatrix} 40 & 0 \\ 0 & 40 \end{bmatrix}$, $K_P = \begin{bmatrix} 180 & 0 \\ 0 & 180 \end{bmatrix}$, $K_D = \begin{bmatrix} 60 & 0 \\ 0 & 60 \end{bmatrix}$. It is observed that the convergence speed is very quick.

Furthermore, as mentioned in Remark 1, the regulation problem (asymptotical stabilization to the origin) could be solved with very good performance by setting $\boldsymbol{x}_1^* = \mathbf{0}$. (Figure is omitted for page limitation.)

Meanwhile, the P·SPR·D control with D=0, i.e.,

$$\dot{\boldsymbol{\xi}} = \boldsymbol{x}_1^* - \boldsymbol{x}_1$$
$$\boldsymbol{\tau} = K_P(\boldsymbol{x}_1^* - \boldsymbol{x}_1) + K_S\boldsymbol{\xi} - K_D\boldsymbol{x}_2$$

becomes ordinary PID control. It is observed that though the convergence was attained by the ordinary PID control (see Fig.3), its performance is inferior to Fig.2. Of course the control performance changes dependent on $K_P,\ K_S,\ K_D,\ D$. However, it was seen that the P·SPR·D control attained always much better performance than the ordinary PID control. Furthermore, Fig.4 shows the simulation results for a control law in which the so-called manual reset quantity $\boldsymbol{m}_0 = \boldsymbol{g}(\boldsymbol{x}_1^*)$ is added to the ordinary PID control. Comparing these three cases, we can say that the P·SPR·D control is the best in regard to both response speed and overshoot. This indicates that the P· SPR· D control possesses a possivility of a new and effective control scheme.

Note that nothing has been mentioned on the controller parameter adjustment. Of course the control performance depends on the parameter values. The values of $K_P,\ K_S,\ K_D$ used in the simulation is the almost optimum values which was obtained by trial and error for the usual PID controller under the condition of diagonal matrices, and the same values are used also for the P· SPR· D control. Although there is a room of argument and improvement as a robot control, the parameter adjustment is left as a future topic.

VI. Conclusion

Based on the passivity theory and LaSalle's invariance principle, we investigaed on the regulation problem by the P · SPR· D control and the L_2 disturbance attenuation problem for the affine nonlinear system. Further we studied the setpoint servo problem for the robot manipulator.

The P· SPR· D control is a new general control scheme and the use of SPR element as a part of the controller possesses an advantage from a passivity based design point of view. Although a number of adjustable parameters increases compared to PID, it implies also to increase a freedom for the design. The optimum parameter adjustment is left as a future topic.

Implementation of the P· SPR· D control is not difficult with a digital processor.

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VII. Appendix A Ordinary PID Control

Consider the robot manipulator (17),(18). the robot manipulator is passive with respect to input $\boldsymbol{\tau}$ and output $\boldsymbol{y}$, and hence K-Y-P property (21) holds.

Let us consider a set-point servo problem with the desired set-point $(\boldsymbol{x}_1^*,\mathbf{0})$. We connect an ordinary PID controller

$$\dot{\boldsymbol{w}} = (\boldsymbol{x}_1^* - \boldsymbol{x}_1) \tag{37}$$
$$\boldsymbol{\tau} = K_P(\boldsymbol{x}_1^* - \boldsymbol{x}_1) + K_I\boldsymbol{w} - K_D\boldsymbol{x}_2 \tag{38}$$

where $K_P,\ K_I,\ K_D$ are the positive-definite diagonal matrices.

Below we prove the asymptotical stability, applying LaSalle's invariance principle.

Since an equilibrium of the closed-loop system (17),(37),(38) satisfies

$$\begin{aligned} \mathbf{0} &= \boldsymbol{x}_{2e} \\ \mathbf{0} &= -\boldsymbol{g}(\boldsymbol{x}_{1e}) + K_P(\boldsymbol{x}_1^* - \boldsymbol{x}_{1e}) + K_I\boldsymbol{w}_e \\ \mathbf{0} &= (\boldsymbol{x}_1^* - \boldsymbol{x}_{1e}) \end{aligned} \tag{39}$$

$(\boldsymbol{x}_{1e} = \boldsymbol{x}_1^*,\ \boldsymbol{x}_{2e} = \mathbf{0},\ \boldsymbol{w}_e = \overline{\boldsymbol{w}} = K_I^{-1}\boldsymbol{g}(\boldsymbol{x}_1^*))$ becomes the equilibrium point.

Now consider a Lyapunov function candidate

$$\begin{aligned} V(\boldsymbol{x},\boldsymbol{w}) &= W(\boldsymbol{x}) + \boldsymbol{g}(\boldsymbol{x}_1^*)^T(\boldsymbol{x}_1^* - \boldsymbol{x}_1) \\ &+\frac{1}{2}(\boldsymbol{x}_1^* - \boldsymbol{x}_1)^T K_P(\boldsymbol{x}_1^* - \boldsymbol{x}_1) \\ &+(\boldsymbol{x}_1^* - \boldsymbol{x}_1)^T K_I(\boldsymbol{w} - \overline{\boldsymbol{w}}) + \frac{1}{2}\alpha(\boldsymbol{w} - \overline{\boldsymbol{w}})^T K_I(\boldsymbol{w} - \overline{\boldsymbol{w}}) \\ &-\alpha(\boldsymbol{x}_1^* - \boldsymbol{x}_1)^T M(\boldsymbol{x}_1)\boldsymbol{x}_2 \end{aligned} \tag{40}$$

$where\ W(\boldsymbol{x}) = \frac{1}{2}\boldsymbol{x}_2^T M(\boldsymbol{x}_1)\boldsymbol{x}_2 + U(\boldsymbol{x}_1) - U(\boldsymbol{x}_1^*),\ \alpha > 0$

We can prove that $V(\boldsymbol{x},\boldsymbol{w})$ is a function bounded below in the neighborhood of $(\boldsymbol{x}^*,\mathbf{0},\overline{\boldsymbol{w}})$.

Take its time derivative along (17),(18),(37),(38), using the K-Y-P property (21), to obtain

$$\begin{aligned}
&\dot{V}(\boldsymbol{x},\boldsymbol{w})\\
&= W_{\boldsymbol{x}_1}(\boldsymbol{x})\boldsymbol{x}_2 + W_{\boldsymbol{x}_2}(\boldsymbol{x})\{\boldsymbol{f}_2(\boldsymbol{x}_1,\boldsymbol{x}_2)+G_2(\boldsymbol{x}_1)\boldsymbol{\tau}\}\\
&\quad -\boldsymbol{g}(\boldsymbol{x}_1^*)^T\boldsymbol{x}_2 + (\boldsymbol{x}_1^*-\boldsymbol{x}_1)^T K_P(-\boldsymbol{x}_2)\\
&\quad -\boldsymbol{x}_2^T K_I(\boldsymbol{w}-\overline{\boldsymbol{w}}) + (\boldsymbol{x}_1^*-\boldsymbol{x}_1)^T K_I\dot{\boldsymbol{w}}\\
&+\alpha(\boldsymbol{w}-\overline{\boldsymbol{w}})^T K_I\dot{\boldsymbol{w}} + \alpha\boldsymbol{x}_2^T M(\boldsymbol{x}_1)\boldsymbol{x}_2 - \alpha(\boldsymbol{x}_1^*-\boldsymbol{x}_1)^T\dot{M}(\boldsymbol{x}_1)\boldsymbol{x}_2\\
&\quad -\alpha(\boldsymbol{x}_1^*-\boldsymbol{x}_1)^T M(\boldsymbol{x}_1)\dot{\boldsymbol{x}}_2\\
&\le \boldsymbol{y}^T\boldsymbol{\tau} - \boldsymbol{g}(\boldsymbol{x}_1^*)^T\boldsymbol{x}_2 - (\boldsymbol{x}_1^*-\boldsymbol{x}_1)^T K_P\boldsymbol{x}_2 - \boldsymbol{x}_2^T K_I(\boldsymbol{w}-\overline{\boldsymbol{w}})\\
&\quad +(\boldsymbol{x}_1^*-\boldsymbol{x}_1)^T K_I(\boldsymbol{x}_1^*-\boldsymbol{x}_1) + \alpha(\boldsymbol{w}-\overline{\boldsymbol{w}})^T K_I(\boldsymbol{x}_1^*-\boldsymbol{x}_1)\\
&\quad +\alpha\boldsymbol{x}_2^T M(\boldsymbol{x}_1)\boldsymbol{x}_2 - \alpha(\boldsymbol{x}_1^*-\boldsymbol{x}_1)^T\dot{M}(\boldsymbol{x}_1)\boldsymbol{x}_2\\
&\quad -\alpha(\boldsymbol{x}_1^*-\boldsymbol{x}_1)^T M(\boldsymbol{x}_1)\{\boldsymbol{f}_2(\boldsymbol{x}_1,\boldsymbol{x}_2)+G_2(\boldsymbol{x}_1)\boldsymbol{\tau}\}\\
&= \boldsymbol{x}_2^T(K_P(\boldsymbol{x}_1^*-\boldsymbol{x}_1) + K_I\boldsymbol{w} - K_D\boldsymbol{x}_2) - \boldsymbol{g}(\boldsymbol{x}_1^*)^T\boldsymbol{x}_2\\
&\quad -(\boldsymbol{x}_1^*-\boldsymbol{x}_1)^T K_P\boldsymbol{x}_2 - \boldsymbol{x}_2^T K_I(\boldsymbol{w}-\overline{\boldsymbol{w}})\\
&\quad +(\boldsymbol{x}_1^*-\boldsymbol{x}_1)^T K_I(\boldsymbol{x}_1^*-\boldsymbol{x}_1) + \alpha(\boldsymbol{w}-\overline{\boldsymbol{w}})^T K_I(\boldsymbol{x}_1^*-\boldsymbol{x}_1)\\
&\quad +\alpha\boldsymbol{x}_2^T M(\boldsymbol{x}_1)\boldsymbol{x}_2 - \alpha(\boldsymbol{x}_1^*-\boldsymbol{x}_1)^T\dot{M}(\boldsymbol{x}_1)\boldsymbol{x}_2\\
&\quad -\alpha(\boldsymbol{x}_1^*-\boldsymbol{x}_1)^T\{\frac{1}{2}\dot{M}(\boldsymbol{x}_1)\boldsymbol{x}_2 + S(\boldsymbol{x}_1,\boldsymbol{x}_2)\boldsymbol{x}_2 + \boldsymbol{g}(\boldsymbol{x}_1)\}\\
&\quad -\alpha(\boldsymbol{x}_1^*-\boldsymbol{x}_1)^T(K_P(\boldsymbol{x}_1^*-\boldsymbol{x}_1) + K_I\boldsymbol{w} - K_D\boldsymbol{x}_2)\\
&= -\boldsymbol{x}_2^T(K_D-\alpha M(\boldsymbol{x}_1))\boldsymbol{x}_2\\
&\quad -(\boldsymbol{x}_1^*-\boldsymbol{x}_1)^T(\alpha K_P - K_I)(\boldsymbol{x}_1^*-\boldsymbol{x}_1)\\
&\quad -\alpha\boldsymbol{g}(\boldsymbol{x}_1^*)^T(\boldsymbol{x}_1^*-\boldsymbol{x}_1) - \alpha(\boldsymbol{x}_1^*-\boldsymbol{x}_1)^T\dot{M}(\boldsymbol{x}_1)\boldsymbol{x}_2\\
&\quad +\alpha(\boldsymbol{x}_1^*-\boldsymbol{x}_1)^T\{\frac{1}{2}\dot{M}(\boldsymbol{x}_1)\boldsymbol{x}_2 + S(\boldsymbol{x}_1,\boldsymbol{x}_2)\boldsymbol{x}_2 + \boldsymbol{g}(\boldsymbol{x}_1)\}\\
&\quad +\alpha(\boldsymbol{x}_1^*-\boldsymbol{x}_1)^T K_D\boldsymbol{x}_2\\
&= -\boldsymbol{x}_2^T(K_D-\alpha M(\boldsymbol{x}_1))\boldsymbol{x}_2\\
&\quad -(\boldsymbol{x}_1^*-\boldsymbol{x}_1)^T(\alpha K_P - K_I)(\boldsymbol{x}_1^*-\boldsymbol{x}_1)\\
&\quad +\alpha(\boldsymbol{g}(\boldsymbol{x}_1)-\boldsymbol{g}(\boldsymbol{x}_1^*))^T(\boldsymbol{x}_1^*-\boldsymbol{x}_1)\\
&\quad +\alpha(\boldsymbol{x}_1^*-\boldsymbol{x}_1)^T Q(\boldsymbol{x}_1,\boldsymbol{x}_2;K_D)\boldsymbol{x}_2
\end{aligned}$$

where $Q(\boldsymbol{x}_1,\boldsymbol{x}_2;K_D) \stackrel{\triangle}{=} -\frac{1}{2}\dot{M}(\boldsymbol{x}_1) + S(\boldsymbol{x}_1,\boldsymbol{x}_2) + K_D$
Here assume for $\beta > 1$

$$(\boldsymbol{x}_1^*-\boldsymbol{x}_1)^T K_P(\boldsymbol{x}_1^*-\boldsymbol{x}_1) \ge \beta(\boldsymbol{g}(\boldsymbol{x}_1)-\boldsymbol{g}(\boldsymbol{x}_1^*))^T(\boldsymbol{x}_1^*-\boldsymbol{x}_1),$$

then we have

$$\begin{aligned}
&\dot{V}(\boldsymbol{x},\boldsymbol{w})\\
&\le -\boldsymbol{x}_2^T(K_D-\alpha M(\boldsymbol{x}_1))\boldsymbol{x}_2\\
&\quad -(\boldsymbol{x}_1^*-\boldsymbol{x}_1)^T(\alpha K_P - K_I)(\boldsymbol{x}_1^*-\boldsymbol{x}_1)\\
&\quad +\frac{\alpha}{\beta}(\boldsymbol{x}_1^*-\boldsymbol{x}_1)^T K_P(\boldsymbol{x}_1^*-\boldsymbol{x}_1)\\
&\quad +\alpha(\boldsymbol{x}_1^*-\boldsymbol{x}_1)^T Q(\boldsymbol{x}_1,\boldsymbol{x}_2;K_D)\boldsymbol{x}_2\\
&= -\boldsymbol{x}_2^T(K_D-\alpha M(\boldsymbol{x}_1))\boldsymbol{x}_2\\
&\quad -\begin{bmatrix}(\boldsymbol{x}_1^*-\boldsymbol{x}_1)\\ \boldsymbol{x}_2\end{bmatrix}^T\begin{bmatrix}A_1 & A_2\\ A_3 & A_4\end{bmatrix}\begin{bmatrix}\boldsymbol{x}_1^*-\boldsymbol{x}_1\\ \boldsymbol{x}_2\end{bmatrix} \le 0 \quad (41)
\end{aligned}$$

$$\textit{where}\ A_1 = (\alpha - \frac{\alpha}{\beta})K_P - K_I,\ A_2 = -\frac{1}{2}\alpha Q(\boldsymbol{x}_1,\boldsymbol{x}_2;K_D)$$
$$A_3 = -\frac{1}{2}\alpha Q(\boldsymbol{x}_1,\boldsymbol{x}_2;K_D)^T,\quad A_4 = K_D - \alpha M(\boldsymbol{x}_1)$$

By supposing that $\boldsymbol{x}_2$ exists in the neighborhood of $\boldsymbol{x}_2 = \boldsymbol{0}$, spectral radius of $Q(\boldsymbol{x}_1,\boldsymbol{x}_2;K_D)$ can be considered whithin a certain value. When $\boldsymbol{x}_2$ exists within that bounds, by taking α suffisiently small and $K_I > 0$ appropriatly small for the given β, we can make the matrix

$$\begin{bmatrix}(\alpha-\frac{\alpha}{\beta})K_P - K_I & -\frac{1}{2}\alpha Q(\boldsymbol{x}_1,\boldsymbol{x}_2;K_D)\\ -\frac{1}{2}\alpha Q(\boldsymbol{x}_1,\boldsymbol{x}_2;K_D)^T & K_D - \alpha M(\boldsymbol{x}_1)\end{bmatrix}$$

and $K_D - \alpha M(\boldsymbol{x}_1)$ be positive definite by choosing $K_P > 0$ and $K_D > 0$ large enough. In other words, if $K_P > 0$ and $K_D > 0$ are large enough and $K_I > 0$ is small, there exists α such that the above matrix and $K_D - \alpha M(\boldsymbol{x}_1)$ become positive definite for the given β.

Let $\Omega_c = \{(\boldsymbol{x},\boldsymbol{w}) \mid V(\boldsymbol{x},\boldsymbol{w}) \le c\}$ and suppose Ω_c is bounded and $\dot{V}(\boldsymbol{x},\boldsymbol{w}) \le 0$ in Ω_c (c is a positive number such that $\dot{V}(\boldsymbol{x},\boldsymbol{w}) \le 0$). Here define Ω_E as a set of all points of Ω_c satisfying $\dot{V}(\boldsymbol{x},\boldsymbol{w}) = 0$ and put

$$\Omega_E = \{(\boldsymbol{x},\boldsymbol{w}) \mid \dot{V}(\boldsymbol{x},\boldsymbol{w}) = 0,\ (\boldsymbol{x},\boldsymbol{w}) \in \Omega_c\} \qquad (42)$$

From (41),(17),(18) and (38) $(\boldsymbol{x},\boldsymbol{w})$ satisfying $\dot{V}(\boldsymbol{x},\boldsymbol{w}) = 0$ is given as $\boldsymbol{x}_1^* - \boldsymbol{x}_1 = \boldsymbol{0},\ \boldsymbol{x}_2 = \boldsymbol{0},\ \boldsymbol{w} = \overline{\boldsymbol{w}}$, namely, a point $(\boldsymbol{x}_1,\boldsymbol{x}_2,\boldsymbol{w}) = (\boldsymbol{x}_1^*,\boldsymbol{0},\overline{\boldsymbol{w}})$. Accordingly, we know from (17),(37),(38) that $(\boldsymbol{x},\boldsymbol{w})$ in Ω_E consists of only the equilibrium point $(\boldsymbol{x}_{1e},\boldsymbol{x}_{2e},\boldsymbol{w}_e) = (\boldsymbol{x}_1^*,\boldsymbol{0},\overline{\boldsymbol{w}})$ when $\boldsymbol{\tau} = K_I\overline{\boldsymbol{w}} = \boldsymbol{g}(\boldsymbol{x}_1^*)$. Thus, the largest invariance set Ω_M in Ω_E consists of only the equilibrium point $(\boldsymbol{x}_1^*,\boldsymbol{0},\overline{\boldsymbol{w}})$. Therefore, by LaSalle's invariance principle all trajectories in Ω_c converges to Ω_M, i.e. to $(\boldsymbol{x}_1^*,\boldsymbol{0},\overline{\boldsymbol{w}})$ as $t \to \infty$. Thus $\boldsymbol{x} = (\boldsymbol{x}_1^*,\boldsymbol{0})$ is aymptotically stable. Q.E.D