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State Feedback Synthesis for Polynomial Systems with Bounded Disturbances

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Abstract— **This paper deals with a state feedback synthesis for polynomial systems in the presence of disturbances with bounded peak or bounded energy. Positively invariant sets are composed of level sets of polynomial Lyapunov functions and are included in the region of the inputs and the state constraints under the disturbances. A two-step non-iterative design procedure is available by using matrix sum of squares (SOS) relaxations and semidefinite programming. At the first step, the matrix SOS technique is applied. Then to remove one of the causes of conservativeness of the first step, the polynomial annihilators are utilized in the second step. Numerical examples illustrate the presented design procedure.**

I. Introduction

Problems of stability and stabilization for systems with disturbances have been discussed from computational viewpoint. In particular, for systems imposed constraints on the state, analysis under bounded disturbances has important applications such as process control and vibration control. What is more, synthesis problems sometimes need constraints on the control input. Since unbounded disturbances cause huge control input signals to stabilize the systems, disturbances must be bounded in such synthesis problems. Analysis and synthesis for linear systems have been discussed under bounded peak disturbances or bounded energy disturbances using the linear matrix inequality (LMI) [1], [2].

As far as nonlinear systems concerned, analysis and synthesis for polynomial nonlinear systems have been discussed from computational viewpoint. A technique of solving a finite number of LMIs from state-dependent LMI (SDLMI) conditions has been discussed to analyze the region of attraction [3], [4]. The technique using SDLMI remains a conservativeness, which is introduced by deriving the SDLMI from the Lyapunov function derivative inequalities. A linear annihilator [4] reduces the conservativeness for analysis problems. Recent techniques using sum of squares (SOS) [5] and complete square matricial representaion (CSMR) [6], [7] for positive polynomials have been applied to analysis and design of polynomial control systems [8], [9], [10], [11], [12]. The work [8], for example, proposes analysis for peak-gain and induced $\mathcal{L}_2$-gain and state feedback synthesis of expanding a region of attraction for polynomials systems with input saturation using SOS conditions. These analysis and synthesis need iterative algorithms because each of the problems includes a bilinear or a trilinear SOS condition. Although these problems are reduced to a bilinear and a trilinear SDP, the non-convex SDP is difficult to solve in general. To cope with the non-convex issue, two strategies have been developed. One of them is usage of a local bilinear SDP solver [9], however, it may still needs iterative algorithms for trilinear SOS conditions. Another one is scalarizing the SDLMIs after dilating the matrix inequalities [10], [11], which enables us to reduce the non-convex SOS problem to a convex one. Unfortunately, the scalarizing technique needs a large number of redundant variables for multiplying the state-dependent matrix polynomials by these variables to be scalar polynomials. The CSMR technique is also applied to output feedback synthesis for polynomial systems [12] with given Lyapunov functions.

To be the scalarizing strategy for the non-convex issue more simply and directly, we have recommended another computational approaches for polynomial systems [13], [14], [15] with polynomial Lyapunov functions by using matrix SOS of positive semidefinite matrix polynomials [16], [17], [18]. One of the advantages of adopting this approach is less computational effort from needless of scalarizing variables. Also, it is known that there exist polynomial Lyapunov functions on bounded regions for a class of stable nonlinear systems [19]. The work [13] gives a guaranteed cost state feedback synthesis and proposes a polynomial annihilator to decrease the conservativeness. Our technique has another applications to stabilization of polynomial systems with input saturations by state feedback control [14] and to filter and observer designs [15]. As far as we know, there has not been any state feedback synthesis for polynomial systems with the bounded by SDLMI formulation without iterative scheme.

In this paper, we propose a two-step non-iterative procedure for state feedback synthesis for polynomial systems with the bounded disturbances. To realize the procedure, we introduce the matrix SOS relaxation and the polynomial annihilator. At the first step, the matrix SOS polynomials relax the SDLMIs, which result in a conservative invariant set and a state feedback controller. At the second step, the polynomial annihilators are utilized based on the first step result to remove the conservativeness. The invariant set becomes larger than that of the first step. Then the corresponding state feedback controller to the second step can be available. The problems in each step are reduced to semidefinite programming (SDP). Numerical examples are shown to illustrate the proposed design procedure.

The paper is organized as follows. Section II describes the problems we are interested in. Section III presents an invariant set analysis of polynomial systems with disturbances. Section IV shows a design procedure for state feedback synthesis under some mild assumptions. Section V illustrates examples. Lastly, section VI concludes with remarks.

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Notation: The notation used is standard. $\|x\|_2 = (x^T x)^{\frac{1}{2}}$, $\|x\|_{\mathcal{L}_2} = \left(\int_0^\infty \|x\|_2^2 dt\right)^{\frac{1}{2}}$, $\mathrm{He}\{A\} = A + A^T$, $\mathbb{S}^n$ means real symmetric matrices of the dimension n. $\mathbb{R}[x]^{n\times m}$ means matrix polynomials of the size $n\times m$. $\Sigma[x]^n$ means matrix SOS polynomials of the dimension n. $\Sigma[x]$ means $\Sigma[x]^1$. $A^T(\star)$ means $A^T A$. Similarly, $B(\star)^T$ means BB^T.

II. Problems

Consider the linear-like representation of polynomial systems [10]:

$$\dot{x} = A(x)Z(x) + B_1(x)u + B_2(x)w \tag{1}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^{m_u}$ is the control input, $w \in \mathbb{R}^{m_w}$ is the disturbance. $A(x)$, $B_1(x)$ and $B_2(x)$ are matrix polynomials of suitable dimension. $Z(x) \in \mathbb{R}[x]^{n_Z}$ is a vector of polynomials in the state $x \in \mathbb{R}^n$ and satisfies the assumption, $Z(x) = 0$ if and only if $x = 0$. Note that, if $n_Z = n$, then it is possible to take $Z(x)$ such that $Z(x) = W(x)x$, $W(x) \in \mathbb{R}[x]^{n\times n}$ and $\det W(x) = 1$ [11]. Let $A(x)Z(x)$ be $f(x)$. For example, if we have an expression $f(x) = \tilde{A}(x)x$, it is simple to write $A(x) = \tilde{A}(x)W(x)^{-1}$. Note also that $A(x)$ always exists for given $f(x)$ and is not unique because all the expressions of $f(x)$ can be written as $(A(x) + \mathcal{N}_A(x))Z(x)$, where $\mathcal{N}_A(x) \in \mathbb{R}[x]^{n\times n_Z}$ satisfies $\mathcal{N}_A(x)Z(x) = 0$ for all $x \in \mathbb{R}^n$. $\mathcal{N}_A(x)$ is called as a polynomial annihilator of $Z(x)$, which will appear in the sequel.

We also consider an auxiliary output:

$$z^c = C_1(x)Z(x) + D_{11}(x)u \tag{2}$$

where $z^c \in \mathbb{R}^{p_{z^c}}$ represents the input and the state constraints, $C_1(x)$ and $D_{11}(x)$ are matrix polynomials of suitable dimension. The constraints on the output are

$$|z_i^c| \le 1 \quad (i = 1, \ldots, p_{z^c}). \tag{3}$$

For $F(x) \in \mathbb{R}[x]^{m_u\times n_Z}$, the state feedback law is

$$u = F(x)Z(x). \tag{4}$$

The two kinds of disturbance signal are as follows:

$$\mathcal{W}_{\mathcal{L}_\infty}(\beta_\infty) = \left\{ w \in \mathbb{R}^{m_w} \mid \|w\|_{\mathcal{L}_\infty}^2 < \beta_\infty \right\} \tag{5}$$

$$\mathcal{W}_{\mathcal{L}_2}(\beta_2) = \left\{ w \in \mathbb{R}^{m_w} \mid \|w\|_{\mathcal{L}_2}^2 < \beta_2 \right\} \tag{6}$$

Let $\mathcal{V}(\alpha)$ be an initial state region depending on a parameter $\alpha(\ge 0)$ and including the origin. We assume that $x(0) \in \mathcal{V}(\alpha)$.

Problem 1: Determine a state feedback law (4) stabilizing the system (1) and (2) with the constraints (3) and the bounded *peak* disturbance (5) where $x(0) \in \mathcal{V}(\alpha)$.

Problem 2: Determine a state feedback law (4) stabilizing the system (1) and (2) with the constraints (3) and the bounded *energy* disturbance (6) where $x(0) \in \mathcal{V}(\alpha)$.

III. Invariant Set Analysis

This section gives invariant set analysis of the closed-loop system under the disturbances (5) and (6). Consider the candidate Lyapunov function as

$$V(x) = Z(x)^T Q^{-1} Z(x), \quad Q \succ 0.$$

To express the gradient of $V(x)$, we define $M(x) \in \mathbb{R}[x]^{n_Z\times n}$ whose (i, j)-element is given by

$$M_{ij}(x) = \frac{\partial Z_i}{\partial x_j}(x)$$

for $i = 1, \ldots, n_Z$ and $j = 1, \ldots, n$. Then the gradient of $V(x)$ can be obtained by the formula

$$\frac{\partial V}{\partial x}(x) = 2Z(x)^T Q^{-1} M(x).$$

A level set of the Lyapunov function defined by

$$\mathcal{E}(Q^{-1}, \rho) = \left\{ x \in \mathbb{R}^n \mid Z(x)^T Q^{-1} Z(x) \le \rho \right\}$$

is an invariant set where $\rho > 0$. $\mathcal{E}(Q^{-1}, 1)$ is abbreviated as $\mathcal{E}(Q^{-1})$. Let $\mathcal{L}(C_1(x) + D_{11}(x)F(x))$ be the region $x \in \mathbb{R}^n$ satisfying the constraints (3), that is,

$$|(c_{1(j)}(x) + d_{11(j)}(x)F(x))Z(x)| \le 1 \ \ (j = 1, \ldots, p_{z^c})$$

where $c_{1(j)}(x)$ and $d_{11(j)}(x)$ are the j-th row vector polynomials of $C_1(x)$ and $D_{11}(x)$, respectively. The invariant set $\mathcal{E}(Q^{-1}, \rho)$ should be inside of the region $\mathcal{L}(C_1(x) + D_{11}(x)F(x))$ under the disturbances.

To simplify the problem, we make a mild assumption on the initial state. Then we give two lemmas about the invariant sets under the disturbances.

Assumption 1: $\mathcal{V}(\alpha) = \mathcal{E}(Q^{-1}, \alpha)$.

Lemma 1: Given β_∞ and $\alpha(\le \rho)$, consider the closed-loop system (1), (2) and (4). If there exists $Q(\succ 0) \in \mathbb{S}^{n_Z}$ satisfying

$$\dot{V}(x) < 0 \quad \forall x \in \partial\mathcal{E}(Q^{-1}, \rho) \ \ \forall w \in \mathcal{W}_{\mathcal{L}_\infty}(\beta_\infty) \tag{7}$$

$$\mathcal{E}(Q^{-1}, \rho) \subseteq \mathcal{L}(C_1(x) + D_{11}(x)F(x)) \tag{8}$$

then the trajectory starting at $x(0) \in \mathcal{V}(\alpha)$ stays in the constraints (3) under the disturbance $w \in \mathcal{W}_{\mathcal{L}_\infty}(\beta_\infty)$.

Proof: From (7) and $V(x(t)) < V(x(0)) \le \alpha \le \rho$, the trajectory starting at $x(0) \in \mathcal{V}(\alpha)$ stays in $\mathcal{E}(Q^{-1}, \rho)$. On the other hand, $\mathcal{E}(Q^{-1}, \rho)$ is inside of $\mathcal{L}(C_1(x)+D_{11}(x)F(x))$ from (8). ■

Lemma 2: Given β_2 and $\alpha(< \beta_2)$, consider the closed-loop system (1), (2) and (4). If there exists $Q(\succ 0) \in \mathbb{S}^{n_Z}$ satisfying

$$\dot{V}(x) < \|w\|_2^2 \quad \forall x \in \mathcal{E}(Q^{-1}, \alpha + \beta_2) \ \ \forall w \in \mathcal{W}_{\mathcal{L}_2}(\beta_2) \tag{9}$$

$$\mathcal{E}(Q^{-1}, \alpha + \beta_2) \subseteq \mathcal{L}(C_1(x) + D_{11}(x)F(x)) \tag{10}$$

then the trajectory starting at $x(0) \in \mathcal{V}(\alpha)$ stays in the constraints (3) under the disturbance $w \in \mathcal{W}_{\mathcal{L}_2}(\beta_2)$.

Proof: From (9) and $V(x(t)) < V(x(0)) + \|w\|_{\mathcal{L}_2}^2 < \alpha + \beta_2$, the trajectory starting at $x(0) \in \mathcal{V}(\alpha)$ stays in $\mathcal{E}(Q^{-1}, \alpha + \beta_2)$. On the other hand, $\mathcal{E}(Q^{-1}, \alpha + \beta_2)$ is inside of $\mathcal{L}(C_1(x) + D_{11}(x)F(x))$ from (10). ■

IV. State Feedback Synthesis

From the lemmas in section III, this section derives a computational method for polynomial systems with the disturbances. We discuss a two-step non-iterative procedure for a state feedback synthesis.

A. The First Step Design

To construct a convex computational method, we accept two assumptions on the output for the constraints (2).

Assumption 2:

$$\begin{bmatrix} C_1(x) & D_{11}(x) \end{bmatrix} = \begin{bmatrix} \bar{C}_1(x) & 0 \\ 0 & \bar{D}_{11}(x) \end{bmatrix}.$$

Assumption 3: $\mathcal{L}(\bar{C}_1(x))$ is a compact set.

Assumption 2 means that the constraints on between the state and the inputs are separable from each other, that is,

$$\mathcal{L}(C_1(x) + D_{11}(x)F(x)) = \mathcal{L}(\bar{C}_1(x)) \cap \mathcal{L}(\bar{D}_{11}(x)F(x)).$$

Then we have a relation:

$$\mathcal{L}(\bar{C}_1(x)) \cap \mathcal{L}(\bar{D}_{11}(x)F(x)) \subset \mathcal{L}(\bar{C}_1(x))$$

We denote that

$$\mathcal{L}(\bar{C}_1(x)) = \left\{ x \in \mathbb{R}^n \mid g_j(x) \geq 0 \ (j = 1, \ldots, n_g) \right\}$$

where $g_j(x) = 1 - Z(x)^T c_{1(j)}^T(x) c_{1(j)}(x) Z(x)$ and n_g is the number of row of $\bar{C}_1(x)$.

Remark 1: We have relaxed the region $\mathcal{L}(C_1(x) + D_{11}(x)F(x))$ into $\mathcal{L}(\bar{C}_1(x))$. The relaxation may cause some conservativeness, however, the relaxation does *not* mean that the input constraints (8) and (10) are ignored.

Then we have the following theorems. The proofs are given in the appendix.

Theorem 1: Consider the closed-loop system (1), (2) and (4). Given β_∞, $\alpha(\leq 1)$ and $\varepsilon(> 0)$, the trajectory starting at $x(0) \in \mathcal{V}(\alpha)$ stays in the constraints (3) under the disturbance $w \in \mathcal{W}_{\mathcal{L}_\infty}(\beta_\infty)$ if there exist $Q(> 0) \in \mathbb{S}^n$, $K(x) \in \mathbb{R}[x]^{m_u \times n_Z}$, $S_{10}(x)$, $S_{1i}(x)$, $S_{2i}(x) \in \Sigma[x]^{n_Z}$ and $S_{20}(x) \in \Sigma[x]^{n_Z+1}$ satisfying

$$\begin{aligned} &\mathrm{He}\{M(x)[A(x)Q + B_1(x)K(x)]\} \\ &\quad + M(x)B_2(x)B_2(x)^T M(x)^T \\ &\quad + \beta_\infty Q + \textstyle\sum_{i=1}^{n_g} S_{1i}(x)g_i(x) + \varepsilon I = -S_{10}(x) \end{aligned} \tag{11}$$

$$\mathrm{He}\begin{bmatrix} 1/2 & c_{1(j)}(x)Q + d_{11(j)}(x)K(x) \\ 0 & Q/2 - \sum_{i=1}^{n_g} S_{2i}(x)g_i(x) \end{bmatrix} = S_{20}(x) \quad (j = 1, \ldots, p_{z^c}) \tag{12}$$

for all $x \in \mathbb{R}^n$. In this case, $F(x) = K(x)Q^{-1}$.

Theorem 2: Consider the closed-loop system (1), (2) and (4). Given β_2, $\alpha(< \beta_2)$ and $\varepsilon(> 0)$, the trajectory starting at $x(0) \in \mathcal{V}(\alpha)$ stays in the constraints (3) under the disturbance $w \in \mathcal{W}_{\mathcal{L}_2}(\beta_2)$ if there exist $Q(> 0) \in \mathbb{S}^n$, $K(x) \in \mathbb{R}[x]^{m_u \times n_Z}$, $S_{10}(x)$, $S_{1i}(x)$, $S_{2i}(x) \in \Sigma[x]^{n_Z}$ and $S_{20}(x) \in \Sigma[x]^{n_Z+1}$ satisfying

$$\begin{aligned} &\mathrm{He}\{M(x)[A(x)Q + B_1(x)K(x)]\} \\ &\quad + M(x)B_2(x)B_2(x)^T M(x)^T \\ &\quad + \textstyle\sum_{i=1}^{n_g} S_{1i}(x)g_i(x) + \varepsilon I = -S_{10}(x) \end{aligned} \tag{13}$$

$$\mathrm{He}\begin{bmatrix} \frac{1}{2(\alpha+\beta_2)} & c_{1(j)}(x)Q + d_{11(j)}(x)K(x) \\ 0 & Q/2 - \sum_{i=1}^{n_g} S_{2i}(x)g_i(x) \end{bmatrix} = S_{20}(x) \quad (j = 1, \ldots, p_{z^c}) \tag{14}$$

for all $x \in \mathbb{R}^n$. In this case, $F(x) = K(x)Q^{-1}$.

Remark 2: The identical equations in each theorem produce linear simultaneous equations whose decision variables have semidefinite constraints, that is, SDP. Thus Theorem 1 and 2 could be answers of Problem 1 and 2,respectively.

One of the causes of conservativeness in the above theorems is elimination of $Z(x)$ to derive the SDLMIs from the Lyapunov derivative inequalities. Note the relationship: $\dot{V}(x) = Z(x)^T Q^{-1} H(x) Q^{-1} Z(x) < 0 \ \Leftarrow \ H(x) \prec 0$ and see the gap between the Lyapunov derivative and the SDLMI.

B. The Second Step Design

To decrease the conservativeness of the first step design, we give the second step design using using the polynomial annihilator, which is a polynomial version of the linear annihilator [4]. The polynomial annihilator, in the situation, is a free multiplier of matrix polynomials $\mathcal{N}(x) \in \mathbb{R}[x]^{n_Z \times n_Z}$ associated with the constraint $\mathcal{N}(x)Q^{-1}Z(x) = 0$ for all $x \in \mathbb{R}^n$. We have the following lemma using Finsler's lemma.

Lemma 3: Let $H(x) \in \mathbb{S}[x]^{n_Z}$, $\mathcal{N}(x) \in \mathbb{R}[x]^{n_Z \times n_Z}$, $Z(x) \in \mathbb{R}[x]^{n_Z}$ and $Q(> 0) \in \mathbb{S}^{n_Z}$ satisfying $\mathcal{N}(x)Q^{-1}Z(x) = 0$ and $\operatorname{rank} \mathcal{N}(x) < n_Z$ for all $x \in \mathbb{R}^n$. The following two statements are equivalent:

i) $Z(x)^T Q^{-1} H(x) Q^{-1} Z(x) < 0 \quad \forall x \in \mathbb{R}^n \setminus \{0\}$.
ii) $H(x) + \mathrm{He}\{\mathcal{N}(x)\} \prec 0 \quad \forall x \in \mathbb{R}^n$.

Proof: We can take $\mathcal{B}$ and $\mathcal{B}^\perp$ in Lemma 7 as $\tilde{\mathcal{N}}(x) \in \mathbb{R}[x]^{m \times n_Z}$ and $Q^{-1}Z(x)$, respectively. Then, $\tilde{\mathcal{N}}(x)Q^{-1}Z(x) = 0$ for all $x \in \mathbb{R}^n$. Let $\mathcal{N}(x)$ be equal to $\mathcal{X}(x)\tilde{\mathcal{N}}(x)$. Since $\operatorname{rank} \tilde{\mathcal{N}}(x) = r < n_Z$ and $\operatorname{rank} \mathcal{X}(x) \leq n_Z$, $\operatorname{rank} \mathcal{N}(x) \leq \min \left\{ \operatorname{rank} \tilde{\mathcal{N}}(x), \operatorname{rank} \mathcal{X}(x) \right\} < n_Z$. Hence, i) and ii) are equivalent by Lemma 7. ■

To find $\mathcal{N}(x)$ for given Q, we may solve a linear programming problem, a class of SDP. Note that an annihilator of $Q^{-1}Z(x)$ can be also an annihilator of $(\mu Q)^{-1}Z(x)$ where $\mu(\neq 0)$ is scalar. We have another two theorems applying Lemma 3 for Lemma 1 and 2. They are correspond to Theorem 1 and 2, respectively.

Theorem 3: Consider the closed-loop system (1), (2) and (4). Given $Q(> 0) \in \mathbb{S}^n$, β_∞, $\alpha(\leq 1)$ and $\varepsilon(> 0)$, the trajectory starting at $x(0) \in \mathcal{V}(\alpha)$ stays in the constraints (3) under the disturbance $w \in \mathcal{W}_{\mathcal{L}_\infty}(\beta_\infty)$ if there exist $\mu(> 0)$, $K(x) \in \mathbb{R}[x]^{m_u \times n_Z}$, $\mathcal{N}(x)$, $\mathcal{N}_A(x) \in \mathbb{R}[x]^{n_Z \times n_Z}$, $S_{10}(x)$, $S_{1i}(x)$, $S_{2i}(x) \in$

$\Sigma[x]^{nz}$ and $S_{20}(x) \in \Sigma[x]^{nz+1}$ satisfying

$$\begin{aligned}&\mathrm{He}\{M(x)[A(x)(\mu Q) + B_1(x)K(x)]\}\\&\quad + M(x)B_2(x)B_2(x)^T M(x)^T + \beta_\infty(\mu Q)\\&\quad + \mathrm{He}\{\mathcal{N}(x)\} + \mathrm{He}\{M(x)\mathcal{N}_A(x)Q\}\\&\quad + \textstyle\sum_{i=1}^{n_g} S_{1i}(x)g_i(x) + \varepsilon I = -S_{10}(x) \qquad (15)\end{aligned}$$

$$\mathrm{He}\begin{bmatrix} 1/2 & c_{1(j)}(x)(\mu Q) + d_{11(j)}(x)K(x) \\ 0 & (\mu Q)/2 - \sum_{i=1}^{n_g} S_{2i}(x)g_i(x) \end{bmatrix} = S_{20}(x)$$
$$(j = 1, \ldots, p_{z^c}) \qquad (16)$$

$\mathcal{N}(x)Q^{-1}Z(x) = 0$ and $\mathcal{N}_A(x)Z(x) = 0$ for all $x \in \mathbb{R}^n$. In this case, $F(x) = K(x)(\mu Q)^{-1}$.

Theorem 4: Consider the closed-loop system (1), (2) and (4). Given $Q(> 0) \in \mathbb{S}^n$, β_2, $\alpha(< \beta_2)$ and $\varepsilon(> 0)$, the trajectory starting at $x(0) \in \mathcal{V}(\alpha)$ stays in the constraints (3) under the disturbance $w \in \mathcal{W}_{\mathcal{L}_2}(\beta_2)$ if there exist $\mu(> 0)$, $K(x) \in \mathbb{R}[x]^{m_u \times nz}$, $\mathcal{N}(x)$, $\mathcal{N}_A(x) \in \mathbb{R}[x]^{nz \times nz}$, $S_{10}(x)$, $S_{1i}(x)$, $S_{2i}(x) \in \Sigma[x]^{nz}$ and $S_{20}(x) \in \Sigma[x]^{nz+1}$ satisfying

$$\begin{aligned}&\mathrm{He}\{M(x)[A(x)(\mu Q) + B_1(x)K(x)]\}\\&\quad + M(x)B_2(x)B_2(x)^T M(x)^T\\&\quad + \mathrm{He}\{\mathcal{N}(x)\} + \mathrm{He}\{M(x)\mathcal{N}_A(x)Q\}\\&\quad + \textstyle\sum_{i=1}^{n_g} S_{1i}(x)g_i(x) + \varepsilon I = -S_{10}(x) \qquad (17)\end{aligned}$$

$$\mathrm{He}\begin{bmatrix} \frac{1}{2(\alpha+\beta_2)} & c_{1(j)}(x)(\mu Q) + d_{11(j)}(x)K(x) \\ 0 & (\mu Q)/2 - \sum_{i=1}^{n_g} S_{2i}(x)g_i(x) \end{bmatrix} = S_{20}(x)$$
$$(j = 1, \ldots, p_{z^c}) \qquad (18)$$

$\mathcal{N}(x)Q^{-1}Z(x) = 0$ and $\mathcal{N}_A(x)Z(x) = 0$ for all $x \in \mathbb{R}^n$. In this case, $F(x) = K(x)(\mu Q)^{-1}$.

C. Design Procedure

The design procedure we propose is the following:

`Step 1)` Determine Q and $F(x)$ satisfying the conditions in Theorem 1 (Theorem 2).

`Step 2)` For fixed Q, determine μ and $F(x)$ satisfying the conditions in Theorem 3 (Theorem 4).

Once `Step 2` is performed in somewhat optimization scheme, for the fixed Lyapunov variable μQ, there does not exist a better μ satisfying the conditions in Theorem 3 (Theorem 4). In this meaning, the design procedure does not need any iteration.

V. Numerical Examples

In this section, we illustrate the results of the previous sections. The examples are computed by using Matlab, YALMIP [20] and SeDuMi [21]. We consider the system (1) with the matrices

$$A(x) = \begin{bmatrix} x_1 & 1 + x_2/5 \\ 0 & -x_2 - x_1x_2 \end{bmatrix} W(x)^{-1}, \quad B_1(x) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$B_2(x) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C_1(x) = \begin{bmatrix} 1.25 & 0 & 0 \\ 0 & 0.4 & 0 \end{bmatrix}^T W(x)^{-1}$$

$$D_{11}(x) = \begin{bmatrix} 0 & 0 & 0.12 \end{bmatrix}^T, \quad Z(x) = W(x)x$$

$$W(x) = \begin{bmatrix} 1 & 0 \\ 1 + x_2/5 & 1 \end{bmatrix}, \quad W(x)^{-1} = \begin{bmatrix} 1 & 0 \\ -1 - x_2/5 & 1 \end{bmatrix}.$$

Firstly, we will stabilize the system under a bounded peak disturbance based on Theorem 1 and Theorem 3 along the design procedure stated at the previous section. In `Step 1`, $\alpha(\le \rho = 1)$ and β_∞ are given by 0.9 and 0.1. The degree of $F(x)$ is fixed at 2. To perform optimization, we take a reference point at $x_{ref} = \begin{bmatrix} 1 & 1 \end{bmatrix}^T$ and add a constraint such as $\eta \cdot x_{ref} \in \mathcal{E}(Q^{-1}, \alpha)$ to the constraints in Theorem 1 and maximize η. Then we have $\eta = 0.3092$.
The invariant set is shown in Fig. 1. In `Step 2`, we will redesign for the above Q with maximization of μ by Theorem 3. Then we have $\mu = 1.3043$ and the feedback gain $F(x)$ is

$$\begin{bmatrix} -8.12 + 1.08x_1 - 0.36x_2 - 3.21x_1x_2 + 0.77x_2^2 + 0.61x_1^2 \\ -4.22 - 0.24x_1 + 0.09x_2 - 0.30x_1x_2 - 0.03x_2^2 + 1.65x_1^2 \end{bmatrix}^T$$

and the Lyapunov function $V(x)$ is

$$4.40x_1^2 + 1.87x_1x_2 + 0.40x_2^2 + 0.37x_1^2x_2 + 0.16x_1x_2^2 + 0.02x_1^2x_2^2$$

where all monomials whose coefficients are less than 0.001 are presented in truncated form. The invariant set is shown in Fig. 2. The set becomes larger than that of the first design.

Next, we will stabilize the system under a bounded energy disturbance based on Theorem 2 and Theorem 4 along the design procedure. In `Step 1`, α and β_2 are given by 0.5 and 1. The degree of $F(x)$ and the constraint for optimization are the same as the above. Under maximization of η in Theorem 2, we have $\eta = 0.2066$. The invariant set is shown in Fig. 3. In `Step 2`, we will redesign for the above Q with maximization of μ by Theorem 4. Then we have $\mu = 1.2021$ and $F(x)$ is

$$\begin{bmatrix} -6.35 + 0.88x_1 - 0.21x_2 - 3.01x_1x_2 + 0.59x_2^2 + 0.25x_1^2 \\ -4.56 - 0.26x_1 + 0.09x_2 - 0.23x_1x_2 + 0.03x_2^2 + 1.57x_1^2 \end{bmatrix}^T$$

and the Lyapunov function $V(x)$ is

$$5.70x_1^2 + 2.64x_1x_2 + 0.61x_2^2 + 0.53x_1^2x_2 + 0.24x_1x_2^2 + 0.02x_1^2x_2^2.$$

The invariant set is shown in Fig. 4. The set also becomes larger than that of the first design.

VI. Conclusions

A state feedback synthesis has been discussed for polynomial systems in the presence of disturbances with bounded peak or bounded energy from computational viewpoint. The proposed design procedure is two-step non-iterative. At the first step, the matrix SOS relaxation is directly applied to the problems. To remove a conservativeness of the first step, the polynomial annihilators are utilized in the second step. Numerical examples have illustrated the proposed design procedure. The invariant set is in the region of the inputs and the state constraints. This method has applications to, for example, mixed bounded peak and energy disturbance rejection, guaranteed cost control via $\dot{V}(x) + \|z^e\|_2^2 < 0$ instead of (7) in Lemma 1, and $\mathcal{L}_2$ gain control via $\dot{V}(x) + \|z^e\|_2^2 - \|w\|_2^2 < 0$ instead of (9) in Lemma 2 where $z^e \in \mathbb{R}^{p_{z^e}}$ is the control output for evaluation such as $z^e = C_2(x)Z(x) + D_{21}(x)u$.

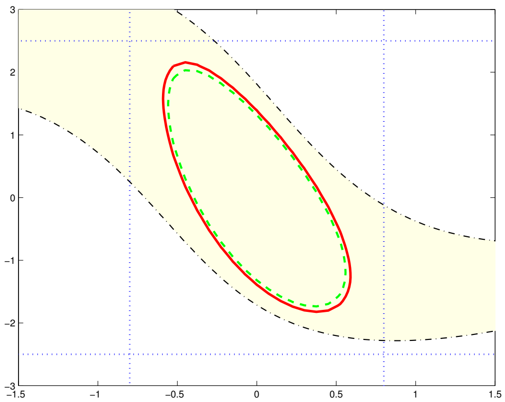

Fig. 1. The invariant set of the first design by Theorem 1: The solid ellipsoid is $\mathcal{E}(Q^{-1})$, and the dashed ellipsoid is $\mathcal{V}(\alpha)$, and the dotted rectangle bounded region is $\mathcal{L}(\bar{C}_1(x))$, and the region between the dashed-dotted curves is $\mathcal{L}(\bar{D}_{11}(x)F(x))$.

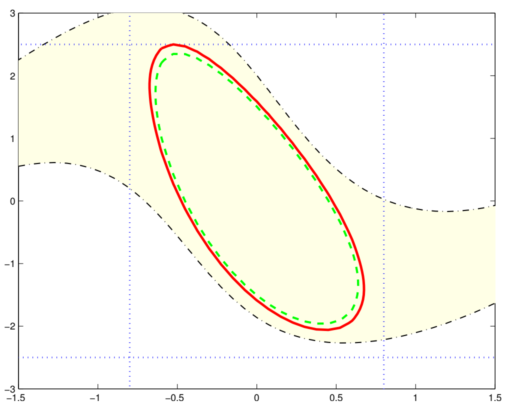

Fig. 2. The invariant set of the second design by Theorem 3: The solid ellipsoid is $\mathcal{E}((\mu Q)^{-1})$, and the dashed ellipsoid is $\mathcal{V}(\mu\alpha)$, and the dotted rectangle bounded region is $\mathcal{L}(\bar{C}_1(x))$, and the region between the dashed-dotted curves is $\mathcal{L}(\bar{D}_{11}(x)F(x))$ by the redesigned $F(x)$.

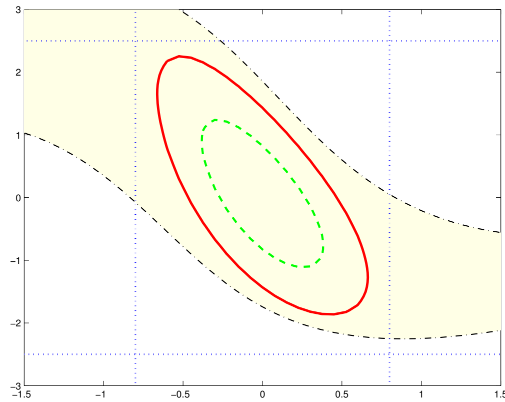

Fig. 3. The invariant set of the first design by Theorem 2: The solid ellipsoid is $\mathcal{E}(Q^{-1}, \alpha+\beta_2)$, and the dashed ellipsoid is $\mathcal{V}(\alpha)$, and the dotted rectangle bounded region is $\mathcal{L}(\bar{C}_1(x))$, and the region between the dashed-dotted curves is $\mathcal{L}(\bar{D}_{11}(x)F(x))$.

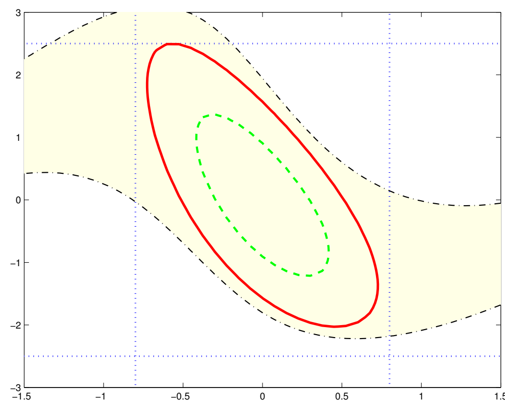

Fig. 4. The invariant set of the second design by Theorem 4: The solid ellipsoid is $\mathcal{E}((\mu Q)^{-1}, \alpha+\beta_2)$, and the dashed ellipsoid is $\mathcal{V}(\mu\alpha)$, and the dotted rectangle bounded region is $\mathcal{L}(\bar{C}_1(x))$, and the region between the dashed-dotted curves is $\mathcal{L}(\bar{D}_{11}(x)F(x))$ by the redesigned $F(x)$.

Appendix

Proof of Theorem 1

Assume (11) and (12) hold. We first show that (7) in Lemma 1 is satisfied. The derivative of the candidate Lyapunov function is

$$\begin{aligned}\dot{V}(x) &= \mathrm{He}\left\{Z(x)^T Q^{-1} M(x)\dot{x}\right\} \\ &= \mathrm{He}\left\{Z(x)^T Q^{-1} M(x)[A(x) + B_1(x)F(x)]Z(x) \right. \\ &\qquad \left. + Z(x)^T Q^{-1} M(x) B_2(x) w\right\}\end{aligned}$$

for all $x \in \mathbb{R}^n$ and $w \in \mathcal{W}_{\mathcal{L}_\infty}(\beta_\infty)$. Using Lemma 4, we have

$$\begin{aligned}\dot{V}(x) &\leq \mathrm{He}\left\{Z(x)^T Q^{-1} M(x)[A(x) + B_1(x)F(x)]Z(x)\right\} \\ &\qquad + (Z(x)^T Q^{-1} M(x) B_2(x))(\star)^T + w^T w \\ &\leq Z(x)^T Q^{-1}[\mathrm{He}\{M(x)[A(x)Q + B_1(x)K(x)]\} \\ &\qquad + (M(x)B_2(x))(\star)^T]Q^{-1}Z(x) + \beta_\infty\end{aligned}$$

for all $x \in \mathbb{R}^n$. Since (11) means

$$\begin{aligned}&\mathrm{He}\{M(x)[A(x)Q + B_1(x)K(x)]\} \\ &\quad + (M(x)B_2(x))(\star)^T + \beta_\infty Q < 0 \quad \forall x \in \mathcal{L}(\bar{C}_1(x)),\end{aligned}$$

$\dot{V}(x) < \beta_\infty(1 - Z(x)^T Q^{-1} Z(x))$ is satisfied for all $x \in \mathcal{L}(\bar{C}_1(x))$. Thus $\dot{V}(x) < 0$ holds for all $x \in \mathcal{L}(\bar{C}_1(x)) \setminus \bar{\mathcal{E}}(Q^{-1}, \rho)$ where $\bar{\mathcal{E}}(Q^{-1}, \rho)$ is $\mathcal{E}(Q^{-1}, \rho) \setminus \partial\mathcal{E}(Q^{-1}, \rho)$, which is a sufficient condition for (7) by Lemma 5. Secondly, we will show (8) in Lemma 1 is satisfied. Starting from (12), we have the following relations:

$$\begin{aligned}(12) \Leftrightarrow\ & Q - \textstyle\sum_{i=1}^{n_g} 2S_{2i}(x)g_i(x) \\ &\qquad \geq (c_{1(j)}(x)Q + d_{11(j)}(x)K(x))^T(\star) \ \ \forall x \in \mathbb{R}^n \\ \Rightarrow\ & Q \geq (c_{1(j)}(x)Q + d_{11(j)}(x)K(x))^T(\star) \ \ \forall x \in \mathcal{L}(\bar{C}_1(x)) \\ \Rightarrow\ & Q \geq (c_{1(j)}(x)Q + d_{11(j)}(x)K(x))^T(\star) \ \ \forall x \in \mathcal{E}(Q^{-1}) \\ \Leftrightarrow\ & 1 \geq \lambda_{\max}(Q^{1/2}(c_{1(j)}(x)Q + d_{11(j)}(x)K(x))^T(\star)Q^{1/2}) \\ &\qquad \forall x \in \mathcal{E}(Q^{-1}) \\ \Leftrightarrow\ & 1 \geq \max_{\substack{\|y\| = 1 \\ x \in \mathcal{E}(Q^{-1})}} \frac{\|(c_{1(j)}(x)Q + d_{11(j)}(x)K(x))Q^{1/2}y\|^2}{\|y\|^2} \\ \Leftrightarrow\ & 1 \geq \max_{\substack{\|y\| \leq 1 \\ x \in \mathcal{E}(Q^{-1})}} \|(c_{1(j)}(x)Q + d_{11(j)}(x)K(x))Q^{1/2}y\|^2 \\ \Leftrightarrow\ & 1 \geq \max_{\substack{\|Q^{-1/2}Z(x)\| \leq 1 \\ x \in \mathcal{E}(Q^{-1})}} \|(c_{1(j)}(x)Q + d_{11(j)}(x)K(x))Z(x)\|^2 \\ \Leftrightarrow\ & 1 \geq \max_{x \in \mathcal{E}(Q^{-1})} \|(c_{1(j)}(x)Q + d_{11(j)}(x)K(x))Z(x)\|^2 \\ \Leftrightarrow\ & 1 \geq |(c_{1(j)}(x)Q + d_{11(j)}(x)K(x))Z(x)| \ \ \forall x \in \mathcal{E}(Q^{-1})\end{aligned}$$

The most right hand for all $j = 1, \ldots, p_{z^c}$ means $\mathcal{E}(Q^{-1}) \subseteq \mathcal{L}(C_1(x) + D_{11}(x)F(x))$.

Proof of Theorem 2

Assume (13) and (14) holds. We will show firstly that (9) in Lemma 2 is satisfied. Using Lemma 4, we have

$$\begin{aligned} & \dot{V}(x) - w^T w \\ \leq\ & \mathrm{He}\left\{Z(x)^T Q^{-1} M(x)[A(x) + B_1(x)F(x)]Z(x)\right\} \\ & +(Z(x)^T Q^{-1} M(x) B_2(x))(\star)^T \\ \leq\ & Z(x)^T Q^{-1}[\mathrm{He}\{M(x)[A(x)Q + B_1(x)K(x)]\} \\ & +(M(x)B_2(x))(\star)^T]Q^{-1}Z(x) \end{aligned}$$

for all $x \in \mathbb{R}^n$ and $w \in \mathcal{W}_{\mathcal{L}_2}(\beta_2)$. Since (13) means

$$\begin{aligned} & \mathrm{He}\{M(x)[A(x)Q + B_1(x)K(x)]\} \\ & \quad +(M(x)B_2(x))(\star)^T \prec 0 \quad \forall x \in \mathcal{L}(\bar{C}_1(x)), \end{aligned}$$

$\dot{V}(x) - w^T w < 0$ is satisfied for all $x \in \mathcal{L}(\bar{C}_1(x))$ and for all $w \in \mathcal{W}_{\mathcal{L}_2}(\beta_2)$, which is a sufficient condition for (9) by Lemma 6. The latter part is omitted.

Lemma 4: For any $\eta(x) \in \Sigma[x]$,

$$\begin{aligned} & \mathrm{He}\{Z(x)^T Q^{-1} M(x) B_2(x) w\} \leq \eta(x) w^T w \\ & \quad +\eta(x)^{-1}(Z(x)^T Q^{-1} M(x) B_2(x))(\star)^T \end{aligned}$$

is satisfied for all $x \in \mathbb{R}^n$ and $w \in \mathbb{R}^{m_w}$.

Proof: Let $\eta(x)$ be $\bar{\eta}(x)^2$ where $\bar{\eta}(x) \in \mathbb{R}[x]$. From the relation

$$(\bar{\eta}(x)^{-1} Z(x)^T Q^{-1} M(x) B_2(x) + \bar{\eta}(x) w^T)(\star)^T \geq 0$$

we have the assertion. ∎

Lemma 5: If the statement

$$\dot{V}(x) < 0 \quad \forall x \in \mathcal{L}(\bar{C}_1(x)) \setminus \bar{\mathcal{E}}(Q^{-1}, \rho) \ \ \forall w \in \mathcal{W}_{\mathcal{L}_\infty}(\beta_\infty)$$

holds, then (7) in Lemma 1 holds where

$$\bar{\mathcal{E}}(Q^{-1}, \rho) = \mathcal{E}(Q^{-1}, \rho) \setminus \partial\mathcal{E}(Q^{-1}, \rho).$$

Proof: Since $\partial\mathcal{E}(Q^{-1}, \rho) \subset \mathcal{L}(\bar{C}_1(x)) \setminus \bar{\mathcal{E}}(Q^{-1}, \rho)$, we have the assertion. ∎

Lemma 6: If the statement

$$\dot{V}(x) < \|w\|_2^2 \quad \forall x \in \mathcal{L}(\bar{C}_1(x)) \ \ \forall w \in \mathcal{W}_{\mathcal{L}_2}(\beta_2)$$

holds, then (9) in Lemma 2 holds.

Proof: Since $\mathcal{E}(Q^{-1}, \alpha + \beta_2) \subset \mathcal{L}(\bar{C}_1(x))$, we have the assertion. ∎

Lemma 7: Let $x \in \mathbb{R}^n$, $H(x) \in \mathbb{S}[x]^{n_Z}$, and $\mathcal{B}(x) \in \mathbb{R}[x]^{m \times n_Z}$ such that $\mathrm{rank}(\mathcal{B}(x)) = r < n_Z$ for all $x \in \mathbb{R}^n \setminus \{0\}$. $\mathcal{B}(x)^\perp \in \mathbb{R}[x]^{n_Z \times (n_Z - r)}$ satisfies $\mathcal{B}(x)\mathcal{B}(x)^\perp = 0$ for all $x \in \mathbb{R}^n$. The following are equivalent.

a) $(\mathcal{B}(x)^\perp)^T H(x) \mathcal{B}(x)^\perp \prec 0 \ \ \forall x \in \mathbb{R}^n \setminus \{0\}$.
b) $\exists\ \mu(x) \in \mathbb{R}[x] : \ H(x) - \mu(x)\mathcal{B}(x)^T\mathcal{B}(x) \prec 0 \ \ \forall x \in \mathbb{R}^n$.
c) $\exists\ \mathcal{X}(x) \in \mathbb{R}[x]^{n_Z \times m} : H(x) + \mathrm{He}\{\mathcal{X}(x)\mathcal{B}(x)\} \prec 0 \ \forall x \in \mathbb{R}^n$.

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