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Fast-Lifting Approach to Time-Delay Systems: Fundamental Framework

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Abstract— **This paper gives a novel approach to time-delay systems consisting of a linear time-invariant (LTI) system and a pure delay. The fast-lifting technique, introduced recently in the context of the study of sampled-data systems, is applied to the monodromy operator of a time-delay system to define the fast-lifted monodromy operator, and a stability condition is given in terms of the spectral radius of the latter operator. Then, by investigating the properties of this operator, an operator class described by two finite-dimensional matrices is introduced as candidates for a solution to the associated operator Lyapunov inequality. It is then established that the analysis restricted to such a class gives an asymptotically exact and nonconservative method for stability analysis if the integer N for fast-lifting is taken sufficiently large.**

I. Introduction

A. Background on and New Approach to Time-Delay Systems

Time-delay systems (TDS) are very commonly encountered in engineering and sciences, e.g., in engineering systems with transportation and communication delays, as well as in biology, physiology and economics. There hence exists a quite long and deep history of study on this subject, and just a small list of books, monographs and survey papers includes [1]–[10], in which a lot of examples of TDS's are also given. Common approaches to such systems include treatment as differential equations on abstract infinite-dimensional linear spaces (e.g., [3]), functional differential equations ([1],[2], [6]) and differential equations over rings of operators (e.g., [11]). This paper, however, is stimulated by the recent study in [12] that has employed the lifting technique [13]–[16] developed and widely used in the area of sampled-data systems. The idea therein is to view the behavior of the TDS Σ shown in Fig. 1 consisting of the finite-dimensional linear time-invariant (FDLTI) system F and the delay H with the interval h on the intervals $[kh,(k+1)h)$ where k is an integer (i.e., apply the lifting at the interval h), and consider the transition between consecutive intervals. Such a new approach naturally has close relationship to each of the above three approaches, but compared with the first one, this new approach could be said to be some sort of its discretized counterpart and seems to be much simpler in the sense that we do not have to deal with unbounded differential operators but have only to deal with bounded integral operators. Furthermore, the underlying idea is quite simple but seems promising, and the present paper, together with [17], aims at extending this new approach in various aspects and developing a novel approach to TDS's as described below.

B. Extended Treatment Developed in the Present Paper

First of all, the study in [12], together with a relevant paper [18], was mainly interested in relating the characteristic roots of a TDS with the spectrum of what we call the monodromy operator in this paper and approximately computing that spectrum numerically. Thus, that approach gave an approximate method for stability analysis of TDS's. The present paper, together with our paper [17], on the other hand, aims at providing a method for stability analysis that is ensured to be "asymptotically exact (i.e., necessary and sufficient)" and is readily applicable to discrete-time controller design for TDS's. To this end, we employ what we call fast-lifting, which is recently introduced for less conservative robust stability analysis of sampled-data systems [19],[20]. This technique is associated with a positive integer N, and, roughly speaking, corresponds to viewing the system behavior at every interval h/N shorter than h. With the fast-lifting technique, we introduce what we call fast-lifted monodromy operator $\mathbf{T}_N$ of Σ and relate its spectrum with exponential stability of Σ. We then discuss stability of Σ via an operator Lyapunov inequality about the fast-lifted monodromy operator $\mathbf{T}_N$, based on the relationship between the spectrum of $\mathbf{T}_N$ and the solution of the Lyapunov inequality. This operator Lyapunov inequality, together with some important property of the solution, plays a key role in our study. The role of the integer N for fast-lifting, from theoretical point of view discussed in this paper, is to help constructing a "piecewise-constant like" solution to the operator Lyapunov inequality, and to ensure the asymptotic exactness of the

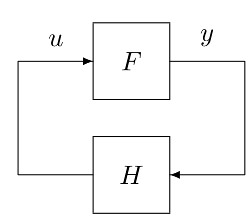

Fig. 1. Feedback system Σ with delay H.

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fast-lifting approach to TDS's. More precisely, the present paper establishes that if N is sufficiently large, an operator with a restricted but tractable form adequately constructed via fast-lifting at the parameter N does solve the operator Lyapunov inequality whenever Σ is stable. Hence, fast-lifting plays a key role in asymptotically exact stability analysis at least in the theoretical aspect. Moreover, from numerical point of view to be discussed in [17], increasing the integer N for fast-lifting plays a significant role also in reducing the errors associated with what is called quasi-finite-rank approximation of the fast-lifted monodromy operator $\mathbf{T}_N$. This facilitates numerical computations for the solution of the operator Lyapunov inequality and thus asymptotically exact stability analysis becomes possible also numerically. This is the first study ensuring such a significant property in stability analysis of TDS's, to the best knowledge of the author.

C. Promising Properties of Our Approach

Interestingly enough, the numerical computation for the solution of the operator Lyapunov inequality mentioned above reduces to solving a discrete-time LMI (a variant of the discrete-time Lyapunov inequality), due to the discrete-time characteristics of the lifting and fast-lifting technique employed. It turns out that the approach developed in our study can readily be applied also to stability analysis of TDS's with discrete-time controllers, and moreover, to discrete-time controller design for TDS's. This is a very important advantage of our approach because almost all LMI conditions for stability of TDS's are in continuous-time (see, e.g., [4]–[6] and the references therein) and thus are not compatible with discrete-time controller design, in spite of general preference for discrete-time controllers and nontrivial problems of discretizing continuous-time controllers for TDS's while retaining closed-loop stability and performance. In the above connection, it is worth stressing that our study provides a framework in which an FDLTI system F with a direct feedthrough matrix D can also be accommodated in Σ. Even though the study of [12] is restricted to retarded TDS's and thus corresponds to the case with $D = 0$, our study can allow nonzero D, and hence the delay interval h can be fractioned into subintervals of equal length and then they can be concatenated with each other with the aid of D to recover the original interval. With such a treatment, the sampling period of the discrete-time controller can be any integer fraction of the delay interval h. For the same reason, our study covers neutral time-delay systems with commensurate delays, too.

The contents of this paper are as follows. In Section II, we introduce the fast-lifted monodromy operator of Σ, which plays a fundamental role in this study. Section III studies some fundamental properties of the fast-lifted monodromy operator $\mathbf{T}_N$, applies them to the stability analysis of Σ, and derives a stability condition based on the spectral radius of $\mathbf{T}_N$. Section IV relates this stability condition to an operator Lyapunov inequality and establishes that, whenever Σ is stable and provided that N is large enough, a solution to this inequality exists within some restricted and tractable class of operators that can be handled "in a finite-dimensional treatment." Section V remarks that the approach can readily be extended to robust stability analysis with respect to the uncertainties in the FDLTI part of Σ. Section VI summarizes the arguments of the paper and gives some remarks about the relationship to some relevant studies as well as the further studies in the companion paper [17]. All the proofs are omitted due to limited space.

The following notation is used in this paper. $\sigma(\cdot)$, $\sigma_p(\cdot)$ and $\sigma_e(\cdot)$ denote the spectrum, point spectrum and essential spectrum of an operator, respectively, while $\lambda(\cdot)$ denotes the set of eigenvalues of a matrix. We simply say that an operator $(\cdot)$ is invertible if it has a bounded inverse. The spectral radius of an operator or a matrix is denoted by $\rho(\cdot)$, while $\rho_e(\cdot) = \sup_{\gamma\in\sigma_e(\cdot)} |\gamma|$ denotes the essential spectral radius of an operator. $\mathbb{R}$ and $\mathbb{N}$ denote the sets of real numbers and positive integers, respectively. $\mathcal{K}_m$ is a shorthand notation for the Hilbert space $(L_2([0,h);\mathbb{R}))^m$ with an underlying $h > 0$, and $\mathcal{F}_n$ denotes the n-dimensional Euclidean space $\mathbb{R}^n$.

II. Fast-Lifted Monodromy Operator of Time-Delay Systems

We consider the feedback system in Fig. 1, denoted by Σ, consisting of the finite-dimensional linear time-invariant (FDLTI) system F and the pure delay H. We then introduce what we call the monodromy operator and fast-lifted monodromy operator associated with the system Σ, and suggest that such operators, together with their straightforward extensions, can play a fundamental role in dealing with analysis and design problems of time-delay systems subject to continuous-time or sampled-data control.

Regarding the system Σ in Fig. 1, we assume that F has the state-space representation

$$\frac{dx}{dt} = Ax + Bu,\ y = Cx + Du \tag{1}$$

with $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times\mu}$, $C \in \mathbb{R}^{\mu\times n}$ and $D \in \mathbb{R}^{\mu\times\mu}$, and the input-output relation of the pure delay H is given by

$$u(t) = y(t-h), \quad h > 0 \tag{2}$$

Since we can handle $D \neq 0$ in the system F, the arguments of this paper will be readily applicable to systems with commensurate delays.

A. Monodromy Operator

We assume that the initial conditions of the TDS Σ are given by $x(0) = x_0$ and $y(\theta - h) = \widehat{u}_0(\theta),\ 0 \le \theta < h$ with some $\widehat{u}_0 \in \mathcal{K}_\mu$. This in particular implies that $u(\theta) = \widehat{u}_0(\theta)$ $(0 \le \theta < h)$, and thus we can denote, in a non-inconsistent fashion, the lifted representation [13]–[16] of u by $\{\widehat{u}_k\}_{k=0}^{\infty}$, where $\widehat{u}_k(\theta) = u(kh+\theta)$. Let us denote $x(kh)$ simply by x_k. By (1), we have

$$x(kh+\theta) = \exp(A\theta)x_k + \int_0^\theta \exp(A(\theta-\tau))B\widehat{u}_k(\tau)d\tau \quad (3)$$

$$\begin{aligned}\widehat{u}_{k+1}(\theta) &= y(kh+\theta) = Cx(kh+\theta) + Du(kh+\theta)\\ &= C\exp(A\theta)x_k + \int_0^\theta C\exp(A(\theta-\tau))B\widehat{u}_k(\tau)d\tau\\ &\quad + D\widehat{u}_k(\theta)\end{aligned} \quad (4)$$

Hence, Σ can be represented by

$$\begin{bmatrix} x_{k+1} \\ \widehat{u}_{k+1} \end{bmatrix} = \begin{bmatrix} A_d & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} \begin{bmatrix} x_k \\ \widehat{u}_k \end{bmatrix} \quad (5)$$

where the matrix $A_d \in \mathbb{R}^{n\times n}$ and the operators $\mathbf{B} : \mathcal{K}_\mu \to \mathcal{F}_n$, $\mathbf{C} : \mathcal{F}_n \to \mathcal{K}_\mu$ and $\mathbf{D} : \mathcal{K}_\mu \to \mathcal{K}_\mu$ are defined as follows.

$$A_d = \exp(Ah) \quad (6)$$

$$\mathbf{B}f = \int_0^h \exp(A(h-\tau))Bf(\tau)d\tau \quad (7)$$

$$(\mathbf{C}v)(\theta) = C\exp(A\theta)v \quad (8)$$

$$(\mathbf{D}f)(\theta) = \int_0^\theta C\exp(A(\theta-\tau))Bf(\tau)d\tau + Df(\theta) \quad (9)$$

With a slight abuse of notation, the operator of multiplication by the matrix D is also represented by the same symbol D; the distinction whether D represents an operator or its underlying matrix will be clear from the context. Then, $\mathbf{D}_0 := \mathbf{D} - D$ is a compact operator.

A similar representation to (5) has been given in [12] by taking the lifted representation $\widehat{x}_k$ of the state of F, in lieu of $\widehat{u}_k$. This implies that in our treatment, the second entry $\widehat{u}_k$ in the state of (5) is regarded as independent of $x(t)$ (even though $\widehat{x}_k$ is obviously not), and this will provide us with a chance for extending the treatment readily to the case with, e.g., a digital controller that obviously has independent discrete-time state variables. That is, our treatment is well suited to immediate generalization to the setting of TDS's under sampled-data control. We believe that this is a very important feature of the present approach and it will be straightforward to see that the results provided in this paper can readily be extended to sampled-data controller design problems. However, we will leave all such further topics to future independent papers and we here confine ourselves to manifesting the fundamental effectiveness and advantages of the present approach. In view of (5), we call the operator

$$\mathbf{T} = \begin{bmatrix} A_d & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} : \mathcal{M} \to \mathcal{M} \quad (10)$$

in (5) the monodromy operator of the system Σ, where $\mathcal{M} := \mathcal{F}_n \oplus \mathcal{K}_\mu$; recall that $\mathcal{F}_n$ and $\mathcal{K}_\mu$ are shorthand notations for $\mathbb{R}^n$ and $(L_2([0,h);\mathbb{R}))^\mu$, respectively, and the symbol $\oplus$ denotes the direct sum of Hilbert spaces, which generates a new Hilbert space [21].

B. Fast-Lifted Monodromy Operator

The monodromy operator introduced above plays a very important role, but from the viewpoint of numerical computation, it is not always easy to deal with. Hence we here prepare for some sort of discretization, even though no approximation is actually introduced yet at this stage. That is, we take a positive integer N and consider the fast-lifting $\mathbf{L}_N : \mathcal{K}_m \to (\mathcal{K}'_m)^N$ [19],[20], where $\mathcal{K}'_m$ denotes the Hilbert space $\mathcal{K}_m$ with h replaced by $h' := h/N$. By definition, if we apply $\mathbf{L}_N$ to $\widehat{u}_k$, we have

$$(\mathbf{L}_N\widehat{u}_k)(\theta') = \check{u}_k(\theta') = \begin{bmatrix} \widehat{u}_k(\theta') \\ \widehat{u}_k(h'+\theta') \\ \vdots \\ \widehat{u}_k((N-1)h'+\theta') \end{bmatrix}, \quad 0 \le \theta' < h' \quad (11)$$

Note that $\mathbf{L}_N$ is invertible and thus nothing will be approximated or lost by applying $\mathbf{L}_N$. Under the notation $\mathbf{I}(\cdot) = \mathrm{diag}[I,\ (\cdot)]$ for an operator $(\cdot)$, i.e., $\mathbf{I}(\mathbf{L}_N) = \mathrm{diag}[I,\ \mathbf{L}_N]$, where I on the right hand side denotes the identity matrix on $\mathcal{F}_n$, let us define

$$\mathbf{T}_N = \mathbf{I}(\mathbf{L}_N)\mathbf{T}\,\mathbf{I}(\mathbf{L}_N)^{-1} =: \begin{bmatrix} A_d & \mathbf{B}_N \\ \mathbf{C}_N & \mathbf{D}_N \end{bmatrix} : \mathcal{M}'_N \to \mathcal{M}'_N \quad (12)$$

where $\mathcal{M}'_N$ is a shorthand notation for $\mathcal{F}_n \oplus (\mathcal{K}'_\mu)^N$. Then, it is easy to see that (5) can be rewritten as

$$\begin{bmatrix} x_{k+1} \\ \check{u}_{k+1} \end{bmatrix} = \mathbf{T}_N \begin{bmatrix} x_k \\ \check{u}_k \end{bmatrix} \quad (13)$$

and hence $\mathbf{T}_N$ given by (12) is called the fast-lifted monodromy operator of the system Σ.

Let us introduce A'_d, $\mathbf{B}'$, $\mathbf{C}'$, $\mathbf{D}'$ and $\mathbf{D}'_0$ defined as A_d, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$ and $\mathbf{D}_0$, respectively, with the horizon $[0,h)$ replaced by $[0,h')$. Then, regarding the representation (12), we readily have (see, e.g., [20])

$$\mathbf{B}_N = \begin{bmatrix} (A'_d)^{N-1}\mathbf{B}' & \cdots & A'_d\mathbf{B}' & \mathbf{B}' \end{bmatrix} \quad (14)$$

$$\mathbf{C}_N = \begin{bmatrix} \mathbf{C}' \\ \mathbf{C}'A'_d \\ \vdots \\ \mathbf{C}'(A'_d)^{N-1} \end{bmatrix} \quad (15)$$

$$\mathbf{D}_N = \begin{bmatrix} \mathbf{D}' & 0 & \cdots & 0 \\ \mathbf{C}'\mathbf{B}' & \mathbf{D}' & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ \mathbf{C}'(A'_d)^{N-2}\mathbf{B}' & \cdots & \mathbf{C}'\mathbf{B}' & \mathbf{D}' \end{bmatrix} \quad (16)$$

$$\mathbf{D}' = \mathbf{D}'_0 + D : \mathcal{K}'_\mu \to \mathcal{K}'_\mu \quad (17)$$

III. Stability Analysis with Fast-Lifted Monodromy Operator

This section studies some important properties of the fast-lifted monodromy operator, and applies them to stability analysis of the system Σ. The spectral discussions here might be more or less related to those found in, e.g., [7], but our arguments are simpler in the sense that we do not have to deal with unbounded operators. More importantly, our arguments proceed in a discrete-time fashion in spite of the continuous-time nature of the system Σ, and this feature will allow us to employ, in later sections, similar ideas to what has led us to novel techniques in the robust stability analysis of sampled-data systems [19],[22]. With such ideas, we will eventually have a discretization method for stability analysis ([17]) such that (i) for each discretization parameter N, it gives a sufficient condition for stability, and (ii) as N tends to infinity, the sufficient condition converges to a necessary and sufficient condition.

A. Fundamental Properties of Fast-Lifted Monodromy Operator

We give some fundamental properties of the fast-lifted monodromy operator in this subsection. More specifically, we introduce the class of operators that we denote by $\mathcal{B}_{\mathbf{F}}$, to which the fast-lifted monodromy operator $\mathbf{T}_N$ belongs, and study the properties of that class. We begin with the following result.

Lemma 1 Consider the linear bounded operator $\mathbf{F}$ on $\mathcal{M}'_N = \mathcal{F}_n \oplus (\mathcal{K}'_\mu)^N$ such that

$$\mathbf{F} = \begin{bmatrix} \mathbf{F}_{00} & \mathbf{F}_{01} \\ \mathbf{F}_{10} & \mathbf{F}_{11} \end{bmatrix} = \begin{bmatrix} \mathbf{F}_{00} & \mathbf{F}_{01} \\ \mathbf{F}_{10} & \mathbf{F}_{110} + F_{11} \end{bmatrix}$$
$$= \mathbf{F}_0 + F_1, \quad F_1 = \begin{bmatrix} 0 & 0 \\ 0 & F_{11} \end{bmatrix} \tag{18}$$

where $\mathbf{F}_{110}$ (and thus $\mathbf{F}_0$) is a compact operator, and F_{11} is the operator of multiplication on $(\mathcal{K}'_\mu)^N$ defined by the constant matrix $F_{11} \in \mathbb{R}^{\mu N \times \mu N}$. Then, we have

(i) $\sigma_e(\mathbf{F}) = \sigma_e(F_1) = \sigma_e(F_{11}) = \sigma(F_{11}) = \sigma_p(F_{11}) = \lambda(F_{11})$

(ii) $\sigma(F_1) = \sigma_p(F_1) = \lambda(F_{11}) \cup \{0\} = \lambda(F_1)$, where F_1 is viewed as a matrix in $\mathbb{R}^{(n+\mu N)\times(n+\mu N)}$ when $\lambda(F_1)$ is referred to.

We henceforth denote the class of linear bounded operators on $\mathcal{M}'_N$ by $\mathcal{B}^{(N)}$, and the class of the operators $\mathbf{F}$ of the form (18) by $\mathcal{B}_{\mathbf{F}}^{(N)} \subset \mathcal{B}^{(N)}$; we can indeed show that $\mathcal{B}_{\mathbf{F}}^{(N)}$ is a proper subset of $\mathcal{B}^{(N)}$. When the underlying N is obvious or is of no particular concern, we simply denote $\mathcal{B}_{\mathbf{F}}^{(N)}$ by $\mathcal{B}_{\mathbf{F}}$ and $\mathcal{B}^{(N)}$ by $\mathcal{B}$. We also have the following results regarding the properties of $\mathcal{B}_{\mathbf{F}}$.

Lemma 2 Suppose $\mathbf{F} \in \mathcal{B}_{\mathbf{F}}$. If $\gamma \in \sigma(\mathbf{F})$, $\gamma \neq 0$ and $\gamma \notin \lambda(F_{11})$, then $\gamma \in \sigma_p(\mathbf{F})$.

Remark 1 We can also show that every $\gamma \in \sigma(\mathbf{F})$ such that $\gamma \neq 0$ and $\gamma \notin \lambda(F_{11})$ is an *isolated point* of $\sigma(\mathbf{F})$ and an eigenvalue of $\mathbf{F}$ with *finite multiplicity*.

Lemma 3 The class $\mathcal{B}_{\mathbf{F}}$ has the following properties.

(i) $\mathcal{B}_{\mathbf{F}}$ is a linear space over $\mathbb{R}$ and constitutes a ring with respect to operator addition and composition. More specifically, $\mathcal{B}_{\mathbf{F}}$ is a Banach algebra [23] with identity under the norm

$$\|\mathbf{F}\| = \sup_{v \in \mathcal{M}'_N \setminus \{0\}} \frac{\|\mathbf{F}v\|_{\mathcal{M}'_N}}{\|v\|_{\mathcal{M}'_N}} \tag{19}$$

(ii) $\mathbf{F}^* \in \mathcal{B}_{\mathbf{F}}$ whenever $\mathbf{F} \in \mathcal{B}_{\mathbf{F}}$.

(iii) If $\mathbf{F} \in \mathcal{B}_{\mathbf{F}}$ and $0 \notin \sigma(\mathbf{F})$, then $\mathbf{F}^{-1} \in \mathcal{B}_{\mathbf{F}}$.

B. Spectral Radius of Fast-Lifted Monodromy Operator and Stability

Based on the above lemmas, we can show the following theorem; some related arguments for the case $D = 0$ (i.e., the retarded case) and $r_0 = 1$ can be found in [12].

Theorem 1 Let $\mathbf{F} \in \mathcal{B}_{\mathbf{F}}$. Then, there exists $M > 0$ and $0 < r < 1$ such that $\|\mathbf{F}^k\| \leq Mr^k, \ \forall k \in \mathbb{N}$ if and only if $\rho(\mathbf{F}) < 1$. More generally, given $r_0 > 0$, there exists $M > 0$ and $0 < r < r_0$ such that $\|\mathbf{F}^k\| \leq Mr^k, \ \forall k \in \mathbb{N}$ if and only if $\rho(\mathbf{F}) < r_0$.

We can easily see that the fast-lifted monodromy operator $\mathbf{T}_N$ given by (12) belongs to the class $\mathcal{B}_{\mathbf{F}}$, and thus we can use Theorem 1 for the stability analysis of the TDS Σ. More precisely, we are readily led to the following theorem.

Theorem 2 Let N be an arbitrary fixed positive integer. The system Σ is exponentially stable if and only if the associated fast-lifted monodromy operator $\mathbf{T}_N$ satisfies $\rho(\mathbf{T}_N) < 1$.

In the above theorem, exponential stability of Σ is defined as follows.

Definition 1 The TDS Σ is said to be exponentially stable if there exist $M > 0$ and $\alpha > 0$ such that

$$\left\| \begin{bmatrix} x(t) \\ u(t+\cdot) \end{bmatrix} \right\|_{\mathcal{M}} \leq M \exp(-\alpha t) \left\| \begin{bmatrix} x(0) \\ u(0+\cdot) \end{bmatrix} \right\|_{\mathcal{M}}$$
$$\forall t \geq 0, \ \forall x(0), \ \forall u(0+\cdot) \tag{20}$$

where $u(\tau + \cdot)$ denotes $u(t), \ \tau \leq t < \tau + h$.

Remark 2 A somewhat similar result to Theorem 2 can be found also in, e.g., [7], but the operator involved in the condition therein (which is the infinitesimal generator of Σ, and thus is unbounded) is different from ours due to the difference in the treatment and thus the underlying function space. Noting that $\mathbf{T}_N$ is precisely $\mathbf{T}$ when $N = 1$, we can easily see that the well-known necessary condition $\rho(D) < 1$ for stability of Σ follows readily from the above theorem, Lemma 1, and the fact that $\sigma_e(\cdot) \subset \sigma(\cdot)$.

In conjunction with the above theorem, the following result is an immediate consequence of (12), which will be used in the following discussions.

Lemma 4 For any $N \in \mathbb{N}$, we have $\sigma(\mathbf{T}_N) = \sigma(\mathbf{T})$ and thus $\rho(\mathbf{T}_N) = \rho(\mathbf{T})$. In particular, $\sigma(\mathbf{T}_{\nu N}) = \sigma(\mathbf{T}_N)$ and $\rho(\mathbf{T}_{\nu N}) = \rho(\mathbf{T}_N)$ for any $\nu \in \mathbb{N}$.

As far as Theorem 2 and Lemma 4 are concerned, it might look $N = 1$ suffices and there is no meaning for taking $N \geq 2$. For the sake of numerical computations, however, we will introduce quasi-finite-rank approximation of $\mathbf{T}_N$, or finite-rank approximation of $\mathbf{D}'_0$ in (17), together with some scaling approach (the details will be discussed in [17]). In that context, taking $N \geq 2$ will be very important in reducing approximation errors and thus arriving at a less conservative (actually asymptotically exact) result. Fundamental theoretical arguments supporting such a direction with the use of N will be developed in the following section.

IV. Operator Lyapunov Inequalities and Scaling Treatment for Stability Analysis

This section relates the stability condition given in Theorem 2 to a discrete-time operator Lyapunov inequality and establishes that a solution to such an inequality exists within some prescribed class of operators. Roughly speaking, we show via some scaling treatment that if N is sufficiently large, such a class can be described by two finite-dimensional matrices. The fast-lifted monodromy operator, however, is cumbersome due to its "mixed nature" with respect to $\mathcal{F}_n$ and $(\mathcal{K}'_\mu)^N$ that constitute $\mathcal{M}'_N$. Hence a key technique of "embedding the vector space $\mathcal{F}_n$ to a function space" is also developed.

A. Lyapunov Inequalities and Their Solutions

Here we give some preliminary results that relate the stability analysis of Σ to operator Lyapunov inequalities. We begin with some pertinent definitions.

Definition 2 For $N \in \mathbb{N}$, the class $\mathfrak{B}_{\mathbf{F}}^{(N)}$ of operators on $\mathcal{M}$ is defined as $\mathfrak{B}_{\mathbf{F}}^{(N)} := \{\mathbf{I}(\mathbf{L}_N)^{-1}\mathbf{F}\,\mathbf{I}(\mathbf{L}_N)\,|\,\mathbf{F} \in \mathcal{B}_{\mathbf{F}}^{(N)}\}$. The class of linear bounded operators on $\mathcal{M}$ is denoted by $\mathfrak{B}$.

Note that when N is different, the underlying space $\mathcal{M}'_N$ for $\mathcal{B}_{\mathbf{F}}^{(N)}$ is different, while the underlying space for $\mathfrak{B}_{\mathbf{F}}^{(N)}$ is $\mathcal{M}$ for all N. We obviously have $\mathfrak{B}_{\mathbf{F}}^{(N)} \subset \mathfrak{B}$ for all $N \in \mathbb{N}$. Furthermore, since the lifted representation of an operator of multiplication by a matrix has block diagonal structure, it is easy to see that $\mathfrak{B}_{\mathbf{F}}^{(N)} \subset \mathfrak{B}_{\mathbf{F}}^{(\nu N)} \subset \mathfrak{B}$ for all $N \in \mathbb{N}$ and $\nu \in \mathbb{N}$, where the inclusion relations are strict.

Definition 3 Let $\mathcal{Z}$ be a Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and let $\mathbf{G}$ be a linear self-adjoint bounded operator on $\mathcal{Z}$. We say that $\mathbf{G}$ is positive definite (resp. positive semi-definite) and denote $\mathbf{G} > 0$ (resp. $\mathbf{G} \geq 0$) if $\langle \mathbf{G}v, v\rangle > 0$ (resp. ≥ 0) whenever $v \neq 0$. We say that $\mathbf{G}$ is strictly positive definite and denote $\mathbf{G} \succ 0$ if there exists a scalar $\delta > 0$ such that $\mathbf{G} - \delta I \geq 0$. Furthermore, we say that $\mathbf{G}$ is negative definite (resp. negative semi-definite, or strictly negative definite) and denote $\mathbf{G} < 0$ (resp. $\mathbf{G} \leq 0$, or $\mathbf{G} \prec 0$) if $-\mathbf{G}$ is positive definite (resp. positive semi-definite, or strictly positive definite).
As in the standard convention, $\mathbf{G}_1 \succ \mathbf{G}_2$ (or $\mathbf{G}_2 \prec \mathbf{G}_1$) is a shorthand notation for $\mathbf{G}_1 - \mathbf{G}_2 \succ 0$. The following result is also standard.

Lemma 5 $\mathbf{G} \succ 0$ if and only if $\sigma(\mathbf{G})$ is contained in a closed interval on the positive real axis. In particular, $\mathbf{G} \succ 0$ only if $\sigma_e(\mathbf{G})$ is contained in a closed interval on the positive real axis.

We are in a position to state the following key results.

Proposition 1 Let $\mathbf{G}$ be a linear bounded operator on a Hilbert space $\mathcal{Z}$. Then, $\rho(\mathbf{G}) < 1$ if and only if there exists an operator $\mathbf{X} \succ 0$ on $\mathcal{Z}$ such that

$$\mathbf{G}^*\mathbf{X}\mathbf{G} - \mathbf{X} \prec 0 \tag{21}$$

Furthermore, if $\mathcal{Z} = \mathcal{M}'_N$, $\mathbf{G} \in \mathcal{B}_{\mathbf{F}}$ and $\rho(\mathbf{G}) < 1$, then there exists $\mathbf{X} \in \mathcal{B}_{\mathbf{F}}$ satisfying $\mathbf{X} \succ 0$ and (21).

Corollary 1 Let $\mathbf{G}$ be a linear bounded operator on the Hilbert space $\mathcal{M}$. Then, $\rho(\mathbf{G}) < 1$ if and only if there exists an operator $\mathbf{X} \succ 0$ on $\mathcal{M}$ such that (21) holds. Furthermore, if $\mathbf{G} \in \mathfrak{B}_{\mathbf{F}}^{(N)}$ and $\rho(\mathbf{G}) < 1$ for some $N \in \mathbb{N}$, then there exists $\mathbf{X} \in \mathfrak{B}_{\mathbf{F}}^{(N)}$ satisfying $\mathbf{X} \succ 0$ and (21).

Proposition 1 given above corresponds to a discrete-time operator Lyapunov inequality, and the first assertion can be found, e.g., in [24],[25]. The second assertion, which follows immediately from the proof in [25] together with Lemma 3, is very important in our subsequent arguments. Corollary 1 follows readily from Proposition 1 and the definition of $\mathfrak{B}_{\mathbf{F}}^{(N)}$.

B. Modified Fast-Lifted Monodromy Operator

The fast-lifted monodromy operator $\mathbf{T}_N$ plays a central role in the following discussions, but is rather cumbersome due to the "mixed nature" of its underlying space $\mathcal{M}'_N = \mathcal{F}_n \oplus (\mathcal{K}'_\mu)^N$. It would be easier if we could instead work on an operator defined on a "pure function space" that is free of the finite-dimensional vector space $\mathcal{F}_n$. Here, we introduce such an operator $\widetilde{\mathbf{T}}_N$ called the modified fast-lifted monodromy operator. In fact, it is defined as the operator $\widetilde{\mathbf{T}}_N = \mathbf{J}_H\mathbf{T}_N\mathbf{J}_S$ on the space $\widetilde{\mathcal{M}}'_N := \widetilde{\mathcal{F}}'_n \oplus (\mathcal{K}'_\mu)^N$ with appropriately defined operators $\mathbf{J}_H$ and $\mathbf{J}_S$ and space $\widetilde{\mathcal{F}}'_n$. The definitions of these operators and space are as follows.

For $m \in \mathbb{N}$, we introduce the ("sampling and hold") operators $\mathbf{J}_{S0} : \mathcal{K}'_m \to \mathcal{F}_m$ and $\mathbf{J}_{H0} : \mathcal{F}_m \to \mathcal{K}'_m$ defined by

$$\mathbf{J}_{S0}f = \frac{1}{\sqrt{h'}}\int_0^{h'} f(\theta')d\theta',$$
$$(\mathbf{J}_{H0}v)(\theta') = \frac{1}{\sqrt{h'}}v,\ 0 \leq \theta' < h' \tag{22}$$

Note that $\mathbf{J}_{S0}$ and $\mathbf{J}_{H0}$ have similar structure to $\mathbf{B}'$ and $\mathbf{C}'$, respectively, which will turn out to be crucial in the numerical computations (to be discussed in [17]). Here we define $\widetilde{\mathcal{F}}'_m \subset \mathcal{K}'_m$ as the space of constant functions $\{v_0\mathbf{1}\,|\,v_0 \in \mathcal{F}_m\}$ with $\mathbf{1}(\theta') = 1$ $(0 \leq \theta' < h')$. It follows readily that $\widetilde{\mathcal{F}}'_m$ is a Hilbert space with the inner product on $\mathcal{K}'_m$ restricted to constant functions. Furthermore, we can consider the restriction of $\mathbf{J}_{S0}$ to $\widetilde{\mathcal{F}}'_m$, which we also denote by the same symbol $\mathbf{J}_{S0}$. Similarly, we can view $\mathbf{J}_{H0}$ also as a mapping from $\mathcal{F}_m$ to $\widetilde{\mathcal{F}}'_m$, which is again denoted by the same symbol $\mathbf{J}_{H0}$. With respect to the inner product on $\widetilde{\mathcal{F}}'_m$ mentioned above and that on $\mathcal{F}_m$, we have $\mathbf{J}_{H0} = \mathbf{J}_{S0}^*$, and also $\mathbf{J}_{S0}\mathbf{J}_{H0} = I$ on $\mathcal{F}_m$ and $\mathbf{J}_{H0}\mathbf{J}_{S0} = I$ on $\widetilde{\mathcal{F}}'_m$. It should be noted, however, that $\mathbf{J}_{H0}\mathbf{J}_{S0} \neq I$ on $\mathcal{K}'_m$. Hence, it is extremely important to pay attention on which Hilbert space we are working on, and for this reason some of our arguments will be carried out in such a way that might look verbose if one would forget about this crucial issue.

We further introduce $\mathbf{J}_S = \mathrm{diag}[\mathbf{J}_{S0},\ I] : \widetilde{\mathcal{M}}'_N \to \mathcal{M}'_N$ and $\mathbf{J}_H = \mathrm{diag}[\mathbf{J}_{H0},\ I] = \mathbf{J}_S^* : \mathcal{M}'_N \to \widetilde{\mathcal{M}}'_N$ (where $\widetilde{\mathcal{M}}'_N = \widetilde{\mathcal{F}}'_n \oplus (\mathcal{K}'_\mu)^N$, and $\mathbf{J}_{S0}$ and $\mathbf{J}_{H0}$ act on $\widetilde{\mathcal{F}}'_n$ and $\mathcal{F}_n$, respectively, while I acts on $(\mathcal{K}'_\mu)^N$). Then, we have $\mathbf{J}_S\mathbf{J}_H = I$ on $\mathcal{M}'_N$ and $\mathbf{J}_H\mathbf{J}_S = I$ on $\widetilde{\mathcal{M}}'_N$. We further introduce the operator class $\widetilde{\mathcal{B}}_{\mathbf{F}} := \{\mathbf{J}_H\mathbf{F}\mathbf{J}_S \,|\, \mathbf{F} \in \mathcal{B}_{\mathbf{F}}\}$ and the modified fast-lifted monodromy operator

$$\widetilde{\mathbf{T}}_N = \mathbf{J}_H\mathbf{T}_N\mathbf{J}_S \in \widetilde{\mathcal{B}}_{\mathbf{F}} \tag{23}$$

defined on $\widetilde{\mathcal{M}}'_N$. Then, it is obvious that $\rho(\mathbf{T}_N) = \rho(\mathbf{J}_S\mathbf{J}_H\mathbf{T}_N) = \rho(\widetilde{\mathbf{T}}_N)$. We next relate the stability condition $\rho(\mathbf{T}_N) < 1$ or $\rho(\widetilde{\mathbf{T}}_N) < 1$ to a discrete-time operator Lyapunov inequality in the following subsection so that we can develop a novel scaling approach to the stability analysis of Σ.

C. Scaling Idea to Stability Analysis and a Class of Solutions to Lyapunov Inequalities

For our later purposes, it is more convenient to convert the analysis with Lyapunov inequalities to the scaling treatment of operators. Indeed, from Proposition 1 and Corollary 1, we can claim that when N is large enough, checking $\rho(\mathbf{T}) < 1$, or equivalently, checking $\rho(\widetilde{\mathbf{T}}_N) < 1$ virtually amounts to finding an invertible operator $\widetilde{\mathbf{S}} \in \widetilde{\mathcal{S}}_N$ such that

$$\|\widetilde{\mathbf{S}}\widetilde{\mathbf{T}}_N\widetilde{\mathbf{S}}^{-1}\| < 1 \tag{24}$$

where $\widetilde{\mathcal{S}}_N$ is some appropriately defined class of operators $\widetilde{\mathbf{S}}$ on $\widetilde{\mathcal{M}}'_N$. To see this, note that the inequality (24) is equivalent to $\left(\widetilde{\mathbf{S}}\widetilde{\mathbf{T}}_N\widetilde{\mathbf{S}}^{-1}\right)^*\widetilde{\mathbf{S}}\widetilde{\mathbf{T}}_N\widetilde{\mathbf{S}}^{-1} \prec I$ and thus if the class $\widetilde{\mathcal{S}}_N$ is such that $\widetilde{\mathbf{S}}^*\widetilde{\mathbf{S}} \succ 0$ whenever $\widetilde{\mathbf{S}} \in \widetilde{\mathcal{S}}_N$, then the condition (24) is equivalent to the existence of $\widetilde{\mathbf{P}}\ (= \widetilde{\mathbf{S}}^*\widetilde{\mathbf{S}}) \succ 0$ such that

$$\widetilde{\mathbf{T}}_N^*\widetilde{\mathbf{P}}\widetilde{\mathbf{T}}_N \prec \widetilde{\mathbf{P}} \tag{25}$$

This Lyapunov inequality then shows stability of Σ immediately from Proposition 1. The above arguments, however, only imply that (24) is sufficient for stability of Σ; what is really important in establishing the claim raised at the beginning of this subsection is to show that restricting to $\widetilde{\mathbf{P}} = \widetilde{\mathbf{S}}^*\widetilde{\mathbf{S}}$ does not lead to conservativeness in the stability analysis, *provided that the class $\widetilde{\mathcal{S}}_N$ is defined appropriately and $N \in \mathbb{N}$ is taken large enough*. In this sense, it is very important how to define the class $\widetilde{\mathcal{S}}_N$, or the class $\widetilde{\mathcal{P}}_N = \{\widetilde{\mathbf{S}}^*\widetilde{\mathbf{S}} \,|\, \widetilde{\mathbf{S}} \in \widetilde{\mathcal{S}}_N\}$. We thus proceed to the definition of $\widetilde{\mathcal{P}}_N$ so that the above claim can indeed be established. We first note the following result† regarding the Lyapunov inequality (25), which essentially follows from (23) and the fact that $\mathbf{J}_S\mathbf{J}_H = I$ on $\mathcal{M}'_N$ and $\mathbf{J}_H\mathbf{J}_S = I$ on $\widetilde{\mathcal{M}}'_N$.

†Note that $\widetilde{\mathbf{P}}$ is an operator on $\widetilde{\mathcal{M}}'_N$ (rather than on $\mathcal{M}'_N$). That is, the domain of $\mathbf{J}_S$ in Lemma 6 is taken to be $\widetilde{\mathcal{M}}'_N$.

Lemma 6 If $\widetilde{\mathbf{T}}_N^*\widetilde{\mathbf{P}}\widetilde{\mathbf{T}}_N \prec \widetilde{\mathbf{P}}$ for $\widetilde{\mathbf{P}} \succ 0$, then $\mathbf{T}_N^*\mathbf{P}\mathbf{T}_N \prec \mathbf{P}$ for $\mathbf{P} = \mathbf{J}_H^*\widetilde{\mathbf{P}}\mathbf{J}_H \succ 0$. Conversely, if $\mathbf{T}_N^*\mathbf{P}\mathbf{T}_N \prec \mathbf{P}$ for $\mathbf{P} \succ 0$, then $\widetilde{\mathbf{T}}_N^*\widetilde{\mathbf{P}}\widetilde{\mathbf{T}}_N \prec \widetilde{\mathbf{P}}$ for $\widetilde{\mathbf{P}} = \mathbf{J}_S^*\mathbf{P}\mathbf{J}_S \succ 0$.

Hence, by the arguments around (25), we see that finding $\widetilde{\mathbf{S}} \in \widetilde{\mathcal{S}}_N$ satisfying (24) is equivalent to finding $\mathbf{P} \succ 0$ of the form $\mathbf{P} = \mathbf{J}_H^*\widetilde{\mathbf{P}}\mathbf{J}_H$, $\widetilde{\mathbf{P}} \in \widetilde{\mathcal{P}}_N$ such that $\mathbf{T}_N^*\mathbf{P}\mathbf{T}_N \prec \mathbf{P}$. Here, let us consider

$$\begin{aligned}\widetilde{\mathbf{P}} &= \begin{bmatrix} \mathbf{J}_{H0}P_{00}\mathbf{J}_{S0} & \mathbf{J}_{H0}P_{01}\mathbf{J}_{S0} & \cdots & \mathbf{J}_{H0}P_{0N}\mathbf{J}_{S0} \\ \mathbf{J}_{H0}P_{01}\mathbf{J}_{S0} & \mathbf{J}_{H0}P_{11}\mathbf{J}_{S0} + \Pi & \cdots & \mathbf{J}_{H0}P_{1N}\mathbf{J}_{S0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{J}_{H0}P_{0N}\mathbf{J}_{S0} & \mathbf{J}_{H0}P_{N1}\mathbf{J}_{S0} & \cdots & \mathbf{J}_{H0}P_{NN}\mathbf{J}_{S0} + \Pi \end{bmatrix} \\ &= \mathrm{diag}[\mathbf{J}_{H0}, \mathbf{J}_{H0}, \cdots, \mathbf{J}_{H0}]P\,\mathrm{diag}[\mathbf{J}_{H0}, \mathbf{J}_{H0}, \cdots, \mathbf{J}_{H0}]^* \\ &\quad +\mathrm{diag}[0, \Pi, \cdots, \Pi] \in \widetilde{\mathcal{B}}_{\mathbf{F}} \end{aligned} \tag{26}$$

where $P := (P_{ij})_{i,j=0}^N \in \mathbb{R}^{(n+\mu N)\times(n+\mu N)}$ and $\Pi \in \mathbb{R}^{\mu\times\mu}$ are matrices such that

$$P + \mathrm{diag}[0, \Pi, \cdots, \Pi] > 0, \quad \Pi > 0 \tag{27}$$

Note that P corresponds to the "compact part" of $\widetilde{\mathbf{P}}$ depending on N while Π to the "noncompact part." Here, we can show the following result on $\widetilde{\mathbf{P}}$.

Proposition 2 The operator $\widetilde{\mathbf{P}}$ given by (26) satisfies $\widetilde{\mathbf{P}} \succ 0$ if and only if (27) holds.

Since the property $\widetilde{\mathbf{P}} \succ 0$ ensured in the above result is consistent with our prior assumption $\widetilde{\mathbf{P}} = \widetilde{\mathbf{S}}^*\widetilde{\mathbf{S}} \succ 0$, we define the class $\widetilde{\mathcal{P}}_N$ as the set of operators $\widetilde{\mathbf{P}}$ given by (26) and (27). The class $\widetilde{\mathcal{S}}_N$ is then defined as the set of operators $\widetilde{\mathbf{S}}$ on $\widetilde{\mathcal{M}}'_N$ such that $\widetilde{\mathbf{S}}^*\widetilde{\mathbf{S}} \in \widetilde{\mathcal{P}}_N$. Now, when $\widetilde{\mathbf{P}} \in \widetilde{\mathcal{P}}_N$, we easily see that $\mathbf{P} = \mathbf{J}_H^*\widetilde{\mathbf{P}}\mathbf{J}_H$ is represented as

$$\begin{aligned}\mathbf{P} &= \mathrm{diag}[I, \mathbf{J}_{H0}, \cdots, \mathbf{J}_{H0}]P\,\mathrm{diag}[I, \mathbf{J}_{H0}, \cdots, \mathbf{J}_{H0}]^* \\ &\quad +\mathrm{diag}[0, \Pi, \cdots, \Pi] \in \mathcal{B}_{\mathbf{F}}\end{aligned} \tag{28}$$

Hence, if we recall the arguments just after Lemma 6, it follows from (12) that finding $\widetilde{\mathbf{S}} \in \widetilde{\mathcal{S}}_N$ satisfying (24) is further equivalent to finding $\mathbf{P}$ given by (27) and (28) such that $\mathbf{T}_N^*\mathbf{P}\mathbf{T}_N \prec \mathbf{P}$, which in turn is equivalent to finding $\mathbf{Q} = \mathbf{I}(\mathbf{L}_N)^{-1}\mathbf{P}\,\mathbf{I}(\mathbf{L}_N) \succ 0$ with $\mathbf{P}$ given by (27) and (28) such that $\mathbf{T}^*\mathbf{Q}\mathbf{T} \prec \mathbf{Q}$. Note that $\mathbf{Q} = \mathbf{I}(\mathbf{L}_N)^{-1}\mathbf{P}\,\mathbf{I}(\mathbf{L}_N)$ introduced here is an operator on $\mathcal{M}$. More specifically, $\mathbf{Q} \in \mathfrak{B}_{\mathbf{F}}^{(1)} \subset \mathfrak{B}_{\mathbf{F}}^{(N)}$ due to the form of the second term on the right hand side of (28).

We are now almost ready to establish the claim raised at the beginning of this subsection. What remains is to show that given any strictly positive definite operator $\mathbf{X} \in \mathfrak{B}_{\mathbf{F}}^{(1)}$, the above special form of $\mathbf{Q} \in \mathfrak{B}_{\mathbf{F}}^{(1)}$ constructed from P and Π satisfying (27) can approximate $\mathbf{X}$ to any degree of accuracy by letting $N \to \infty$. We can indeed establish the following result.

Proposition 3 Suppose that an arbitrary $\mathbf{X} \in \mathfrak{B}_{\mathbf{F}}^{(1)}$ is given. For any $\varepsilon > 0$ there exists $N_0 \in \mathbb{N}$ such that for all $N \geq N_0$, there exists $\mathbf{Q} = \mathbf{I}(\mathbf{L}_N)^{-1}\mathbf{P}\mathbf{I}(\mathbf{L}_N)$ with $\mathbf{P}$ given by (28) such that $\|\mathbf{X} - \mathbf{Q}\| < \varepsilon$. In particular, if $\mathbf{X}$ is strictly positive definite, there exists such $\mathbf{Q}$ that is strictly positive definite.

Since $\mathbf{T} \in \mathcal{B}_{\mathbf{F}}^{(1)} = \mathfrak{B}_{\mathbf{F}}^{(1)}$, it follows that, regarding the existence of a solution to $\mathbf{T}^*\mathbf{X}\mathbf{T} \prec \mathbf{X}$, restricting $\mathbf{X} \succ 0$ to belong to $\mathfrak{B}_{\mathbf{F}}^{(1)}$ does not lead to loss of generality by Corollary 1. Hence, the claim raised in the beginning of this subsection has been established by the above proposition.

Having established the crucial claim, we see that we can have an asymptotically exact (i.e., nonconservative) stability analysis method with the scaling approach by $\widetilde{\mathbf{S}} \in \widetilde{\mathcal{S}}_N$ on $\widetilde{\mathbf{T}}_N$ by letting $N \to \infty$, as far as a theoretical side is concerned. Even though this scaling approach is essentially the same as solving an operator Lyapunov inequality, it leads to an advantage especially with respect to a numerical procedure for stability analysis. As will be studied in [17], we will develop, through quasi-finite-rank approximation of the fast-lifted monodromy operator $\mathbf{T}_N$, a method that leads to an "approximate solution" to an operator Lyapunov inequality, and then give a method, through some error analysis, to check if it indeed solves the Lyapunov inequality. In such a context, scalar inequalities such as (24) turn out to be easier to deal with, and with such a technique, the numerical search for an appropriate $\widetilde{\mathbf{S}}$ (or equivalently, $\widetilde{\mathbf{P}} = \widetilde{\mathbf{S}}^*\widetilde{\mathbf{S}} \succ 0$) can be carried out in an asymptotically exact fashion; we will further discuss the role of N in [17] in connection with quasi-finite-rank approximation, where the parameter N will play an important role also in reducing the approximation error to any degree.

V. Extension to Robust Stability Problem

We have established in the preceding section that stability analysis of Σ amounts to finding an appropriate operator $\widetilde{\mathbf{P}}$ satisfying (25), (26) and (27). Even though it is very important to give a numerical procedure for this purpose, it takes a large space to provide the details about such a procedure. Hence we opt to confine the present paper to theoretical developments about the fast-lifting approach to the time-delay system Σ, and the discussions necessary for developing numerical procedures will be deferred to the paper [17]. In this section, we remark that the fast-lifting approach is not limited to (nominal) stability analysis but also applicable to robust stability analysis with respect to the uncertainties in the FDLTI part of Σ.

We begin with the following result, which corresponds to some sort of the small-gain theorem.

Proposition 4 Suppose that $\mathbf{A} \in \mathcal{B}_{\mathbf{F}}$. Then, $0 \notin \sigma(I - \mathbf{A}\mathbf{U})$ for all $\mathbf{U} \in \mathcal{B}_{\mathbf{F}}$ such that $\|\mathbf{U}\| \le 1$ if and only if $\|\mathbf{A}\| < 1$. Similarly, $0 \notin \sigma(I - \mathbf{A}\mathbf{U})$ for all $\mathbf{U} \in \mathcal{B}_{\mathbf{F}}$ such that $\|\mathbf{U}\| < 1$ if and only if $\|\mathbf{A}\| \le 1$.

To avoid introducing new notations that are necessary for rigorous statements but require a large space, we henceforth confine ourselves to the case $N = 1$, even though the idea can readily be extended to the general case.

We first introduce the notation $\mathcal{B}_{\mathbf{F}}(\mu)$, which is specific as to the underlying μ (i.e., the number of the inputs and outputs of F), to denote the class $\mathcal{B}_{\mathbf{F}}$ given by (18). The class of operators $\mathbf{F}_{11}$ in the bottom-right block of $\mathbf{F} \in \mathcal{B}_{\mathbf{F}}(\mu)$ (i.e., an operator on $\mathcal{K}_\mu$ given as the sum of a compact operator and a multiplication operator) is denoted by $\mathcal{B}_{\mathbf{F}1}(\mu)$. The set of $\mathbf{U} \in \mathcal{B}_{\mathbf{F}}(\mu)$ with $\|\mathbf{U}\| < 1$ (resp. $\|\mathbf{U}\| \le 1$) is denoted by $\mathcal{U}_{\mathbf{F}}(\mu)$ (resp. $\underline{\mathcal{U}}_{\mathbf{F}}(\mu)$), and the set of $\mathbf{U} \in \mathcal{B}_{\mathbf{F}1}(\mu)$ with $\|\mathbf{U}\| < 1$ (resp. $\|\mathbf{U}\| \le 1$) is denoted by $\mathcal{U}_{\mathbf{F}1}(\mu)$ (resp. $\underline{\mathcal{U}}_{\mathbf{F}1}(\mu)$. Suppose that $\mathbf{A} \in \mathcal{B}_{\mathbf{F}}(\mu_1 + \mu_2)$ is represented as

$$\mathbf{A} = \left[\begin{array}{c:cc} \mathbf{A}_{00} & \mathbf{A}_{01} & \mathbf{A}_{02} \\ \hdashline \mathbf{A}_{10} & \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{20} & \mathbf{A}_{21} & \mathbf{A}_{22} \end{array}\right] : \mathcal{F}_n \oplus (\mathcal{K}_{\mu_1} \oplus \mathcal{K}_{\mu_2}) \ni \left[\begin{array}{c} p_0 \\ \hdashline p_1 \\ p_2 \end{array}\right] \mapsto \left[\begin{array}{c} q_0 \\ \hdashline q_1 \\ q_2 \end{array}\right] \in \mathcal{F}_n \oplus (\mathcal{K}_{\mu_1} \oplus \mathcal{K}_{\mu_2}) \quad (29)$$

Then, we denote by $\mathbf{A} \star \mathbf{U}_2$ the mapping from $[p_0^*,\ p_1^*]^*$ to $[q_0^*,\ q_1^*]^*$ when p_2 and q_2 are related by $p_2 = \mathbf{U}_2 q_2$ with $\mathbf{U}_2 \in \mathcal{B}_{\mathbf{F}1}(\mu_2)$. Under these notations, we are immediately led to the following "main loop theorem."

Proposition 5 $\mathbf{A} \star \mathbf{U}_2$ is well-defined and belongs to $\mathcal{B}_{\mathbf{F}}(\mu_1)$ and $\|\mathbf{A} \star \mathbf{U}_2\| \le 1$ for all $\mathbf{U}_2 \in \mathcal{U}_{\mathbf{F}1}(\mu_2)$ if and only if $0 \notin \sigma(I - \mathbf{A}_{22}\mathbf{U}_2)$ and $0 \notin \sigma(I - (\mathbf{A} \star \mathbf{U}_2)\mathbf{U}_1)$ for all $\mathbf{U}_1 \in \mathcal{U}_{\mathbf{F}}(\mu_1)$ and $\mathbf{U}_2 \in \mathcal{U}_{\mathbf{F}1}(\mu_2)$. Similarly, $\mathbf{A} \star \mathbf{U}_2$ is well-defined and belongs to $\mathcal{B}_{\mathbf{F}}(\mu_1)$ and $\|\mathbf{A} \star \mathbf{U}_2\| < 1$ for all $\mathbf{U}_2 \in \underline{\mathcal{U}}_{\mathbf{F}1}(\mu_2)$ if and only if $0 \notin \sigma(I - \mathbf{A}_{22}\mathbf{U}_2)$ and $0 \notin \sigma(I - (\mathbf{A} \star \mathbf{U}_2)\mathbf{U}_1)$ for all $\mathbf{U}_1 \in \underline{\mathcal{U}}_{\mathbf{F}}(\mu_1)$ and $\mathbf{U}_2 \in \underline{\mathcal{U}}_{\mathbf{F}1}(\mu_2)$. In particular, if $\|\mathbf{A}\| \le 1$, then $\|\mathbf{A} \star \mathbf{U}_2\| \le 1$ for all $\mathbf{U}_2 \in \mathcal{U}_{\mathbf{F}1}(\mu_2)$, and if $\|\mathbf{A}\| < 1$, then $\|\mathbf{A} \star \mathbf{U}_2\| < 1$ for all $\mathbf{U}_2 \in \underline{\mathcal{U}}_{\mathbf{F}1}(\mu_2)$.

Based on the above proposition, we can deal with the robust stability problem of the system Σ_Δ shown in Fig. 2 with $u(t),\ y(t) \in \mathbb{R}^{\mu_1}$, $w(t),\ z(t) \in \mathbb{R}^{\mu_2}$, where G is an FDLTI system and Δ denotes the uncertainty represented by an FDLTI system. The details, however, are omitted due to limited space.

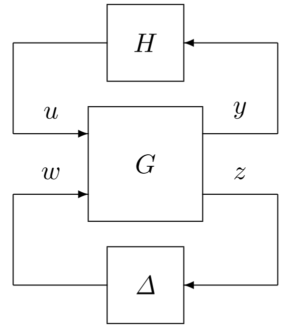

Fig. 2. Uncertain feedback system Σ_Δ with delay H.

VI. Concluding Remarks

Stimulated by the study in [12], we have developed a new approach to time-delay systems. The operator that played the key role in the study of [12], which we call the monodromy operator in this paper, was transformed by applying the fast-lifting technique to define the fast-lifted monodromy operator. Stability of the system Σ was then related to the spectrum of the fast-lifted monodromy operator, and then to the solution to an operator Lyapunov inequality. A class of candidates to the solutions to the Lyapunov inequality was then introduced, which is represented by two finite-dimensional matrices P and Π and thus is easy to deal with. It was further shown that a solution does exist within this class whenever Σ is stable, provided that the integer N for fast-lifting is taken sufficiently large. A basic idea for generalization of this approach to the robust stability problem with respect to the uncertainties in the FDLTI part F was also given. Based on this fundamental but novel theoretical development, we will establish in [17] a numerical method for stability analysis of Σ by deriving a discrete-time LMI condition for stability. A very important contribution therein is that our approach can lead to an asymptotically exact (i.e., non-conservative) method also from the numerical point of view as the approximation parameter N is made sufficiently large. This is the first study ensuring such a significant property, to the best knowledge of the author.

The stability analysis method developed in this paper has some similarity to the discretized Lyapunov functional (DLF) method [26],[27], and it may be possible to clarify their mutual relationship. Unlike in that method, however, our result is more general in that it is not limited to retarded systems with a single delay length but is also applicable to neutral systems with commensurate delays without essential difficulties. Another difference in these two methods is that the DLF method uses piecewise linear approximation while our approach corresponds, roughly speaking, to some sort of piecewise constant approximation, but a more obvious and sharp difference is that the DLF method is based on the Lyapunov-Krasovskii functional and thus is in continuous-time, while ours are based on a new (bounded) operator-theoretic approach that works in discrete-time. It does not seem so hard to allow piecewise linear approximation also in our approach, but it is left to future studies at this moment.

Due to the discrete-time characteristics of our approach, the (asymptotically exact) LMI condition for stability that we derive in the companion paper [17] is a discrete-time one, which, after some straightforward extension, is suited not only for analysis of TDS's with continuous-time or discrete-time controllers but also for design of discrete-time controllers for continuous-time TDS's. This is a very important advantage of our approach because almost all LMI conditions for stability of TDS's are in continuous-time (variants of the continuous-time Lyapunov inequality). It is quite reasonable to design discrete-time controllers nowadays, and continuous-time controllers quite often have to be discretized for the purpose of implementation. It is often the case that continuous-time controllers for TDS's are infinite-dimensional, and their discretization could lead to loss of closed-loop stability. Regarding these issues, our approach provides a clear and significant advantage that it can give a method for direct design of discrete-time controllers with guaranteed closed-loop stability in a delay-dependent fashion, provided that the sampling period is taken as an integer fraction of the delay h or commensurate delays.

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