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Stability of reset control systems with nonzero reference

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Abstract— **This paper is devoted to the stability and performance analysis for reset control systems. The output of the system must track a nonzero reference in presence of external $\mathcal{L}_2$ bounded disturbances. Constructive LMI-based conditions are proposed by using quadratic and piecewise quadratic Lyapunov functions.**

Keywords. Reset control systems, stability, nonzero reference, LMI.

I. Introduction

Over the last decade, in the quest for providing more flexible tools for achieving the stabilization and performance tasks, research efforts have focused on developing control algorithms, using controllers that involve switching or on-line adaptation. This may complicate the analysis of the stability, but may overcome performance limitations of more classical controllers, i.e., regular linear or nonlinear controllers (see, e.g., [11], [14], [4]).

In this paper, the class of hybrid systems under consideration is the one which includes reset control systems, combining continuous dynamics (represented by differential equations) with finite dynamics (instantaneous jumps of variables) [4]. The concept of reset systems was previously introduced in [3] with the so-called Clegg integrator and next generalized to a first order system in [8]. Several works have proposed explicit models for control systems involving Clegg integrator or FORE (First Order Reset Element) to achieve stability and performance using Lyapunov based conditions [1], [7], [16], [5].

In more recent publications, stability and performance of systems which include reset controllers (Clegg integrator or FORE) have been studied. In [14], [10] a more general model is proposed. Authors introduced piecewise quadratic Lyapunov functions to solve the problem of stability and to upperbound the input-output $\mathcal{L}_2$ gain. In [15], a generalization of the FORE allows to guarantee asymptotic tracking of constant references. More recently the $\mathcal{H}_2$ problem was also considered [13]. Better performances conduct to largest control values and these results are extended in [9] to handle saturation at the controller output. But to the best of our knowledge, there does not exist any work studying stability issue of a nonlinear system resulting from a general reset controller tracking a nonzero reference. Hence, we propose in this paper to analyze stability and tracking performance of a SISO system associated to a reset controller in presence of external perturbation.

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In this context, by exploiting properties of suitable Lyapunov functions, LMI-based conditions are proposed to guarantee asymptotic stability and minimize an upperbound of a tracking criterion. For different class of references, we prove the asymptotic stability and that the output of the closed-loop converges to a reference (a priori known) or equivalently that error tends to zero.

Notations. The identity matrix of order n is denoted I_n. The null $m \times n$ matrix is denoted $0_{m\times n}$. When no confusion is possible, identity and null matrices are simply denoted I and 0, respectively. For two symmetric matrices, A and B, $A > B$ means that $A - B$ is positive definite. A' denotes the transpose of A.

II. Problem statement

The plants considered throughout the paper are described by the following SISO system:

$$\begin{array}{rcl}\dot{x}_p & = & A_p x_p + B_p u,\\ y & = & C_p x_p,\end{array} \tag{1}$$

where $x_p \in \mathbb{R}^{n_p}$ is the state vector, $y \in \mathbb{R}$ is the measured output of the plant and $u \in \mathbb{R}$ its input. In (1), $A_p,\ B_p,\ C_p$ are constant matrices of appropriate dimensions.
Associate to system (1), we consider a reset system controller. Based on the use of the framework introduced in [4], we consider the following model of reset controller:

$$\begin{array}{l}\left.\begin{array}{l}\dot{x}_r(t,i) = A_r x_r(t,i) + B_r e(t)\\ y_r(t,i) = C_r x_r(t,i) + D_r e(t)\end{array}\right\} \begin{array}{l}\text{if } e(t)y_r(t,i) \geq 0,\\ \forall t \in [t_i, t_{i+1}]\end{array}\\ x_r(t_{i+1}, i+1) = 0 \quad \text{ if } e(t_{i+1})y_r(t_{i+1}, i) \leq 0,\end{array} \tag{2}$$

where $x_r(t,i) \in \mathbb{R}^{n_r}$ is the state of the controller at time t with i resets occured until time t. Note that $x(t_{i+1}, i)$ and $x(t_{i+1}, i+1)$ are the value of vector x before and after a jump, respectively. The variable $y_r \in \mathbb{R}$ is the controller output and e denotes its input. In (2), A_r, B_r, C_r, D_r are constant matrices of appropriate dimensions.

Throughout the paper, we will often use the notation x for $x(t,i)$ and x^+ for $x(t_{i+1}, i+1)$.

Remark 1: Many choices are possible to define the flow and jump sets. Here, they are based on the sign of the controller input and output. The objective is to limit overshoot (and then settling time) by imposing that the slowest signal changes of sign as the same time as the fastest signal. For example, by considering a "phase delay" controller, the input e changes of sign before the output y_r. The proposed reset condition leads to an acceleration of change of sign of y_r.

The interconnection between system (1) and (2) is as follows:

$$u = y_r,\ e = r - y, \tag{3}$$

where $r \in \mathbb{R}$ is a reference to track and $e \in \mathbb{R}$ is the tracking error (see Figure 1). We also consider an external perturbation d such as the disturbance vector $d : [0, \infty) \mapsto \mathbb{R}^q$ is assumed to be limited in energy, i.e., $d \in \mathcal{L}_2^q$ and for some scalar δ, $0 < \frac{1}{\delta} < \infty$, one gets:

$$\| d \|_2^2 = \int_0^\infty d(\tau)' d(\tau) d\tau \leq \frac{1}{\delta}. \quad (4)$$

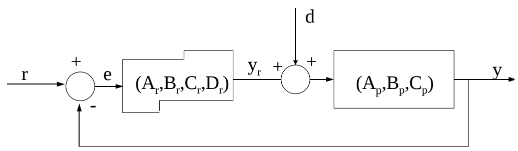

Fig. 1. Closed-loop system

Remark 2: The system (2) may be a reset control system specially designed such that the closed-loop (1)-(2)-(3) is stable and satisfies some constraints (time rising, overshoot ...), for example see the FORE in [6], [14]. Moreover, system (2) may be a classical continuous dynamic controller to which a reset equation is added in order to improve performances.

Let us define the augmented state vector

$$x = [x_p' \ x_r']' \in \mathbb{R}^n, \quad (5)$$

with $n = n_p + n_r$, and the closed-loop system (1)-(2)-(3) under consideration reads:

$$\begin{array}{l} \dot{x} = A_F x + Br + B_d d, \ \text{if } x \in \mathcal{F}, \\ x^+ = A_J x, \ \text{if } x \in \mathcal{J}, \end{array} \quad (6)$$

and the output is defined by $y = Cx$, with

$$A_F = \begin{bmatrix} A_p - B_p D_r C_p & B_p C_r \\ -B_r C_p & A_r \end{bmatrix}, \ A_J = \begin{bmatrix} I_{n_p} & 0 \\ 0 & 0 \end{bmatrix},$$

$$B = \begin{bmatrix} B_p B_r \\ B_r \end{bmatrix}, \ B_d = \begin{bmatrix} B_p \\ 0 \end{bmatrix}, \ C = \begin{bmatrix} C_p & 0 \end{bmatrix}. \quad (7)$$

The flow and jump sets, $\mathcal{F}$ and $\mathcal{J}$ are respectively described by:

$$\begin{array}{l} \mathcal{F} := \{x \in \mathbb{R}^n \ ; x' M x \geq 0\}, \\ \mathcal{J} := \{x \in \mathbb{R}^n \ ; x' M x \leq 0\}, \end{array} \quad (8)$$

where M is a *reset matrix* of appropriate dimensions.

To avoid Zeno solutions (see [4]), we adopt the following assumption (see [10]).

Assumption 1: For system (6), the reset matrix is such that

$$x \in \mathcal{J} \Longrightarrow x^+ \in \mathcal{F}. \quad (9)$$

Note that, for reset systems (2), $D_r \geq 0$ is a sufficient condition to satisfy Assumption 1. Indeed, by noting that $e^+ = e$ and $y_r^+ = D_r e$, the sign of quantity $e^+ y_r^+$, and then sign of $x' M x$, depend on D_r.

The problem we intend to solve is summarized as follows.

Problem 1: Given a reset matrix M, find a Lyapunov function such as

- if $d = 0$, system (6) is asymptotically stable;
- if $d \neq 0$, the tracking error e defined in (3) converges asymptotically to zero, for all admissible disturbances d satisfying (4).

To address Problem 1, let us consider the following lemma.

Lemma 1: Given a vector e, its time-derivative $\dot{e}$ and a finite positive scalar η such that

$$||e||_2^2 + ||\dot{e}||_2^2 \leq \eta, \quad (10)$$

then $\lim_{t \to \infty} e(t) = 0$.

Proof: The following property holds:

$$\begin{array}{rl} \left| \frac{1}{2} \left(e'(t)e(t) - e'(0)e(0) \right) \right| & = \left| \int_0^t e(t)\dot{e}(t) dt \right| \\ & \leq \int_0^t |e(t)\dot{e}(t)| \, dt \\ & \leq \sqrt{\int_0^t |e(t)|^2 dt} \sqrt{\int_0^t |\dot{e}(t)|^2 dt}, \end{array}$$

with $t \to \infty$, one gets with (10)

$$\left| \frac{1}{2} \left(e'(t)e(t) - e'(0)e(0) \right) \right| \quad \leq \quad ||e||_2 ||\dot{e}||_2. \quad (11)$$

Hence, $\int_0^t e(t)\dot{e}(t)dt$ is also finite and $\lim_{t \to \infty} e(t)$ does exist. Moreover using (10) again, we get $\lim_{t \to \infty} e(t) = 0$. ■

Remark 3: The determination of the matrix M is a key tool for modeling and analyzing reset control systems. For the reset system (6) with a non-null reference r, it is obvious that the reset matrix M depends on the reference.

In the sequel, Problem 1 is addressed by considering two kinds of references: constant and decreasing references.

III. CONSTANT REFERENCE

In this section, we assume that the reference is constant $r = r_0$. The closed-loop system becomes, in absence of perturbation:

$$\begin{array}{l} \dot{x} = A_F x + B r_0, \ \text{if } x \in \mathcal{F} \\ x^+ = A_J x, \ \text{if } x \in \mathcal{J} \end{array} \quad (12)$$

To analyze the stability of system (12), let us consider the following change of variables:

$$\tilde{x} = \begin{bmatrix} \tilde{x}_p' & \tilde{x}_r' \end{bmatrix}' = x - x_e, \quad (13)$$

where x_e is the equilibrium state. Due to Assumption 1, we have $x_e \in \mathcal{F}$. Indeed, $x_e \in \mathcal{J}$ implies $x_e^+ = A_J x_e = x_e \in \mathcal{F}$ and the vector x_e satisfies:

$$x_e \quad = \quad -A_F^{-1} B r_0, \quad (14)$$

$$C x_e \quad = \quad r_0 \quad (15)$$

provided that A_F is non-singular. If A_F is singular, x_e if exists, has to satisfy $A_F x_e + B r_0 = 0$ instead of (14). Hence, x_e corresponds to the equilibrium, in the case $d = 0$. Note that relation (15) means that $y_e = r_0$, that is, $e_e = 0$. Then, performing the change of variables (13), (14) and (15), the closed-loop reset system reads:

$$\begin{array}{l} \dot{\tilde{x}} = A_F \tilde{x} + B_d d, \ \text{if } \tilde{x} \in \mathcal{F} \\ \tilde{x}^+ = A_J \tilde{x}, \ \text{if } \tilde{x} \in \mathcal{J} \\ y = C\tilde{x} + r_0 \end{array} \quad (16)$$

with the error written as:

$$e = -C\tilde{x} = -C_p\tilde{x}_p. \quad (17)$$

The flow and jump sets $\mathcal{F}$ and $\mathcal{J}$ may be determined from the sign of the product ey_r with

$$y_r = C_r x_r + D_r e = C_r\tilde{x}_r - D_r C_p\tilde{x}_p. \quad (18)$$

Therefore, the matrix $\tilde{M}$ allowing to define $\mathcal{F}$ and $\mathcal{J}$ similarly to (8) with respect to $\tilde{x}$, is rewritten using (17) and (18) by $\tilde{M} = Q'\mathcal{M}Q$ with $\mathcal{M} = \begin{bmatrix} 0 & -I \\ -I & 0 \end{bmatrix}$ and $Q = \begin{bmatrix} C_p & 0 \\ D_r C_p & C_r \end{bmatrix}$. The following proposition provides constructive conditions to solve Problem 1 by exploiting the properties of quadratic Lyapunov functions.

Theorem 1: If there exist symmetric positive definite matrix $P \in \mathbb{R}^{n\times n}$ and positive scalars τ, τ_F, τ_J and γ satisfying:

$$\begin{bmatrix} A_F'P + PA_F - \tau A_F'A_F + \tau_F\tilde{M} & \tau A_F' & (P - \tau A_F')B_d & C' & 0 \\ \star & -\tau I & \tau B_d & 0 & C' \\ \star & \star & -\tau B_d'B_d - \gamma I & 0 & 0 \\ \star & \star & \star & -\gamma I & 0 \\ \star & \star & \star & \star & -\gamma I \end{bmatrix} \leq 0, \quad (19)$$

$$A_J'PA_J - P - \tau_J\tilde{M} \leq 0, \quad (20)$$

then

- when $d = 0$, the reset control system (16) is globally asymptotically stable;
- when $d \neq 0$, relation (10) holds with

$$\eta = \gamma\tilde{x}_0'P\tilde{x}_0 + \gamma^2||d||_2^2, \quad (21)$$

for all disturbances d satisfying (4) and thus, with Lemma 1, $\lim_{t\to\infty} e(t) = 0$.

Proof: Let us consider the quadratic Lyapunov function

$$V(\tilde{x}) = \tilde{x}'P\tilde{x},\ P = P' > 0. \quad (22)$$

By derivating (22) along continuous trajectories of system (16), we obtain $\dot{V} = \tilde{x}'(A_F'P + PA_F)\tilde{x} + 2d'B_d'P\tilde{x}$. Let us define the quantity L as

$$L = \dot{V}(\tilde{x}) + \frac{1}{\gamma}(e'e + \dot{e}'\dot{e}) - \gamma d'd, \quad (23)$$

where e is defined in (17) and $\dot{e} = -C\dot{\tilde{x}}$. Along continuous trajectories of system (16), we want to satisfy $L < 0$, if $\tilde{x} \in \mathcal{F}$, or equivalently $L \leq 0$, if $\tilde{x}'\tilde{M}\tilde{x} \geq 0$. This condition can be rewritten as follows by using the S-procedure [2]:

$$L + \tau_F\tilde{x}'\tilde{M}\tilde{x} \leq 0 \quad (24)$$

with τ_F a positive scalar. Moreover, the variables $\tilde{x}$, $\dot{\tilde{x}}$, and d satisfy the following equality $N\begin{bmatrix} \tilde{x}' & \dot{\tilde{x}}' & d' \end{bmatrix}' = 0$ with $N = \begin{bmatrix} A_F & -I & B_d \end{bmatrix}$. By applying the Finsler's lemma [2], inequality (24) is equivalent to

$$L + \tau_F\tilde{x}'\tilde{M}\tilde{x} \leq \tau N'N, \quad (25)$$

with τ a positive scalar and L a function of $\tilde{x}$ and d. The satisfaction of relation (19) implies that (25) is satisfied.

Regarding the discrete part, the satisfaction of relation (20) guarantees that the candidate quadratic Lyapunov function (22) is non-increasing along resets:

$$\Delta V(\tilde{x}) \leq 0 \text{ if } \tilde{x}'\tilde{M}\tilde{x} \leq 0. \quad (26)$$

Hence, in the case $d = 0$, the satisfaction of relations (19) and (20) ensures the global asymptotic stability of reset system (16) or equivalently the asymptotic convergence of the error e to 0.

For $d \neq 0$, the satisfaction of relation (19) and (20) implies, in particularly, that $L \leq 0$ and $\Delta V(\tilde{x}) \leq 0$. Let us introduce the following notation. The value of Lyapunov function (22) and of its time-derivative at $\tilde{x}(t_i, i)$ will be noted $V(t_i, i)$ and $\dot{V}(t, i)$, respectively. By integrating relation (23) along the continuous trajectories of the system, it follows:

$$\sum_{i=0}^{T}\int_{t_i}^{t_{i+1}} \dot{V}(t,i)dt + \gamma^{-1}\left(\int_0^\infty e'edt + \int_0^\infty \dot{e}'\dot{e}dt\right) - \gamma\int_0^\infty d'ddt \leq 0 \ . \quad (27)$$

Moreover, note that one gets

$$\sum_{i=0}^{T}\int_{t_i}^{t_i+1} \dot{V}(t,i)dt = V(t_{T+1}, T+1) - V(t_0, 0) - \sum_{i=0}^{T}\left[V(t_{i+1}, i+1) - V(t_i+1, i)\right] \ . \quad (28)$$

From the satisfaction of (26)-(27) and from (28) one gets:

$$V(t_{T+1}, T+1) - V(t_0, 0) + \gamma^{-1}\left(||e||_2^2 + ||\dot{e}||_2^2\right) - \gamma||d||_2^2 \leq 0$$

or equivalently, with $V(t_{T+1}, T+1) > 0$, it follows

$$||e||_2^2 + ||\dot{e}||_2^2 \leq \gamma V(t_0, 0) + \gamma^2||d||_2^2,$$

and by noting $x(T_0, 0) = x_0$, relation (21) is obtained and the proof of Theorem 1 is complete. ■

In the case where matrix A_F is not Hurwitz (specially if the reset controller is not exponentially stable) or to reduce conservatism, piecewise quadratic Lyapunov functions can be considered as in [14]. Let us emphasize that even in this case, we suppose that A_F is non-singular.

Let us consider (without loss of generality) that the system (16) is in observability canonical form. Choose any $N \geq 2$ and define $\theta_i,\ i = 0, \ldots, N$ such that $0 = \theta_0 < \theta_1 < \ldots < \theta_N = \frac{\pi}{2}$ (for example, in our case we select $\theta_i = \frac{i\pi}{2N}$). Define the angle vectors $\Theta_i \in \mathbb{R}^n$ as

$$\Theta_i = \begin{bmatrix} 0_{1\times(n-2)} & \sin(\theta_i) & \cos(\theta_i) \end{bmatrix}',\ i = 0, \ldots, N.$$

and their orthogonal matrices $\Theta_{i\perp}$ ($\Theta'_{i\perp}\Theta_i = 0$) as:

$$\Theta_{i\perp} = \begin{bmatrix} I_{n-2} & 0 & 0 \\ 0_{n-2} & \cos(\theta_i) & -\sin(\theta_i) \end{bmatrix}', i = 0, \dots, N.$$

Define also the sector matrices $S_i = S_i' \in \mathbb{R}^{n\times n}$:

$$\begin{aligned} S_0 &= Q'\left(\Theta_0\Theta_N' + \Theta_N\Theta_0'\right)Q, \\ S_i &= -Q'\left(\Theta_i\Theta_{i-1}' + \Theta_{i-1}\Theta_i'\right)Q,\ i = 1, \dots, N, \end{aligned}$$

and the angular sectors:

$$\Pi_i = \{x \in \mathbb{R}^n;\ x'S_i x \geq 0\},\ i = 0, \dots, N. \tag{29}$$

Theorem 2: If there exist symmetric positive definite matrices $P_i \in \mathbb{R}^{n\times n}$, $i = 0, \dots, N$ and positive numbers τ, τ_i, ρ_i, $i = 0, \dots, N$, and γ satisfying:

$$\begin{bmatrix} A_F'P_i + P_iA_F' - \tau A_F A_F + \tau_i S_i & \tau A_F' & (P_i - \tau A_F')B_d & C' & 0 \\ \star & -\tau I & \tau B_d & 0 & C' \\ \star & \star & -\tau B_d'B_d - \gamma I & 0 & 0 \\ \star & \star & \star & -\gamma I & 0 \\ \star & \star & \star & \star & -\gamma I \end{bmatrix} \leq 0,$$

$$i = 1, \dots, N, \tag{30}$$

$$A_J'P_0A_J - P_0 + \tau_0 S_0 \leq 0, \tag{31}$$
$$\Theta_{i\perp}'Q'^{-1}(P_i - P_{i+1})Q^{-1}\Theta_{i\perp} = 0,\ i = 1, \dots, N-1 \tag{32}$$
$$\Theta_{0\perp}'Q'^{-1}(P_1 - P_0)Q'^{-1}\Theta_{0\perp} = 0, \tag{33}$$
$$\Theta_{N\perp}'Q'^{-1}(P_N - P_0)Q'^{-1}\Theta_{N\perp} = 0, \tag{34}$$
$$P_i - \rho_i S_i > 0,\ i = 0, \dots, N \tag{35}$$

then

- when $d = 0$, the reset control system (16) is globally asymptotically stable;
- when $d \neq 0$, relation (10) holds with

$$\eta = \gamma\tilde{x}_0'P_i\tilde{x}_0 + \gamma^2||d||_2^2,\ \text{if } \tilde{x}_0 \in \Pi_i, \tag{36}$$

for all disturbances d satisfying (4) and thus, with Lemma 1, $\lim_{t\to\infty} e(t) = 0$.

Proof: The two first equations are obtained from Theorem 1 by patching together $N+1$ quadratic functions ($V(\tilde{x}) = \tilde{x}'P_i\tilde{x}$, $i = 1, \dots, N$ for the flow set and $V(\tilde{x}) = \tilde{x}'P_0\tilde{x}$ for the jump set). Note that with the adopted convention, the jump set $\mathcal{J}$ could be written as $\mathcal{J} := \{\tilde{x} \in \mathbb{R}^n; \tilde{x}'S_0\tilde{x} \geq 0\}$ and by patching together angular sectors Π_i, $i = 1, \dots, N$, we can define similarly the flow set $\mathcal{F}$.

Conditions (32)-(34) ensure that the patched Lyapunov function is continuous on the patching surface. For example, the patching surface at the boundary of flow sets Π_1 and Π_2 is defined as the hyperplane $\{\tilde{x} \in \mathbb{R}^n;\ \Theta_{1\perp}'\tilde{x} = 0\}$.

The patched Lyapunov function $V(\tilde{x}) = \tilde{x}'P_i\tilde{x}$, $i = 0, \dots, N$ must be positive, i.e. $\tilde{x}'P_i\tilde{x} > 0$ if $\tilde{x} \in \Pi_i$. By applying the S-procedure again, we obtain relation (35).

Moreover, in the case $d \neq 0$, the satisfaction of relation (30), for $i = 1, \dots, N$, implies:

$$||e||_2^2 + ||\dot{e}||_2^2 \leq \gamma V(\tilde{x}(0)) + \gamma^2||d||_2^2,$$

where $V(\tilde{x}_0) = \tilde{x}_0'P_i\tilde{x}_0$ if $\tilde{x}_0 \in \Pi_i$, with $\tilde{x}(t_0, 0) = \tilde{x}_0$. ■

Remark 4: Equalities (32)-(34) can be treated as inequalities (see [14]).

Remark 5: The proposed refinement is not unique. If no solutions can be found for Theorem 2 for a given partition, it is natural to refine the partition (by increasing N or by considering another distribution for angles θ_i, for example) and to try again. Hence, the proposed approach increases the flexibility of the candidate Lyapunov function $V(\tilde{x}) = \tilde{x}'P_i\tilde{x}$, $i = 0, \dots, N$, if $x \in \Pi_i$ and conducts to solutions where the standard quadratic LMI conditions failed.

IV. Decreasing reference

In this section we consider system (6) with a decreasing reference $r = r_0e^{-\varepsilon t}$. Hence, by defining the augmented state vector $\overline{x} = [x_p'\ x_r'\ r']' = [x'\ r']' \in \mathbb{R}^{n+1}$, the system (6) reads:

$$\begin{aligned} \dot{\overline{x}} &= A_F\overline{x} + B_d d,\ \text{if } \overline{x} \in \mathcal{F}, \\ \overline{x}^+ &= A_J\overline{x},\ \text{if } \overline{x} \in \mathcal{J}, \end{aligned} \tag{37}$$

and the output is defined by $y = C\overline{x}$ and the error as $e = r - y = Z\overline{x}$ with

$$A_F = \begin{bmatrix} A_p - B_pD_rC_p & B_pC_r & B_pD_r \\ -B_rC_p & A_r & B_r \\ 0 & 0 & -\varepsilon \end{bmatrix}, C = \begin{bmatrix} C_p' \\ 0 \\ 0 \end{bmatrix}',$$

$$A_J = \begin{bmatrix} I_{n_p} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, Z' = \begin{bmatrix} -C_p' \\ 0 \\ 1 \end{bmatrix}. \tag{38}$$

Due to the reference r, the flow and jump sets $\mathcal{F}$ and $\mathcal{J}$ have to be rewritten for $(x, r) \in \mathbb{R}^n \times \mathbb{R}$:

$$\begin{aligned} \mathcal{F} &:= \{x'M_1x + 2x'M_2r + r'M_3r \geq 0\}, \\ \mathcal{J} &:= \{x'M_1x + 2x'M_2r + r'M_3r \leq 0\}, \end{aligned} \tag{39}$$

with

$$M_1 = Q'\mathcal{M}_1Q,\ M_2 = \tfrac{1}{2}\begin{bmatrix} -2C_p' \\ C_r' \end{bmatrix},\ M_3 = D_r, \tag{40}$$

with $\mathcal{M}_1 = \begin{bmatrix} 0 & -I \\ -I & 0 \end{bmatrix}$ and $Q = \begin{bmatrix} C_p & 0 \\ D_rC_p & C_r \end{bmatrix}$. Let us define matrix $\mathbb{M}$

$$\mathbb{M} = \begin{bmatrix} M_1 & M_2 \\ M_2' & M_3 \end{bmatrix}. \tag{41}$$

Theorem 1 can be used to solve Problem 1 for system (37) by considering $P \in \mathbb{R}^{(n+1)\times(n+1)}$ and $\mathbb{M}$ instead of matrix M. If the use of a piecewise quadratic function is needed, principal difficulties reside in the definition of angular sectors Π_i in view of (29). From matrix $\mathbb{M}$ defined in (41) and M_1, M_2, M_3 defined in (40), we let $\mathbb{M}_1 = \begin{bmatrix} M_1 & 0 \\ 0 & 0 \end{bmatrix}$, and $\mathbb{M}_2 = \begin{bmatrix} 0 & M_2 \\ M_2' & M_3 \end{bmatrix}$. Due to the structure of A_F and A_J given in (38), we proposed a relaxation method which will only cover the matrix $\mathbb{M}_1$. Thus, the approach proposed in section III can be extended.

Let us consider (without loss of generality) that the system (16) is in observability canonical form. Choose any $N \geq 2$ and define $\theta_i,\ i = 0, \ldots, N$ such that $0 = \theta_0 < \theta_1 < \ldots < \theta_N = \frac{\pi}{2}$ (for example, in our case we select $\theta_i = \frac{i\pi}{2N}$). Define the angle vectors $\overline{\Theta}_i \in \mathbb{R}^{n+1}$ as

$$\overline{\Theta}_i = \left[\begin{array}{cccc} 0_{1\times(n-2)} & \sin(\theta_i) & \cos(\theta_i) & 0 \end{array}\right]', i = 0, \ldots, N.$$

and their orthogonal matrices $\overline{\Theta}_{i\perp}$ ($\overline{\Theta}_{i\perp}'\overline{\Theta}_i = 0$) as:

$$\overline{\Theta}_{i\perp} = \left[\begin{array}{cccc} I_{n-2} & 0 & 0 & 0 \\ 0_{n-2} & \cos(\theta_i) & -\sin(\theta_i) & 0 \end{array}\right]', i = 0, \ldots, N.$$

Define also the sector matrices $\overline{S}_i = \overline{S}_i' \in \mathbb{R}^{n+1\times n+1}$:

$$\begin{array}{rcl} \overline{S}_0 & = & Q'\left(\overline{\Theta}_0\overline{\Theta}_N' + \overline{\Theta}_N\overline{\Theta}_0'\right)Q \\ \overline{S}_i & = & -Q'\left(\overline{\Theta}_i\overline{\Theta}_{i-1}' + \overline{\Theta}_{i-1}\overline{\Theta}_i'\right)Q,\ i = 1, \ldots, N, \end{array}$$

and the angular sectors:

$$\begin{array}{l} \overline{\Pi}_i = \left\{\overline{x} \in \mathbb{R}^{n+1};\ \overline{x}'\left(\overline{S}_i + \mathbb{M}_2\right)\overline{x} \geq 0\right\},\ i = 1, \ldots, N, \\ \overline{\Pi}_0 = \left\{\overline{x} \in \mathbb{R}^{n+1};\ \overline{x}'\left(\overline{S}_0 - \mathbb{M}_2\right)\overline{x} \geq 0\right\}. \end{array}$$

Theorem 3: If there exist symmetric positive definite matrices $P_i \in \mathbb{R}^{(n+1)\times(n+1)}$, $i = 0, \ldots, N$ and positive numbers τ, τ_i, $i = 0, \ldots, N$, and γ satisfying:

$$\left[\begin{array}{cc} A_F'P_i + P_iA_F - \tau A_F'A_F + \tau_i\left(\overline{S}_i + \mathbb{M}_2\right) & \tau A_F' \\ \star & -\tau I \\ \star & \star \\ \star & \star \end{array}\right.$$

$$\left.\begin{array}{ccc} (P_i - \tau A_F')B_d & Z' & 0 \\ \tau B_d & 0 & Z' \\ -\tau B_d'B_d - \gamma I & 0 & 0 \\ \star & -\gamma I & 0 \\ \star & \star & -\gamma I \end{array}\right] \leq 0,$$

$$i = 1, \ldots, N, \quad (42)$$

$$A_J'P_0A_J - P_0 + \tau_0\left(\overline{S}_0 - \mathbb{M}_2\right) \leq 0, \quad (43)$$
$$\overline{\Theta}_{i\perp}'Q^{-1'}(P_i - P_{i+1})Q^{-1}\overline{\Theta}_{i\perp} = 0\ i = 1, \ldots, N-1, \quad (44)$$
$$\overline{\Theta}_{0\perp}'Q^{-1'}(P_1 - P_0)Q^{-1}\overline{\Theta}_{0\perp} = 0, \quad (45)$$
$$\overline{\Theta}_{N\perp}'Q^{-1'}(P_N - P_0)Q^{-1}\overline{\Theta}_{N\perp} = 0, \quad (46)$$
$$P_i - \rho_i\left(\overline{S}_i + \mathbb{M}_2\right) > 0, i = 1, \ldots N, \quad (47)$$
$$P_0 - \rho_0\left(\overline{S}_0 - \mathbb{M}_2\right) > 0, \quad (48)$$

then

- when $d = 0$, the reset control system (37) is globally asymptotically stable;
- when $d \neq 0$, relation (10) holds with

$$\eta = \gamma\overline{x}_0'P_i\overline{x}_0 + \gamma^2||d||_2^2,\ \text{if } \overline{x}_0 \in \overline{\Pi}_i. \quad (49)$$

for all disturbances d satisfying (4) and, thus with Lemma 1, we have $\lim_{t\to\infty} e(t) = 0$.

Proof: The proof of Theorem 3 follows the lines of Theorem 1. With proposed angular sectors, the jump set $\mathcal{J}$ reads $\mathcal{J} := \{\overline{x} \in \mathbb{R}^{n+1},\ \overline{x}'(\overline{S}_0 - \mathbb{M}_2)\overline{x} \geq 0\}$ and $\mathcal{F}$ is the union of sets defined as $\mathcal{F}_i := \{\overline{x} \in \mathbb{R}^{n+1},\ \overline{x}'(\overline{S}_i + \mathbb{M}_2)\overline{x} \geq 0\}$, $i = 1, \ldots,\ N$. ■

Remark 6: Previous results could adapted if system (2) is defined with different flow and jump sets.

V. Simulation results

In this section, illustrations of previous results are proposed. We implement Theorems 2 and 3 in order to minimize the variable γ. Indeed, consider for example Theorem 3, if solutions exist, one gets

$$||e||_2^2 + ||\dot{e}||_2^2 \leq \gamma\overline{x}_0'P_i\overline{x}_0 + \gamma^2||d||_2^2,\ \text{if } \overline{x}_0 \in \overline{\Pi}_i. \quad (50)$$

Then the gain γ could be seen as a performance index: smaller will be γ, smaller will be the energy of $||e||_2^2 + ||\dot{e}||_2^2$ and better will be the closed-loop performances.

A. Constant reference

To illustrate Theorem 2, we consider the example of a second order system controlled by a FORE presented in [10]. We have $A_p = \left[\begin{array}{cc} 0 & 0 \\ 1 & -0.2 \end{array}\right]$, $B_p = \left[\begin{array}{c} 1 \\ 1 \end{array}\right]$, $C_p = \left[\begin{array}{c} 0 \\ 1 \end{array}\right]'$ and the controller FORE is characterized by $A_r = \lambda_r, B_r = 1, C_r = 1$. Figure 2 illustrates that the gain γ of

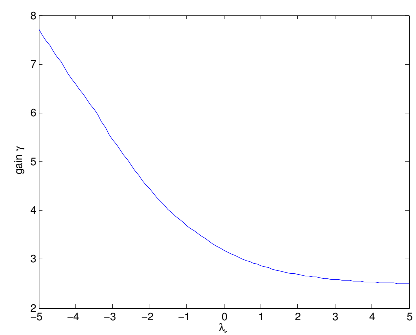

Fig. 2. Gain γ as a function of the pole of the FORE λ_r

the system decreases when the pole λ_r increases and due to the reset action, we can consider positive pole for the controller (which leads to unstable linear-based closed-loop). This theoretical behavior is confirmed by time responses presented in Figure 3, where several responses to a unit step reference are reported. We also considered, at $t = 25s$ a constant perturbation during $2s$ with an amplitude of 0.3. As expected, the closed-loop system has better performances when λ_r is increasing (better overshoot, time rising and less sensitive to the perturbation). This is depicted in Figure 3.

B. Slowly decreasing reference

To illustrate Theorem 3, we choose, as in [12], a reference slowly decreasing $r = e^{-0.01t}$.

Consider the mechanical mode of a DC motor modeled by the following transfer function:

$$F(s) = \frac{k_m}{s(1 + t_m s)}. \quad (51)$$

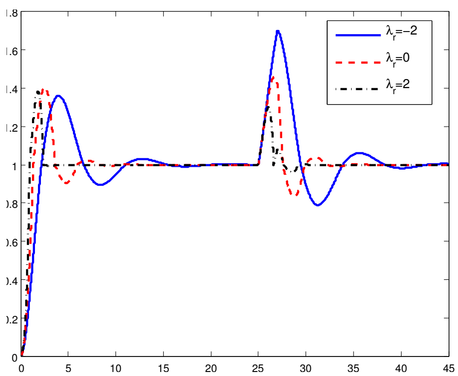

Fig. 3. Step response with perturbation between $t = 25s$ and $t = 27s$ with an amplitude of 0.3

The following controller has been designed to ameliorate the margin phase of (51):

$$A_r = -0.5,\ B_r = 0.1675,\ C_r = 1,\ D_r = 0.1675. \quad (52)$$

By applying Theorem 3, we obtain the following results (see Table I, where N is the number of subregions). We also

	without reset	$N = 2$	$N = 10$	$N = 20$	$N = 30$
γ	16.1447	16.1437	14.23	13.46	13.3

TABLE I

GAIN γ EVALUATION

consider an implementation error. Indeed, we suppose that B_r in (52) is replaced by $B_r = -0.1675$. With a classical linear-based controller, the closed-loop becomes unstable. By exploiting reset advantages, the closed-loop remains stable and with Theorem 3 one gets $\gamma = 20.60$. Figure 4 reports these remarks: performances are better by considering a reset controller. If this controller is uncorrectly implemented, the system output still converges to the reference but is more sensitive to external perturbation.

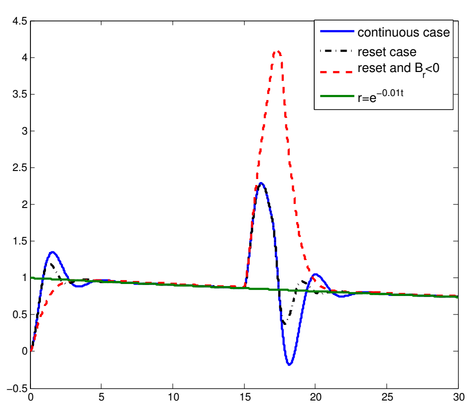

Fig. 4. Temporal response with decreasing reference and a perturbation at $t = 15s$ until $t = 17s$ with an amplitude of 0.3

VI. Conclusion

Given a general reset controller, we studied the stability and tracking problem of the closed-loop system in presence of an external perturbation. For different classes of references, LMI-based conditions are obtained. Numerical examples are presented to illustrate the results and to show improvements that a reset controller could lead to. If a general model for reset controller is considered, the resetting condition is still restrictive and the design of reset conditions should also be investigated. The evaluation of the results on an experimental setup will also be considered in a next future.

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