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$\mathcal{H}_\infty$ and Dwell Time Specifications of Switched Linear Systems

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***Abstract*— This paper deals with continuous-time systems and addresses the problem of characterizing open-loop switching strategies, based on dwell time specification, assuring a pre-specified root mean square gain (RMS). As a natural consequence of treating general systems of this particular class in terms of the order and the number of subsystems, only sufficient conditions are worked out. However, as positive features, they are expressed through linear matrix inequalities (LMI) being thus numerically solvable in polynomial time and allow the treatment of stable switched linear systems which do not admit a common Lyapunov function. The conservativeness of the proposed design method is evaluated by comparison with an easy to calculate lower bound on the minimum dwell time that assures the specified $\mathcal{H}_\infty$ level. An example is included for illustration.**

I. INTRODUCTION

Hybrid and switched dynamic systems have received a great deal of attention in the last decades. The stability analysis of continuous-time switched linear systems have been addressed by many authors, [4], [8], [10], [11], [13], [21] and [23]. General results on this topic are presented in the book [1] and in the survey paper [19]. More specifically, in reference [10] the interested reader can find an interesting discussion on a collection of results on uniform stability of switched systems. The reader is also requested to see [6], [15] and [16] for a rather complete review on stability of continuous-time switched linear systems, where special attention is given to the case of switching between two linear systems. For control synthesis see [12], [17] and [20]. In this paper, the stability conditions for continuous-time linear switched systems provided in [7] are used. In fact, they make possible the determination of an upper bound of the minimum dwell time that preserves stability and are expressed in terms of linear matrix inequalities - LMI plus a scalar variable, being thus solvable in polynomial time (see [3]) by the machinery available in the literature to date.

This paper deals with RMS gain and dwell time specifications for continuous-time switched linear systems of the following form

$$\dot{x} = A_\sigma x + B_\sigma w, \ x(0) = 0 \quad (1)$$
$$y = C_\sigma x + D_\sigma w \quad (2)$$

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defined for all $t \geq 0$ where $x(t) \in \mathbb{R}^n$ is the state, $w(t) \in \mathbb{R}^m$ is the exogenous input and $\sigma(t) : t \geq 0 \to \{1, \cdots, N\}$ is the switching rule. Calling $\mathcal{D}_T$ the set of all switching policies with dwell time greater or equal to T, given a pair of nonnegative real numbers (T, γ) the main goal is to provide conditions to assure that

$$J(\sigma) := \sup_{w \in \mathcal{L}_2} \int_0^\infty (y'y - \gamma^2 w'w) dt \leq 0, \ \forall \ \sigma \in \mathcal{D}_T \quad (3)$$

It is important to recognize that the complete solution to this problem is extremely difficult to obtain and, to our best knowledge only few references are available up to now in the literature where only particular problems of this class are treated, considering a small number ($N = 2$) of reduced order subsystems ($n = 1$). One of the key references on this problem is [9] where the problem (3) with $T = 0$ is solved. In fact in [9], necessary and sufficient conditions are given to ensure that $J(\sigma)$ is nonnegative for arbitrary switching rules $\sigma(t) \in \{1, \cdots, N\}$ defined for $t \geq 0$.

Certainly, the main difficulty to be circumvented stems from the fact that the global solution to the optimal control problem

$$\sup_{\sigma \in \mathcal{D}_T} J(\sigma) = \sup_{w \in \mathcal{L}_2, \sigma \in \mathcal{D}_T} \int_0^\infty (y'y - \gamma^2 w'w) dt \quad (4)$$

although completely characterized through the associated Hamilton-Jacobi-Bellman equation can not be obtained in this way since it is virtually impossible to solve due to the algebraic structure of the set $\mathcal{D}_T$ for any given dwell time $T \geq 0$, [14], [18]. Hence, for the moment, suboptimal solutions easier to calculate are acceptable. This is the main focus of this paper to be developed afterwards. The conservativeness of the proposed conditions to assure the validity of (3) will be tested through comparisons with known results provided by the classical $\mathcal{H}_\infty$ Theory as well as the imposition that (3) has to hold for a subset of $\mathcal{D}_T$ composed by all periodic policies with period $T > 0$ which naturally generates a necessary condition for its validity.

The notation used throughout is standard. Capital letters denote matrices, small letters denote vectors and small Greek letters denote scalars. For matrices or vectors $(')$ indicates transpose. For symmetric matrices, $X > 0$ (≥ 0) indicates that X is positive definite (nonnegative definite). The sets of real and natural numbers are denoted by $\mathbb{R}$ and $\mathbb{N}$ respectively and $\mathbb{K} = \{1, \cdots, N\}$. The squared norm of a trajectory $z(t)$ defined for all $t \geq 0$ equals $\|z(t)\|_2^2 = \int_0^\infty z(t)'z(t)dt$. All trajectories such that $\|z(t)\|_2^2 < \infty$ characterize the set $\mathcal{L}_2$, see [5].

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II. Preliminaries

To be used in the sequel, this section is entirely dedicated to analyze some properties of the finite horizon $\mathcal{H}_\infty$ problem stated as follows

$$V(\xi,\tau) = \sup_{w\in\mathcal{L}_2} \int_0^\tau (y'y - \gamma^2 w'w)dt \tag{5}$$

subject to

$$\begin{aligned} \dot{x} &= Ax + Bw,\ x(0) = \xi & (6) \\ y &= Cx + Dw & (7) \end{aligned}$$

where we notice that all matrices of the state space realization with compatible dimensions, the initial state and the final time τ are given. Obviously, since the system (6)-(7) is time invariant the initial time considered at $t = 0$ can be moved forward, in which occurrence the optimal solution does not change provided the final time is also moved forward accordingly. Under standard assumptions on the system, namely stability and minimality, this problem can be solved with no difficulty (see, for instance, [5]) and its optimal solution provides the cost-to-go function

$$V(\xi,\tau) = \xi' P(0)\xi \tag{8}$$

where $P(t) = P(t)' \in \mathbb{R}^{n\times n}$, defined for all $t \in [0,\ \tau]$ is the unique positive definite solution to the differential Riccati equation

$$\begin{aligned} -\dot{P} &= A'P + PA + C'C + \\ &+ (PB + C'D)(\gamma^2 I - D'D)^{-1}(PB + C'D)' \end{aligned} \tag{9}$$

subject to the final condition $P(\tau) = 0$. Thanks to the assumptions, this equation admits a solution in the time interval $t \in [0,\tau]$ for $\tau < \infty$ whenever $\gamma^2 I - D'D > 0$. For a thorough treatment on the existence of the solution of the $\mathcal{H}_\infty$ differential Riccati equation as a function of the final time, the interested reader is referred to [2] . As for $\tau = +\infty$, under the same assumptions, the existence of a positive definite stabilizing solution of the associated algebraic Riccati equation is in one-to-one correspondence to the contractivity of the open loop transfer function $G(s) = C(sI - A)^{-1}B + D$, i.e. $\|G(s)\|_\infty^2 < \gamma^2$. Moreover, it is interesting to remark that the optimal input (worst input trajectory) is given by $w_\tau(t) = L(t)x_\tau(t)$ where

$$L(t) = (\gamma^2 I - D'D)^{-1}(P(t)B + C'D)' \tag{10}$$

is the time varying feedback gain and $x_\tau(t)$ is the corresponding state trajectory. Hence, if $\xi = 0$ then $V(0,\tau) = 0$ and $w_\tau(t) = 0$ for all $t \in [0,\tau]$. In other words, under zero initial conditions, the null trajectory is the worst input trajectory.

Lemma 1: For every fixed initial condition $\xi \in \mathbb{R}^n$, the cost-to-go function $V(\xi,\tau)$ is nondecreasing in the interval $\tau \in (0,\ +\infty)$.

Proof: Consider $\delta > \tau > 0$ and the input trajectory

$$w(t) = \begin{cases} w_\tau(t) & t \in [0,\tau] \\ 0 & t \in (\tau,\delta] \end{cases} \tag{11}$$

the cost-to-go function as defined in (5) allows us to write

$$\begin{aligned} V(\xi,\delta) &= \sup_{w\in\mathcal{L}_2} \int_0^\delta (y'y - \gamma^2 w'w)dt \\ &\geq \int_0^\tau (y_\tau' y_\tau - \gamma^2 w_\tau' w_\tau)dt + \int_\tau^\delta y'ydt \\ &\geq V(\xi,\tau) + \int_\tau^\delta y'ydt \\ &\geq V(\xi,\tau) \end{aligned} \tag{12}$$

from which the claim follows. ■

This lemma puts in evidence a property of the cost-to-go function, based on which the upper bound

$$V(\xi,\tau) \leq V(\xi,+\infty) = \xi' P\xi, \forall\ \tau > 0 \tag{13}$$

holds where, with a little abuse of notation, $P = P' \in \mathbb{R}^{n\times n}$ denotes the positive definite stabilizing state steady solution ($\dot{P}(t) = 0, \forall t \geq 0$) of the corresponding algebraic Riccati equation derived from (9), that is

$$\begin{aligned} 0 &= A'P + PA + C'C + \\ &+ (PB + C'D)(\gamma^2 I - D'D)^{-1}(PB + C'D)' \end{aligned} \tag{14}$$

It is important to remark that, in this case, the worst input is generated by state feedback with the stationary gain $L = (\gamma^2 I - D'D)^{-1}(PB + C'D)'$. All these results will be used in the sequel to approach the $\mathcal{H}_\infty$ control of switched linear systems previously defined. However, for the moment, let us restate a result of [7] related to the stability of switched linear systems of the form

$$\dot{x} = A_\sigma x \tag{15}$$

where it is assumed that each matrix of the set $\{A_1, \cdots, A_N\}$ is asymptotically stable and

$$\sigma \in \mathcal{D}_T = \{\sigma(t) = i \in \mathbb{K},\ \forall t \in [t_k, t_{k+1})\} \tag{16}$$

where t_k and t_{k+1} are successive switching times satisfying $t_{k+1} - t_k \geq T$ for all $k \in \mathbb{N}$ and the index $i \in \mathbb{K}$ selected at each instant of time $t_k \geq 0$ is arbitrary.

Lemma 2: Assume that, for some $T \geq 0$, there exists a collection of positive definite matrices $\{X_1, \cdots, X_N\}$ of compatible dimensions such that

$$A_i' X_i + X_i A_i < 0,\ \forall\ i \in \mathbb{K} \tag{17}$$

and

$$e^{A_i' T} X_j e^{A_i T} - X_i < 0,\ \forall\ i \neq j \in \mathbb{K} \tag{18}$$

The equilibrium solution $x = 0$ of the switched linear system (15) is globally asymptotically stable for all $\sigma \in \mathcal{D}_T$.

This lemma, to be used afterwards provides a stability condition with two important features. First, it can be handled by any LMI solver whenever $T \geq 0$ is fixed. Second, using a line search procedure, there is no difficulty to calculate the minimum value of $T \geq 0$ preserving feasibility. Clearly, since the stability condition provided by Lemma 2 is only sufficient, the minimization of T gives an upper bound to the true minimum dwell time T_*.

On the other hand, Lemma 2 encompasses also the case of quadratically stable systems. Indeed, if feasibility of (17) and (18) hold for $T \to 0$, then, in the limit, $X_i \to X$, where $X > 0$ satisfies $A_i'X + XA_i < 0$, for each $i \in \mathbb{K}$. This is a well known necessary and sufficient condition for quadratic stability and a sufficient condition for stability under arbitrary switching signals. It is well known that the existence of a common Lyapunov function is a very conservative condition for stability under arbitrary switching.

Now, the question arises of assessing a guaranteed RMS gain under an arbitrary switching law. This problem can be easily handled via LMI. Indeed, consider again the switched linear system

$$\begin{aligned} \dot{x} &= A_\sigma x + B_\sigma w \ , \ x(0) = 0 \qquad (19) \\ y &= C_\sigma x + D_\sigma w \qquad (20) \end{aligned}$$

For simplicity we make the assumption that all linear systems represented by (A_i, B_i, C_i, D_i), $i \in \mathbb{K}$, besides being stable, are in minimal form. Let $\gamma > 0$ such that

$$\gamma^2 > \max_{i \in \mathbb{K}} \| C_i(sI - A_i)^{-1} B_i + D_i \|_\infty^2 := \gamma_c^2$$

Theorem 1: Assume that there exists a positive definite matrix P such that

$$\begin{bmatrix} A_i'P + PA_i & PB_i & C_i' \\ B_i'P & -\gamma^2 I & D_i' \\ C_i & D_i & -I \end{bmatrix} < 0 \ , \ \forall \ i \in \mathbb{K} \qquad (21)$$

then, for each switching signal $\sigma \in \mathbb{K}$, the equilibrium solution $x = 0$ of the switched linear system (19)-(20) is globally asymptotically stable and

$$\sup_{w \in \mathcal{L}_2, w \neq 0} \int_0^\infty (y'y - \gamma^2 w'w) dt < 0 \qquad (22)$$

Proof: First of all notice that (21) is equivalent to $\gamma^2 I - D_i'D_i > 0$ and

$$\begin{aligned} 0 &> A_i'P + PA_i + C_i'C_i + \\ &+ (PB_i + C_i'D_i)(\gamma^2 I - D_i'D_i)^{-1}(PB_i + C_i'D_i)' \end{aligned}$$

for all $i \in \mathbb{K}$. In particular

$$A_i'P + PA_i < 0 \ , \ \forall \ i \in \mathbb{K}$$

so that global asymptotic stability under arbitrary switching is ensured. Also, the state of the system goes to zero for each σ and each input square integrable disturbance w. This means that, taking $V(x) = x'Px$, we have $V(x(\infty)) = 0$.

Now, compute the derivative of $V(x)$ along the trajectories of (19) and (20). Letting

$$w^* = (\gamma^2 I - D_\sigma'D_\sigma)^{-1}(P_\sigma B_\sigma + C_\sigma'D_\sigma)'x$$

from the above matrix inequality it turns out that

$$\begin{aligned} \dot{V}(x) &= x'(A_\sigma'P + PA_\sigma)x + 2x'PB_\sigma w \\ &< -y'y + \gamma^2 w'w - \\ &\quad -(w - w^*)'(\gamma^2 I - D_\sigma'D_\sigma)(w - w^*) \\ &< -y'y + \gamma^2 w'w \end{aligned}$$

Integrating from 0 to $+\infty$ and recalling that $V(x(0)) = V(x(\infty)) = 0$ it follows that

$$\int_0^\infty (y'y - \gamma^2 w'w) dt < 0$$

for all $\sigma \in \mathbb{K}$ and for all $w \neq 0 \in \mathcal{L}_2$. ■

Consider now inequality (21). Taking λ_i, $i \in \mathbb{K}$ in the unitary simplex, i.e. $\lambda_i \geq 0$ and $\sum_{i \in \mathbb{K}} \lambda_i = 1$, one can multiply (21) by λ_i, sum up and use the Schur Complement to obtain

$$\begin{aligned} 0 &> A_\lambda'P + PA_\lambda + C_\lambda'C_\lambda + \\ &+ (PB_\lambda + C_\lambda'D_\lambda)(\gamma^2 I - D_\lambda'D_\lambda)^{-1}(PB_\lambda + C_\lambda'D_\lambda)' \end{aligned}$$

where

$$\begin{bmatrix} A_\lambda & B_\lambda \\ C_\lambda & D_\lambda \end{bmatrix} = \sum_{i \in \mathbb{K}} \lambda_i \begin{bmatrix} A_i & B_i \\ C_i & D_i \end{bmatrix}$$

This means that the polytopic system defined by A_λ, B_λ, C_λ, D_λ has $\mathcal{H}_\infty$ norm less than γ for each choice of λ in the unitary simplex. In conclusion, $\mathcal{H}_\infty$ performances of switched systems under arbitrary switching laws are related to those of polytopic systems. This fact extends a well know result for stability under arbitrary switching, for which quadratic stability is only a conservative sufficient condition. For a thorough discussion on nonconservative solution via polyhedral Lyapunov function, the interested reader is referred to the recent volume [1].

III. $\mathcal{H}_\infty$ AND DWELL TIME SPECIFICATIONS

In this section we turn our attention to the switched linear system

$$\begin{aligned} \dot{x} &= A_\sigma x + B_\sigma w \ , \ x(0) = \xi \qquad (23) \\ y &= C_\sigma x + D_\sigma w \qquad (24) \end{aligned}$$

where the initial condition $\xi \in \mathbb{R}^n$ is arbitrary but fixed. Again we make the assumption that all linear systems (A_i, B_i, C_i, D_i), $i \in \mathbb{K}$ are stable and in minimal form. Moreover, we again assume that $\gamma > 0$ is such that $\gamma > \gamma_c$. Hence, it is well known, see e.g. [5], that $\|D_i\| < \gamma$ and there exist positive definite and stabilizing solutions P_i, $i \in \mathbb{K}$ of the algebraic Riccati equations

$$\begin{aligned} 0 &= A_i'P_i + P_iA_i + C_i'C_i + \\ &+ (P_iB_i + C_i'D_i)(\gamma^2 I - D_i'D_i)^{-1}(P_iB_i + C_i'D_i)' \end{aligned}$$

In this section a suboptimal solution to the $\mathcal{H}_\infty$ problem (3) is provided. To this end, we need to introduce the following matrices

$$\begin{aligned} H_i &= A_i + B_iL_i \qquad (25) \\ Q_i &= (C_i + D_iL_i)'(C_i + D_iL_i) - \gamma^2 L_i'L_i \qquad (26) \\ L_i &= (\gamma^2 I - D_i'D_i)^{-1}(P_iB_i + C_i'D_i)' \qquad (27) \end{aligned}$$

and we left the reader to verify that the above algebraic Riccati equations can be factorized as

$$H_i'P_i + P_iH_i + Q_i = 0 \qquad (28)$$

for all $i \in \mathbb{K}$. As indicated before, noticing that the optimal gain L_i is determined from the unique stabilizing solution to the algebraic Riccati equation (28), matrix H_i is Hurwitz for each $i \in \mathbb{K}$. However, since matrix Q_i for each $i \in \mathbb{K}$ is not positive definite, the stabilizing solution of the Riccati equation is not a Lypunov matrix associated to the closed loop system, a well known fact in $\mathcal{H}_\infty$ theory.

The next lemma is of key importance since it gives an upper bound to the $\mathcal{H}_\infty$ cost appearing in the left hand side of (3).

Lemma 3: For the switched linear system (23)-(24), the following upper bound holds

$$J(\sigma) \leq \sum_{k=0}^{\infty} x(t_k)' P_{\sigma(t_k)} x(t_k), \ \forall \ \sigma \in \mathcal{D}_T \tag{29}$$

where P_i for each $i \in \mathbb{K}$ is the stabilizing positive definite solution to the algebraic Riccati equation (28).

Proof: Taking into account that jumps on $\sigma \in \mathcal{D}_T$ occur at time t_k for $k \geq 0$, we have

$$\begin{aligned} J(\sigma) &= \sup_{w \in \mathcal{L}_2} \int_0^\infty (y'y - \gamma^2 w'w) dt \\ &\leq \sum_{k=0}^{\infty} \sup_{w \in \mathcal{L}_2} \int_{t_k}^{t_{k+1}} (y'y - \gamma^2 w'w) dt \\ &\leq \sum_{k=0}^{\infty} \sup_{w \in \mathcal{L}_2} \int_{t_k}^{\infty} (y'y - \gamma^2 w'w) dt \\ &\leq \sum_{k=0}^{\infty} x(t_k)' P_{\sigma(t_k)} x(t_k) \end{aligned} \tag{30}$$

where the first inequality follows from an obvious property of the $\sup\{\cdot\}$ operator and that for all $t \in [t_k, t_{k+1}]$ the switching rule $\sigma(t_k) \in \mathbb{K}$ remains unchanged. The second inequality is a direct consequence of Lemma 1 and finally the third inequality is provided by the cost-to-go function. This proves the proposed lemma. ∎

From the proof of Lemma 3 it is clear that two different sources of conservatism have been introduced. One is due to the fact that we have replaced the supremum with respect to the input $w \in \mathcal{L}_2$ on the entire time interval $[0, +\infty)$ by the supremum with respect to independent inputs on successive time intervals $[t_k, t_{k+1})$ for all $k \geq 0$. The second is an immediate consequence of Lemma 1. Following the same reasoning adopted in [7] the last source of conservativeness can be significatively attenuated by using the time varying solution of the Riccati equation in a finite time interval such that any $\sigma \in \mathcal{D}_T$ further satisfies the additional constraint $t_{k+1} - t_k \leq \mathcal{T}$ for all $k \geq 0$, for some $\mathcal{T} \geq T \geq 0$ given. The next theorem states the main result of this paper.

Theorem 2: Given $T > 0$. Assume that there exists a collection of positive definite matrices $\{Z_1, \cdots, Z_N\}$ of compatible dimensions such that

$$H_i' Z_i + Z_i H_i + Q_i < 0 \ , \ \forall \ i \in \mathbb{K} \tag{31}$$

and

$$e^{H_i' T} Z_j e^{H_i T} - Z_i + P_i < 0, \ \forall \ i \neq j \in \mathbb{K} \tag{32}$$

The following hold:

a) The equilibrium solution $x = 0$ of the switched linear system (23)-(24) is globally asymptotically stable.
b) Any trajectory of the switched linear system (23)-(24) with zero initial condition satisfies

$$J(\sigma) < 0, \ \forall \ \sigma \in \mathcal{D}_T \tag{33}$$

Proof: For any $\sigma \in \mathcal{D}_T$, between two successive switching times the worst input is applied. Consequently, the closed loop switched system obeys the differential equation $\dot{x} = H_\sigma x$. From (31) and (28) it is seen that the matrix $X_i := Z_i - P_i > 0$ and satisfies

$$H_i' X_i + X_i H_i < 0 \ , \ \forall \ i \in \mathbb{K} \tag{34}$$

On the other hand, using (32) we obtain

$$\begin{aligned} e^{H_i' T} X_j e^{H_i T} - X_i &< -e^{H_i' T} P_j e^{H_i T} \\ &< 0, \ \forall \ i \neq j \in \mathbb{K} \end{aligned} \tag{35}$$

Hence, part a) follows from Lemma 2. To prove part b), we first assume that the initial condition $x(0) = \xi \in \mathbb{R}^n$ is arbitrary. Since by assumption $\sigma \in \mathcal{D}_T$, consider that $\sigma(t) = i \in \mathbb{K}$ for all $t \in [t_k, t_{k+1})$ where $t_{k+1} = t_k + T_k$ with $T_k \geq T > 0$ and that at $t = t_{k+1}$ the switching policy jumps to $\sigma(t) = j \in \mathbb{K}$. Moreover, since (34) implies that for each $i \in \mathbb{K}$ the inequality $e^{H_i' \tau}(Z_i - P_i) e^{H_i \tau} \leq (Z_i - P_i)$ holds for any $\tau \geq 0$, with $v(x(t)) = x(t)' Z_{\sigma(t)} x(t)$ inequalities (35) yield

$$\begin{aligned} v(x(t_{k+1})) &= x(t_{k+1})' Z_j x(t_{k+1}) \\ &< x(t_k)' e^{H_i'(T_k - T)} (Z_i - P_i) e^{H_i (T_k - T)} x(t_k) \\ &< x(t_k)' (Z_i - P_i) x(t_k) \\ &< v(x(t_k)) - x(t_k)' P_{\sigma(t_k)} x(t_k) \end{aligned} \tag{36}$$

which summing up for all $k \geq 0$ and remembering that the switched linear system under consideration is asymptotically stable, we obtain

$$\begin{aligned} \sum_{k=0}^{\infty} x(t_k)' P_{\sigma(t_k)} x(t_k) &< v(x(0)) \\ &< \xi' Z_{\sigma(0)} \xi \\ &< \max_{i \in \mathbb{K}} \xi' Z_i \xi \end{aligned} \tag{37}$$

applying Lemma 3 and making $\xi \to 0$, the claim follows. ∎

The above result deserves a few comments. First, as apparent from (31) and (32), feasibility is always met with for large values of T. Indeed, for $T \to \infty$, the inequalities are satisfied thanks to the fact that $\gamma > \gamma_c$ and $Z_i - P_i > 0$. Moreover, for $\gamma \to \infty$, it follows that

$$\begin{aligned} &Q_i \to C_i' C_i \\ &H_i \to A_i \\ &P_i \to \int_0^\infty e^{A_i' t} C_i' C_i e^{A_i t} dt \end{aligned}$$

so that conditions (31) and (32) of Theorem 2 boil down to those in [7] for the $\mathcal{H}_2$ cost.

It is important to stress that the result of Theorem 2 is based on sufficient conditions and upper bounds determination. Hence, it may be conservative. However, to our best knowledge, this is the first result available in the literature to date capable to deal with switched linear systems of arbitrary dimension n and composed by an arbitrary number of subsystems N. In the next section the conservativeness of the proposed result is evaluated by means of a simple example.

IV. Numerical Issues

In this section we put in evidence some relevant points concerning the result of Theorem 2. First of all, as already mentioned, for $\gamma^2 = +\infty$ it reduces to that of [7] for all $\sigma \in \mathcal{D}_T$ such that $t_{k+1} - t_k \leq \mathcal{T}$ for all $k \geq 0$ and $\mathcal{T} = +\infty$. That is, no upper bound on the dwell time is provided by the designer.

Let us now introduce the following function

$$T(\gamma) = \min_{T>0, Z_1>0, \cdots, Z_N>0} \{T \ : \ (31) - (32)\} \tag{38}$$

which for $\gamma > 0$ fixed, can be calculated as follows:

a) For each subsystem $i \in \mathbb{K}$, the algebraic Riccati equation

$$\begin{aligned} 0 &= A_i'P_i + P_iA_i + C_i'C_i + \\ &+ (P_iB_i + C_i'D_i)(\gamma^2 I - D_i'D_i)^{-1}(P_iB_i + C_i'D_i)' \end{aligned}$$

is solved and matrices H_i, Q_i are determined.

b) The optimal solution of the problem appearing in the right hand side of (38) is solved by any LMI solver and line search.

From (38), a relevant issue is the domain of the function $T(\gamma)$, that is the values of the parameter $\gamma > 0$ assuring that $T(\gamma)$ exists or, in other words the values of $\gamma > 0$ for which the constraints of problem (38) are feasible. Fortunately, the answer to this question is that feasibility of the constraints of problem (38) is assured for all $\gamma > 0$ such that the algebraic Riccati equations (28) admit positive definite stabilizing solutions. Actually, in the affirmative case, considering T big enough inequalities (31)-(32) are satisfied for Z_i closed enough to P_i for all $i \in \mathbb{K}$. Hence, $T(\gamma)$ defined in (38) can be determined for all γ satisfying

$$\gamma > \max_{i \in \mathbb{K}} \|C_i(sI - A_i)^{-1}B_i + D_i\|_\infty = \gamma_c \tag{39}$$

It is important to keep in mind that problem (38) remains solvable even when γ is set arbitrarily close to γ_c. On the other hand, since for any fixed dwell time the switching rule $\sigma(t) = i \in \mathbb{K}$ for all $t \geq 0$ belongs to $\mathcal{D}_T$ then feasibility of inequality (3) also requires (39) to hold. This allows the conclusion that the conditions provided by Theorem 2 are not conservative in the present case.

Following Theorem 1, if there exists a positive definite matrix $P = P' > 0$ satisfying the linear matrix inequalities (21) then (3) holds for all $T \geq 0$. That is, the $\mathcal{H}_\infty$ condition holds for all policies $\sigma(t) \in \mathbb{K}$ with possibly arbitrary fast switching. In the affirmative case, (3) remains feasible for all $\gamma > 0$ satisfying

$$\gamma > \min_{\gamma>0, P>0} \{\gamma \ : \ (21)\} := \gamma_d \tag{40}$$

The fact that $\gamma_d \geq \gamma_c$ is an obvious consequence that in the last case all possible switching rules are taken under consideration which is not true for the former one since the minimum dwell time $T(\gamma)$ must be satisfied for all switching policy belonging to $\mathcal{D}_{T(\gamma)}$. Only a special (and small) class of switched linear systems admits a common positive definite solution to the inequalities (21). However, even for systems of this class and $\gamma > \gamma_d$ we can not obtain $T(\gamma) = 0$ as a solution to the problem (38). This fact can be verified with no difficulty since inequality (32) is infeasible for $T = 0$. This is true because for $T = 0$ the switching can be arbitrarily fast and each switching causes a positive increment on $\mathcal{H}_\infty$ cost, making it unbounded. This puts in evidence the importance of considering switching policies with minimum and maximum dwell time as done in [7], see also [22]. This aspect goes beyond the scope of this paper and so is left to future research.

A. Example

For illustration purpose of the theoretical results obtained so far, let us consider the following example with $N = 2$ already analyzed in [7] for dwell time calculations. The matrices of the switching system (1)-(2) are given by

$$\left[\begin{array}{c|c} A_1 & B_1 \\ \hline C_1 & D_1 \end{array}\right] = \left[\begin{array}{cc|c} 0 & 1 & 0 \\ -10 & -1 & 1 \\ \hline 0.8715 & 0 & -0.8715 \end{array}\right] \tag{41}$$

$$\left[\begin{array}{c|c} A_2 & B_2 \\ \hline C_2 & D_2 \end{array}\right] = \left[\begin{array}{cc|c} 0 & 1 & 0 \\ -0.1 & -0.5 & 1 \\ \hline 0 & 0.3350 & 0.3350 \end{array}\right] \tag{42}$$

and it is important to mention that they are not open loop quadratically stable, in which case the value of γ_d can not be calculated. The output matrices have been determined in such a way that each transfer function has an unitary $\mathcal{H}_\infty$ norm, yielding $\gamma_c = 1$.

Moreover, with $T > 0$ fixed it is always possible to define a time-switching control strategy $\sigma \in \mathcal{D}_T$ such that $H_{\sigma(t)}$ is periodic. As a consequence, a necessary condition for the feasibility of constraints (31) and (32) is

$$\theta(T) = \max_{q=1,\cdots,n} \left| \lambda_q \left(\prod_{p=1}^{N} e^{E_p T} \right) \right| < 1 \tag{43}$$

where $\lambda_q(\cdot)$ denotes a generic eigenvalue of $(\cdot)$ and $\{E_1, \cdots, E_N\}$ are matrices corresponding to any permutation among those of the set $\{H_1, \cdots, H_N\}$. However, since the conditions of Theorem 2 take into account non-periodic policies as well, the necessary condition (43) for the existence of a feasible solution to inequalities (31)-(32), generally does not meet sufficiency. Hence a relevant function to be determined, based on this necessary condition is

$$T_p(\gamma) = \max_{T>0} \{T \ : \ \theta(T) = 1\} \tag{44}$$

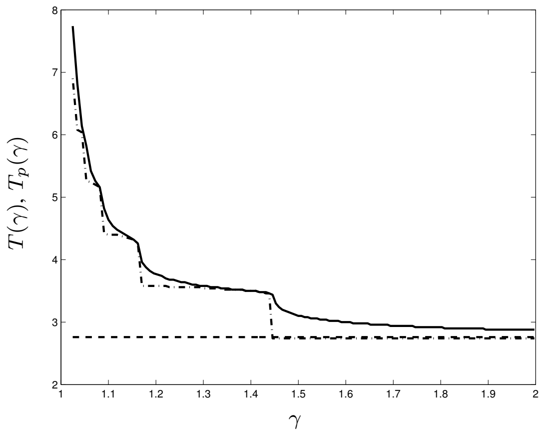

Fig. 1. The functions $T(\gamma)$ and $T_p(\gamma)$

Figure 1 shows in solid line the function $T(\gamma)$, in dashdot line the function $T_p(\gamma)$ against $\gamma \in (1,\ 2]$ and in dashed line the value of $T(\infty)$ which is in accordance to the fact that, for this particular example, the minimum dwell time preserving asymptotical stability belongs to the interval $T_* \in [2.71,\ 2.76]$, for more details see [7]. From this figure it is also confirmed that $T_p(\gamma) \leq T(\gamma)$ for all $\gamma > \gamma_c$ and that both are decreasing functions. The consequence is that the minimum dwell time is associated to $\gamma = +\infty$. This is an expected behavior of the function $T(\gamma)$ since for smaller values of γ, bounded bellow by γ_c, the switched linear system must support richer switching rules without loosing stability. This is compensated by the increasing of the corresponding dwell time $T(\gamma)$. Figure 1 also puts in evidence the good concordance between the functions $T(\gamma)$ obtained from a sufficient condition assuring inequality (3) and $T_p(\gamma)$ obtained from a necessary condition assuring the same inequality. Although mentioned before, this aspect could be improved but, in our opinion, the results reported in this simple example are precise enough to classify the proposed method as a valid procedure for $\mathcal{H}_\infty$ and dwell time specifications.

V. Conclusion

In this paper we have proposed a new design procedure for $\mathcal{H}_\infty$ and dwell time specifications for switched linear systems. It was possible to address the $\mathcal{H}_\infty$ control problem of this class of dynamic systems in a general framework in order to design a minimum dwell time assuring a pre-specified RMS gain.

Although some conservativeness have been introduced in order to obtain solvable problems, the final design method appears to provide good and precise control laws as observed in an academical example. Of course, further research efforts are needed towards its final validation. Special attention was devoted to the numerical solvability of the design problems by means of methods based on linear matrix inequalities and line search.

Several issues raised are left to further research. In our opinion, the most important one is related to the solution of the time varying Riccati equation associated to the $\mathcal{H}_\infty$ control problem with finite time horizon. This feature may have a decisive impact on the determination of more precise dwell time based policies assuring a pre-specified RMS gain level. Taking into account the time varying nature of the involved switching conditions, this point constitutes a real theoretical challenge.

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