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Observability of Vortex Flows

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***Abstract*— We study the observability of one and two point vortex flow from one or two Eulerian or Lagrangian observations. By observability we mean the ability to determine the locations and strengths of the vortices from the time history of the observations. An Eulerian observation is a measurement of the velocity of the flow at a fixed point in the domain of the flow. A Lagrangian observation is the measurement of the position of a particle moving with the fluid. To determine observability we introduce the observability and the strong observability rank conditions and compute them for the various flows and observations. We find that vortex flows with Lagrangian observations tend to be more observable then the same flows with Eulerian observations.**

Keywords: Point vortex flow, observability, Eulerian observation, Lagrangian observation, observability rank condition.

I. Introduction

There has been a tremendous effort put forth to study the controllability of fluids under various forms of activation. Except for the data assimilation community, there has been less effort studying the observability of fluids. This paper is a study of the observability of simple flows, the two dimensional flow induced by one or two point vortices in the plane. Admittedly these are ideal situations but we can't hope to understand the observability of more realistic flows unless we understand the observability of simpler ones. We are following earlier work on filtering of vortex flows found in [9], [4], [5], [10] and [11].

Consider m point vortices in the plane. Corresponding to the j^{th} vortex there are three parameters $x_j = (x_{j1}, x_{j2}, x_{j3})$ that completely determine it. The first two are the coordinates of its center and the third is its strength. The velocity field at $(\xi_1, \xi_2) \in I\!R^2$ induced by this vortex is

$$\vec{u}_j(x_j, \xi) = \frac{x_{j3}}{r_j^2} \begin{bmatrix} x_{j2} - \xi_2 \\ \xi_1 - x_{j1} \end{bmatrix} \tag{I.1}$$

where $r_j^2 = (\xi_1 - x_{j1})^2 + (\xi_2 - x_{j2})^2$. This is an incompressible and irrotational flow with a singularity at $\xi = (x_{j1}, x_{j2})$.

The flow induced by all m vortices is

$$\vec{u}(x, \xi) = \sum_{j=1}^{m} \vec{u}_j(x_j, \xi) \tag{I.2}$$

This is also incompressible and irrotational with m singularities at the centers of the vortices. The center of the k^{th} vortex moves with the flow induced by the remaining $m-1$ vortices and its strength does not change

$$\begin{bmatrix} \dot{x}_{k1} \\ \dot{x}_{k2} \\ \dot{x}_{k3} \end{bmatrix} = f_k(x) = \sum_{j \neq k} \begin{bmatrix} \vec{u}_j(x_j, (x_{k1}, x_{k2})) \\ 0 \end{bmatrix} \tag{I.3}$$

There is a rich literature on vortex flow. For an introduction we refer the reader to the text [1] . For further information, the review article [2] is excellent.

We shall study the observability of vortex flows under two types of measurements. An Eulerian observation is a measurement of the velocity (I.2) of the flow at a fixed point $\xi^i \in I\!R^2$. A Lagrangian observation is a measurement of the location $\xi^i(t)$ of a particle moving with the flow. We may have more than one observation. A flow is said to be observable if the observations uniquely determine it.

Here is an outline of the rest of the paper. In Section II we study the observability of a single vortex under one Eulerian or Lagrangian observation. In Section III we introduce the observability rank condition, a test of the observability of an observed dynamical system. In Section IV this is used to test the observability of a single vortex under either an Eulerian or a Lagrangian observation. The next section discusses two vortex flow. In Section VI we study its observability under one or two Eulerian observations. In Section VII we study the observability of two vortex flow under one or two Lagrangian observations.

II. Observability of One Vortex Flow

Consider a single point vortex of unknown position and magnitude. There are three state variables, the location of the center and the strength of the vortex. The dynamics is trivial as none of these variables change. Suppose we have an Eulerian observation, i.e., we measure the velocity of the flow at some point in the plane. With this observation we cannot determine all three state variables. We know that the center lies on a line perpendicular to the observed velocity but we don't know where on the line it is because we don't know its strength.

If we have Eulerian observations of the velocity at two points in the plane and these points are not collinear with the center of the vortex then we know the center is at the intersection of the perpendiculars to the velocities. Once we know where the center is, we can determine the strength of the vortex from an observed velocity.

If the two observation points are collinear with the center of the vortex then it is still observable but a bit of analysis is needed.

Now consider the flow of a single vortex with a single Lagrangian observation of position. A Lagrangian observation is the position of a fluid particle moving with the flow. From the history of its position we can obtain the velocity of our Lagrangian particle. If we take perpendiculars to the velocities at two different times then they will intersect at the center of the vortex and the magnitude of the velocities determines its strength. Hence one vortex flow with one Lagrangian observation is always observable while one vortex flow with one Eulerian observation is never observable. Two vortex flow with one Eulerian or Lagrangian observation is not always observable as we shall see below.

III. Observability Rank Condition

Consider an observed dynamics

$$\dot{x} = f(x) \tag{III.1}$$
$$y = h(x) \tag{III.2}$$
$$x(0) = x^0 \tag{III.3}$$

The state $x \in I\!R^n$ or is local coordinates on a manifold locally diffeomorphic to $I\!R^n$. For simplicity of exposition we shall assume the former but all our results readily generalize to the latter. We shall also assume that f and g are sufficiently smooth functions. The state is not observed directly but the output $y \in I\!R^p$ is. The system is observable if the map from initial state to output history, $x^0 \mapsto y(0:\infty)$, is one to one. The symbol $y(0:T)$ denotes the trajectory $t \mapsto y(t),\ 0 \leq t < T$

In other words, the observed system (III.1,III.2) is observable if the output time trajectory uniquely determines the initial state. The system is locally observable if this map is locally one to one. In other words, neighboring initial states lead to different output trajectories.

The system is short time observable if the map $x^0 \mapsto y(0:T)$ is one to one for every $T > 0$. In other words an output trajectory immediately distinguishes its initial state. The system is short time, locally observable if this map is locally one to one.

Here is a sufficient condition for short time, local observability. First some notation. The exterior derivative of the function h is the one form

$$dh(x) = \frac{\partial h}{\partial x_j}(x) dx_j$$

with the summation convention on repeated indices understood. If h is column vector valued then dh is a column of one forms.

The Lie derivative of the function h by the vector field f is the function

$$L_f(h)(x) = \frac{\partial h}{\partial x_j}(x) f_j(x)$$

If h is column vector valued then so is $L_f(h)$.

We can iterate this operation

$$L_f^0(h)(x) = h(x)$$
$$L_f^r(h)(x) = \frac{\partial L^{r-1} h}{\partial x_j}(x) f_j(x)$$

for $r = 1, 2, \ldots$

Definition 3.1: The observed system (III.1,III.2) satisfies the observability rank condition (ORC) at x if

$$\{dL_f^r(h)(x) : r = 0, 1, 2, \ldots\} \tag{III.4}$$

contains n linearly independent covectors. The observed system (III.1,III.2) satisfies the observability rank condition if it satisfies the observability rank condition at every $x \in I\!R^n$.

Let $\lceil n/p \rceil$ denote the smallest integer greater than or equal to n/p. The observed system satisfies the strong observability rank condition (SORC) at x if the covectors

$$\left\{dL_f^r(h)(x) : r = 0, 1, 2, \ldots, \lceil n/p \rceil - 1\right\} \tag{III.5}$$

are linearly independent and

$$\left\{dL_f^r(h)(x) : r = 0, 1, 2, \ldots, \lceil n/p \rceil\right\} \tag{III.6}$$

contains n linearly independent covectors. The observed system satisfies the strong observability rank condition if it satisfies the strong observability rank condition at every $x \in I\!R^n$.

Suppose the observed system (III.1,III.2) satisfies the observability rank condition then it is short time, locally observable. The ORC is almost necessary. If the observability rank condition is violated on an open subset of $I\!R^n$ then (III.1,III.2) is not short time, locally observable. For proofs, see [3].

If a system satisfies the strong observability rank condition then we can distinguish points with the least differentiations of the output. The SORC is a sufficient condition for the local convergence of an extended Kalman filter [7].

IV. ORC for Finite Dimensional Flows

A finite dimensional flow in a domain $\Omega \subset I\!R^d$, $d = 2$ or 3, is a finite dimensional dynamics (III.1) whose state $x(t)$ determines the flow field on Ω at time t. Let ξ be coordinates on Ω. Corresponding to each $x \in I\!R^n$ the velocity of the fluid at ξ is $\vec{u}(x, \xi) \in I\!R^d$.

As an example consider m point vortices in the plane. The state x lives in $I\!R^n$ where $n = 3m$ and consists of the locations and strengths of the m vortices. Once we know x, we know the flow field (I.2) induced by these vortices.

We consider two types of observations of the fluid. A set of m Eulerian observations are measurements of the flow velocity at m fixed, distinct locations. It takes the form

$$y^i = h^i(x) = \vec{u}(x, \xi^i) \tag{IV.1}$$

for some fixed $\xi^i \in \Omega$, $i = 1, \ldots, m$. Now each $y^i \in I\!R^d$ and the total number of observations is $p = dm$.

A set of m Lagrangian observations are the positions of m distinct particles moving with the flow. To model it we use the technique of Ide, Kuznetsov and Jones, [6], [8]. We define an extended system with state $z' = \left[\begin{array}{cccc} x' & \xi^{1'} & \ldots & \xi^{m'} \end{array} \right]$ with each $\xi^i \in \Omega \subset I\!R^d$ and dynamics

$$\dot{z} \quad = \quad g(z) = \left[\begin{array}{cccc} f'(x) & \vec{u}'(x,\xi^1) & \ldots & \vec{u}'(x,\xi^m) \end{array} \right]' \tag{IV.2}$$

where $\xi^i(t)$ is the location of the i^{th} Lagrangian observation at time t. The Lagrangian observations are

$$w^i \quad = \quad k^i(z) = \xi^i \tag{IV.3}$$

for $i = 1, \ldots, m$. Each observation $w^i \in I\!R^d$ so the total number of observed variables is $p = dm$.

Notice that $L_g(k^i)(z) = \vec{u}(x,\xi^i)$ so one might expect that the observability rank condition for the system (III.1) with m Eulerian observations (IV.1) is closely related to the observability rank condition for the extended system (IV.2) with m Lagrangian observations (IV.3). This is true up to a point.

We calculate the first few terms of (III.4) for the two systems. For the fluid (III.1) with m Eulerian observations (IV.1) we have

$$dh^i(x) \quad = \quad d\vec{u}(x,\xi^i) = \frac{\partial \vec{u}}{\partial x_j}(x,\xi^i)dx_j \tag{IV.4}$$

$$\begin{aligned} dL_f(h^i)(x) \quad &= \quad dL_f(\vec{u})(x,\xi^i) \\ &= \quad \left(\frac{\partial^2 \vec{u}}{\partial x_j \partial x_l}(x) f_l(x) + \frac{\partial \vec{u}}{\partial x_l}(x) \frac{\partial f_l}{\partial x_j}(x) \right) dx_j \end{aligned} \tag{IV.5}$$

We should note that h^i is d vector valued so dh^i is a d column vector whose components are one forms and so is $dL_f(h^i)$.

Let d_z be the exterior differentiation operator in the z variables, i.e.,

$$d_z k^i(z) = \frac{\partial k^i}{\partial x_j}(x,\xi^1,\ldots,\xi^k)dx_j + \frac{\partial k^i}{\partial \xi^i}(x,\xi^1,\ldots,\xi^k)d\xi^i$$

If $\xi \in I\!R^d$ and k^i takes values in $I\!R^d$ then $\frac{\partial k^i}{\partial \xi^i}(x,\xi^1,\ldots,\xi^m)$ is a $d \times d$ matrix and $d\xi^i$ is a column of d one forms on $I\!R^d$.

For the extended system (IV.2) with m Lagangian observations (IV.3) we have

$$d_z k^i(z) \quad = \quad d\xi^i \tag{IV.6}$$

$$d_z L_g(k^i)(z) \quad = \quad d_z \vec{u}(x,\xi^i) = \frac{\partial \vec{u}}{\partial x_j}(x,\xi^i)dx_j \tag{IV.7}$$

$$\text{mod}\left\{d\xi^1,\ldots,d\xi^m\right\} \tag{IV.8}$$

$$\begin{aligned} d_z L_g^2(k^i)(z) \quad = \quad & \frac{\partial^2 \vec{u}}{\partial x_l \partial x_j}(x) f_l(x) dx_j \\ & + \frac{\partial \vec{u}}{\partial x_l}(x) \frac{\partial f_l}{\partial x_j}(x) dx_j \\ & + \frac{\partial^2 \vec{u}}{\partial x_j \partial \xi}(x,\xi^i)\vec{u}(x,\xi^i)dx_j, \\ & \text{mod}\left\{d\xi^i, dL_g(\xi^i)\right\} \end{aligned} \tag{IV.9}$$

Notice that (IV.6) span the extra dimensions of the extended system. Modulo (IV.6), the one forms (IV.7) span the same dx_j dimensions as (IV.4). So far so good but in general (IV.9) does not span the span the same dx_j dimensions as (IV.5) modulo the one forms (IV.6), (IV.7) because (IV.9) has an extra term that is not present in (IV.5).

The flow of one point vortex with one observation discussed above illustrates this point. Let x_1, x_2 denote the center of the vortex and x_3 its strength. The dynamics (III.1) is trivial,

$$\dot{x} = f(x) = 0$$

The flow field in $I\!R^2$ corresponding to x is

$$\vec{u}(x,\xi) \quad = \quad \frac{x_3}{r^2} \left[\begin{array}{c} \xi_2 - x_2 \\ x_1 - \xi_1 \end{array} \right] \tag{IV.10}$$

where $r^2 = (x_1 - \xi_1)^2 + (x_2 - \xi_2)^2$.

Suppose there is one Eulerian observation, without loss of generality, at the origin $\xi^1 = (0,0)$, then (IV.1) becomes

$$y = h^1(x) = \vec{u}(x,\xi^1) = \frac{x_3}{r^2} \left[\begin{array}{c} -x_2 \\ x_1 \end{array} \right]$$

and

$$dh^1(x)$$

$$= \frac{1}{r^4} \left[\begin{array}{c} 2x_1x_2x_3dx_1 + (x_2^2 - x_1^2)x_3dx_2 - x_2r^2dx_3 \\ (x_2^2 - x_1^2)x_3dx_1 - 2x_1x_2x_3dx_2 + x_1r^2dx_3 \end{array} \right]$$

$$dL_f^r(h^1)(x) = \left[\begin{array}{c} 0 \\ 0 \end{array} \right], \qquad r = 1, 2, \ldots$$

Without loss of generality we can assume that the center of the vortex lies on the x_1 axis, it is not at the origin and it has nonzero strength so $x_1 \neq 0$, $x_2 = 0$ and $x_3 \neq 0$. Then

$$dh^1(x) = \left[\begin{array}{c} -\frac{x_3}{x_1^2}dx_2 \\ \\ -\frac{x_3}{x_1^2}dx_1 + \frac{1}{x_1}dx_3 \end{array} \right]$$

so the rank of (III.4) is 2.

We have only two independent one forms so the three dimensional Eulerian observed system does not satisfy the observability rank condition.

If we have one Lagrangian observation then the extended dynamics is

$$\dot{z} \quad = \quad \left[\begin{array}{c} \dot{x} \\ \\ \dot{\xi}_1^1 \\ \\ \dot{\xi}_2^1 \end{array} \right] = g(z) = \left[\begin{array}{c} 0 \\ \\ \frac{x_3}{r^2}(\xi_2^1 - x_2) \\ \\ \frac{x_3}{r^2}(x_1 - \xi_1^1) \end{array} \right]$$

and the observation is

$$\begin{bmatrix} y_1^1 \\ y_2^1 \end{bmatrix} = k^1(z) = \begin{bmatrix} \xi_1^1 \\ \xi_2^1 \end{bmatrix}$$

If we assume without loss of generality that at some t, the Lagrangian observation is made at the origin, $\xi^1(t) = 0$, and $x_1 \neq 0$, $x_2 = 0$ and $x_3 \neq 0$

$$d_z k^1(z) = \begin{bmatrix} d\xi_1^1 \\ d\xi_2^1 \end{bmatrix} \quad \text{(IV.11)}$$

$$d_z L_g(k^1)(z) = \begin{bmatrix} -\frac{x_3}{x_1^2} dx_2 \\ \\ -\frac{x_3}{x_1^2} dx_1 + \frac{1}{x_1} dx_3 \end{bmatrix} \quad \text{(IV.12)}$$

We need to compute the extra term in (IV.9).

$$\frac{\partial^2 \vec{u}}{\partial x_3 \partial \xi}(x, \xi) = \frac{1}{r^4} \begin{bmatrix} -2(x_1 - \xi_1)(x_2 - \xi_2) & (x_1 - \xi_1)^2 - (x_2 - \xi_2)^2 \\ (x_2 - \xi_2)^2 - (x_1 - \xi_1)^2 & 2(x_1 - \xi_1)(x_2 - \xi_2) \end{bmatrix}$$

We can make some simplifying assumptions because we are not going to differentiate further. We assume we are at a time t where $\xi^1(t) = 0$ and $x_1 \neq 0,\ x_2 = 0,\ x_3 \neq 0$. Then at this t the extra term is

$$\begin{bmatrix} -\frac{2x_3^2}{x_1^4} dx_1 + \frac{x_3}{x_1^3} dx_3 \\ \\ -\frac{2x_3^2}{x_1^4} dx_2 \end{bmatrix} \quad \text{(IV.13)}$$

The observability rank condition is satisfied because there are three linearly independent one form among (IV.12) and (IV.13). Hence the strong observability rank condition holds for the flow of one vortex with one Lagrangian observation. As we have seen, the observability rank condition does not hold for the flow of one vortex with one Eulerian observation.

V. Two Vortex Flow

Two vortex flow can be quite complicated but the motion of the centers of the vortices is relativity simple. The system is six dimensional, the center of one vortex is at x_{11}, x_{12} and its strength is x_{13} while the center of the other vortex is at x_{21}, x_{22} and its strength is x_{23}. The dynamics is

$$\begin{bmatrix} \dot{x}_{11} \\ \dot{x}_{12} \\ \dot{x}_{13} \\ \dot{x}_{21} \\ \dot{x}_{22} \\ \dot{x}_{23} \end{bmatrix} = f(x) = \begin{bmatrix} \frac{x_{23}}{r^2}(x_{22} - x_{12}) \\ \frac{x_{23}}{r^2}(x_{11} - x_{21}) \\ 0 \\ \frac{x_{13}}{r^2}(x_{12} - x_{22}) \\ \frac{x_{13}}{r^2}(x_{21} - x_{11}) \\ 0 \end{bmatrix} \quad \text{(V.1)}$$

where $r^2 = (x_{11} - x_{21})^2 + (x_{12} - x_{22})^2$. The distance r between the centers remains constant because each center moves perpendicular to the line between them.

When the strengths are of different magnitudes, $|x_{13}| \neq |x_{23}|$, the centers of the two vortices move on two concentric circles in the plane. If the vortices are of same orientation the centers will stay as far away as possible on the concentric circles. When the they are of opposite orientation, they will stay as close as possible.

If the strengths are equal, $x_{13} = x_{23}$, then the centers will rotate around a single circle staying as far away as possible. If the strengths are opposite, $x_{13} = -x_{23}$, then the two centers will fly off to infinity along two parallel lines.

Suppose that the strengths are not opposite, $x_{13} \neq -x_{23}$, and without loss of generality the vortices start at $(x_{11}(0), x_{12}(0)) = (1, 0)$ and $(x_{21}(0), x_{22}(0)) = (-1, 0)$ then the two vortices will rotate around the point $\xi^c = (\xi_1^c, \xi_2^c) = (\frac{x_{13}-x_{23}}{x_{13}+x_{23}}, 0)$ with angular velocity $\omega = \frac{x_{13}+x_{23}}{4}$. The induced flow will be momentarily stagnant at $\xi^s = (\frac{x_{23}-x_{13}}{x_{13}+x_{23}}, 0) = -\xi^c$ but generally this stagnation point will rotate with the vortices remaining on the line between their centers. The one exception is when the strengths are equal, $x_{13} = x_{23}$, for then the stagnation point is the center of rotation at $(0, 0)$ and remains there.

When the vortices rotate on a circle or on a pair of concentric circles, it is informative to consider the flow in the frame that co-rotates with the vortices. A co-rotating point is one where the flow appears stationary in this co-rotating frame. For reasons that will become apparent later we are particularly interested in co-rotating points that are collinear with the centers of the vortices.

Suppose that the strengths are not opposite, $x_{13} \neq -x_{23}$, and without loss of generality the vortices are momentarily at $(x_{11}(0), x_{12}(0)) = (1, 0)$ and $(x_{21}(0), x_{22}(0)) = (-1, 0)$. Then at this moment the collinear, co-rotating points are at $(\xi_1, 0)$ where ξ_1 is a root of the cubic

$$\omega(\xi_1 - \xi_1^c)(\xi_1^2 - 1) = x_{13}(\xi_1 + 1) + x_{23}(\xi_1 - 1)$$

When the orientations of the vortices are the same, $x_{13}x_{23} > 0$, there are always three co-rotating points that are collinear with the vortex centers. One lies between the centers and the other two lie to either side of the centers.

When the orientations of the vortices are opposite, $x_{13}x_{23} < 0$, there is only one co-rotating point that is collinear with the vortex centers. It lies outside the centers in the direction of the stronger vortex.

VI. Eulerian Observability of Two Vortex Flow

Suppose there are two vortices of unknown positions and magnitudes and there is one Eulerian observation, the velocity at a fixed point, without loss of generality, the origin. The dynamics is (V.1) and the observation is

$$y = \begin{bmatrix} \frac{x_{12}x_{13}}{r_1^2} + \frac{x_{22}x_{23}}{r_2^2} \\ \\ -\frac{x_{11}x_{13}}{r_1^2} - \frac{x_{12}x_{23}}{r_2^2} \end{bmatrix} \quad \text{(VI.1)}$$

where $r_i^2 = x_{i1}^2 + x_{i2}^2$.

Is this system always observable? The answer is clearly no. We can interchange the vortices and not change the output trajectory. But clearly this is a problem with the way we are modeling things. We should take as state space not $\mathbb{R}^6$ but $\mathbb{R}^6$ mod the equivalence relation of interchanging the vortices.

More precisely we must exclude from the state space the possibility that the centers of vortices coincide $(x_{11}, x_{12}) \neq (x_{21}, x_{22})$ and that one of the strengths is zero $x_{13} \neq 0, x_{23} \neq 0$. We also assume that the Eulerian observation at the origin does not coincide with a center of a vortex $(x_{11}, x_{12}) \neq (0,0)$, $(x_{21}, x_{22}) \neq (0,0)$. This defines an open subset of $\mathbb{R}^6$ and we identify points of this subset that satisfy the equivalence relation (**??**). The resulting state space is a six dimensional manifold.

But even if we redefine the state space in this fashion, the system may be unobservable. Consider two equal vortices symmetrically placed with respect to the observer. Without loss of generality we can assume the observer is at the origin. A symmetric configuration is one satisfying

$$x_{11} = -x_{21}, \quad x_{12} = -x_{22}, \quad x_{13} = x_{23} \tag{VI.2}$$

The vortices will rotate in a circle around the origin where the observed velocity is identically zero. Therefore we cannot infer anything about their locations and strengths except that the configuration is symmetric.

But this is a very special configuration of the vortices and the observer, perhaps most configurations are observable. To test this we wrote software to compute the SORC for two vortices and one Eulerian or Lagrangian observation.

For one Eulerian observation, the dimension of the state space is six and the dimension of the output is two. The computed rank of

$$\begin{bmatrix} dh(x) \\ dL_f(h)(x) \\ dL_f^2(h)(x) \end{bmatrix} \tag{VI.3}$$

is six if the observation is not collinear with the centers of the vortices. Hence the SORC is satisfied at almost all x.

Assume that the configuration of the vortices and the observer is not one of the symmetric ones discussed above (VI.2). If the Eulerian observation is collinear with the centers of the vortices then the computed rank of (VI.3) is five and the SORC does not hold. Unless the observation is at the center of rotation of the vortices, the Eulerian observation does not stay collinear with their centers and the computed rank increases to six. so the ORC is satisfied for such configurations.

If the observation is at the center of rotation then the Eulerian observation stays collinear with their centers so the computed rank of (VI.3) remains five and SORC remains unsatisfied. The direction that is orthogonal to the one forms (VI.3) is that of moving the vortices away from the observer while maintaining collinearity and increasing their strength. The line between the centers is rotating so the rank of (III.4) is six and the ORC is satisfied.

Consider a symmetric configuration (VI.2) with the Eulerian observation at the origin. The two vortices will rotate around the origin maintaining these relations. The computed rank of (VI.3) is three as expected and so the SORC does not hold. One expects the rank to be three as there are three directions to change the six dimensional state while still satisfying the relations (VI.2). Moreover the rank of (III.4) is also three so the ORC does not hold also.

Now consider two vortex flow with two Eulerian observations at different locations. The observation y is four dimensional. If the vortices are of different strengths, $x_{13} \neq x_{23}$, then the computed rank of

$$\begin{bmatrix} dh^1(x) \\ dh^2(x) \end{bmatrix} \tag{VI.4}$$

is four and the computed rank of

$$\begin{bmatrix} dh^1(x) \\ dh^2(x) \\ dL_f(h^1)(x) \\ dL_f(h^2)(x) \end{bmatrix} \tag{VI.5}$$

is six so the SORC holds.

If the vortices are of same strength, $x_{13} = x_{23}$, but at least one of the observations is not collinear with the centers of the vortices and the other observation is not half way between them, then the rank of (VI.4) is four and the rank of (VI.5) is six so the SORC holds.

If the vortices are of same strength, $x_{13} = x_{23}$, and both observations are collinear with the centers of the vortices then the rank of (VI.4) is four but the rank of (VI.5) is five so the SORC does not hold. However the ORC does hold.

If the vortices are of same strength, $x_{13} = x_{23}$, and one observation is halfway between the centers of the vortices then the rank of (VI.4) is four but the rank of (VI.5) is five so the SORC does not hold. However the ORC does hold.

So two vortex flow with two Eulerian observations is always short time locally observable.

VII. Lagrangian Observability of Two Vortex Flow

Now we consider two vortex flow with one Lagrangian observation. It need not be observable. Consider the symmetric configuration (VI.2). The velocity of the flow field is zero at the origin. So a Lagrangian observation starting at the origin will remain at the origin and the flow is not observable.

For two vortices and one Lagrangian observation, the extended state space is eight dimensional. If the observation is not collinear with the centers of the vortices then the computed rank of

$$\begin{bmatrix} dk(x) \\ dL_g(k)(x) \\ dL_g^2(k)(x) \\ dL_g^3(k)(x) \end{bmatrix} \tag{VII.1}$$

is eight so the SORC holds.

If the Lagrangian observer is collinear with the centers of the vortices but not half way between two vortices of equal strength then the rank is seven so the SORC is not satisfied. If the Lagrangian observer is not at a co-rotating, collinear point then the rank of (VII.1) immediately becomes eight so the ORC is satisfied. If the observer is at a co-rotating, collinear point then the rank of (VII.1) remains seven so the SORC remains unsatisfied. But the computed rank of (III.4) is eight so the ORC is satisfied.

Now consider a Lagrangian observer halfway between two vortices of the same strength, (VI.2). Then the observation is made at a stagnation point of the flow and so it remains there. The computed rank of (VII.1) is five as expected. There are three directions to change the vortices which leave the observation halfway between two vortices of the same strength.

The extended state of two vortex flow with two Lagrangian observations is ten dimensional and the observation is four dimensional. It always satisfies the SORC, the computed rank of

$$\begin{bmatrix} dk(x) \\ dL_g(k)(x) \end{bmatrix} \tag{VII.2}$$

is eight and the computed rank of

$$\begin{bmatrix} dk(x) \\ dL_g(k)(x) \\ dL_g^2(k)(x) \end{bmatrix} \tag{VII.3}$$

is ten. Hence the system is short time locally observable.

VIII. Conclusion

We have studied the observability of one and two point vortex flow under one or two Eulerian or Lagrangian observations. Although we are not able to prove it in all cases, apparently the extra term in the Observability Rank Condition for Lagrangian observations has a positive impact on the observability of the flow.

The next step is to extend these results to multi vortex and other higher dimensional flows. There are similar symmetric configurations in multi vortex flow where the vortices rotate in a symmetric fashion around the origin which is a stagnation point of the flow [1], p. 184. One expects that such configurations will be unobservable under one Eulerian or Lagrangian observation at the origin. Furthermore it is known that the dynamics of four point vortices can be chaotic [2]. It would be interesting to study the observability of such a system. Other interesting cases are spatially discretized Euler or Navier Stokes equations. These latter studies are probably not possible analytically but could be done numerically.

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