Proceedings of the
47th IEEE Conference on Decision and Control
Cancun, Mexico, Dec. 9-11, 2008

Elimination of Fixed Modes of a Decentralized Distributed Estimator with Inter-Agent Communication

Maxim V. Subbotin, Roy S. Smith

***Abstract*— We consider a decentralized distributed estimator for a formation of agents described with a discrete-time LTI system. Due to the decentralized structure, the estimator may contain poles invariant with respect to local agents' observer gains. We prove that under mild observability assumptions there exists a communication topology and transmitter-receiver gains which allow assigning all the poles of the distributed estimator. We also propose an algorithm for the design of a minimal communication topology which guarantees closed-loop pole assignability.**

I. INTRODUCTION

Decentralized and distributed estimation is one of the long-standing problems in engineering. Being closely connected to a decentralized control problem it has been a topic of research for many decades. In recent years the problem has been addressed with new interest and enthusiasm due to advances in communication and sensor technologies. The areas of application, just to mention a few, span from formation and vehicle platoon control problems [3], [4], [6] to sensor networks and distributed power systems [5], [11]. One of the main issues in the analysis and design of the decentralized and distributed control and estimation systems was shown to be the structure of information flow and properties of the communication between various components of a system. The communication constraints are critical in defining stability properties of a system and limitations on the achievable performance [3], [6], [11].

In this paper we consider the problem of the design of a distributed decentralized estimator for a class of discrete linear time-invariant systems introduced by Smith and Hadaegh in [6]. In our previous papers [7], [8] we have developed techniques for synthesis of the distributed decentralized estimator. In this paper we address the question of feasibility of the design problem and analyze structural properties of the considered class of systems. Due to the constraints of the decentralized distributed structure, the closed-loop system consisting of local agents' state feedbacks and estimators may contain fixed modes - eigenvalues of the system which are invariant under the structured feedback. Existence of fixed modes in the system results in infeasibility of the design problem if the modes are unstable or imposes constraints on the system performance if they are stable. We show that we can eliminate the fixed modes by introducing communication between agents and propose a method which allows us to do this with minimal communication.

Maxim V. Subbotin and Roy S. Smith, Department of Electrical and Computer Engineering, University of California, Santa Barbara, CA 93106-9560, USA, subbotin@engineering.ucsb.edu, roy@ece.ucsb.edu

The main result of the paper is a proof that under natural observability assumptions it is always possible to design a communication topology and choose transmitter and receiver gains for inter-agent communication links that guarantee assignability of all eigenvalues of the distributed decentralized estimator. A condition required to ensure assignability of the poles is that the formation dynamics should be observable from the collective measurements of all agents. This is a fairly mild and practical condition since agents usually have an access only to some local measurements, each agent's own position and velocity, for example, and cannot estimate the collective formation dynamics from only their own local measurements. At the same time if the formation dynamics is not observable from all agents collectively even a centralized estimator is not guaranteed to be stable. To establish our result we use a method developed by Anderson and Clements [1] for algebraic characterization of fixed modes in the system with decentralized output feedback and exploit the structure of the proposed decentralized estimator.

Our main proof leads naturally to the procedure for elimination of the fixed modes. We develop a method which allows elimination of all fixed modes in the dynamics of the distributed estimator by introducing communication between agents with the minimum number of communication links in a topology and the minimum dimension of the communication signals. The elimination procedure consists of several steps with algebraic conditions which can be easily programmed in an automatic routine. To illustrate the application of the proposed method we develop an example with a formation of two agents. The distributed estimator of the example system has an unstable fixed mode when agents are using only their own local measurements for the estimation. The introduction of a communication link eliminates the fixed mode.

In next section we describe the class of systems we work with and introduce variables and notation necessary for further analysis. Later we introduce the communication into the system and derive our main results. In the last two sections of the paper we describe the proposed elimination algorithm and show an illustrative example. Throughout the paper we use the following notation. The identity matrix with dimension $n \times n$ is defined as I_n and a column vector with the dimension n and all elements equal to 1 is defined as 1_n. A block diagonal matrix B with submatrices $B_i,\ i = 1, ..., n$ on the diagonal is denoted by $B = \mathrm{diag}(B_1,\ ...,\ B_n)$. The matrix B' is the transpose of B. The symbol $\otimes$ is used to denote the Kronecker product for matrices.

II. Problem Formulation

We consider discrete LTI systems described by

$$x(k+1) \quad = \quad Ax(k) + B_u u(k), \tag{1}$$

where $x(k) \in I\!R^{n_x}$ is the system state, $u(k) \in I\!R^{m_u}$ is the actuation input. The state dynamics (1) represent the collective formation dynamics of N vehicles; the agents of the formation. The control input is composed of individual control inputs of each agent, $u(k) = \sum_{i=1}^{N} u_i(k)$, and i^{th} agent's control signal is defined by $u_i(k) = \Pi_i u(k) \in I\!R^{m_u}$, where Π_i is the projection matrix and $\sum_{i=1}^{N} \Pi_i = I$. Note that this definition of the joint control input, $u(k)$, captures the situation when $u(k) = [\bar{u}_1(k)'\ \bar{u}_2(k)'\ ...\ \bar{u}_N(k)']'$ - a stack of control inputs from all agents, $\bar{u}_i(k) \in I\!R^{m_i}$, $i = 1, ..., N$. Each agent measures a signal, $y_i(k) = C_i x(k) \in I\!R^{k_{y_i}}$ – the system output available to agent i. The collected measurement signal is hence represented by,

$$y(k) \quad = \quad Cx(k), \tag{2}$$

where $y(k) = [y_1(k)'\ y_2(k)'\ ...\ y_N(k)']'$ and $C = [C_1'\ C_2'\ ...\ C_N']'$. Note that the system described by (1), (2) is similar to the system analyzed in [7] where we proposed a solution to the synthesis part of the problem. The difference between these two systems is in the absence of a process noise in (1) and a measurement noise in (2). Since the presence of noises in the system model does not influence our further analysis we have dropped the noises in (1), (2) as well as a noise in communication signals introduced later. In the following we introduce a decentralized distributed estimation structure defined by Smith and Hadaegh in [6].

We assume that a stabilizing state feedback, $u(k) = -Kx(k)$, which satisfies a formation-wide objective function, is given and specifies the desired closed-loop dynamics of the formation through the matrix $A_{clp} = A - B_u K$. Since we focus on the estimation part of the problem, we do not consider a particular method for choice of K. The formation control law, $u(k)$, is calculated and implemented by each agent individually using available measurements and information transmitted from other agents, hence resulting in a decentralized and distributed architecture. The i^{th} agent's control system consists of a combination of a full order formation state estimator which provides local agent's estimate of the formation state, $\hat{x}_i(k) \in I\!R^{n_x}$, and state feedback for the calculation of $u_i(k)$. As a result, the i^{th} agent's contribution to the control input is given by,

$$u_i(k) \quad = \quad -\Pi_i K \hat{x}_i(k). \tag{3}$$

To be able to approach the distributed estimator design problem, we first derive the equations describing the complete closed-loop formation dynamics. For simplicity, we start our derivations for the case where each agent's estimator uses only agent's local measurements to update its formation state estimate, and then introduce communication in the estimation structure. If there is no communication between agents, the state estimator of the i^{th} agent, used to provide formation state estimate, $\hat{x}_i(k)$, is given by,

$$\hat{x}_i(k+1) = A_{clp}\hat{x}_i(k) + L_i(y_i(k) - C_i\hat{x}_i(k)), \tag{4}$$

where L_i is the agent's estimator gain. We can define an estimation error for each agent,

$$e_i(k) \quad = \quad x(k) - \hat{x}_i(k).$$

Then the closed-loop plant dynamics are given by,

$$x(k+1) = A_{clp}x(k) + B_u \sum_{i=1}^{N} \Pi_i K e_i(k). \tag{5}$$

And the i^{th} estimator error dynamics are,

$$e_i(k+1) = x(k+1) - \hat{x}_i(k+1) =$$
$$(A_{clp} - L_i C_i)e_i(k) + B_u \sum_{i=1}^{N} \Pi_i K e_i(k). \tag{6}$$

If we collect estimation errors in one vector, $e(k) = [e_1(k)'\ e_2(k)'\ ...\ e_N(k)']' \in I\!R^{Nn_x}$, we can write the collected estimation error dynamics which together with (5) describe the complete dynamics of the closed-loop system,

$$e(k+1) = (A_M - L_f C_f)e(k), \tag{7}$$

where $A_M = I_N \otimes A_{clp} + M_N$, $M_N = 1_N \otimes [B_u\Pi_1 K\ ...\ B_u\Pi_N K]$, $L_f = \mathrm{diag}(L_1, ..., L_N)$, $C_f = \mathrm{diag}(C_1, ..., C_N)$.

Analyzing (5) and (7) we see that closed-loop plant dynamics are driven by the estimation error and are specified by the choice of the state feedback gain K in A_{clp}. The estimation error dynamics (7) are decoupled from (5). The stability of the collected estimation error and, as a consequence, the complete closed-loop system depends on the properties of the matrix $A_M - L_f C_f$. The block-diagonal matrix $L_f C_f$ represents an output feedback term which we may use to ensure stability of the collected estimation error dynamics (7). Due to its block-diagonal structure $L_f C_f$ is naturally limited in its ability to assign eigenvalues of $A_M - L_f C_f$. The system (7) may contain fixed modes, eigenvalues of the matrix A_M which are not affected by the decentralized output feedback $L_f C_f$. Fixed modes are a structural property of a given system matrix, A_M, and decentralized output feedback, $L_f C_f$ [9], [10]. It is essential to determine the presence of fixed modes in our decentralized estimation error dynamics since, in the best case, they impose the constraints on the best achievable performance of the decentralized estimator. In the worst situation, when the fixed modes are unstable, they result in infeasibility of the estimator design problem. There are several methods present in the literature which allow characterization of the fixed modes for the system (7) [1], [2], [9]. We can apply one of these methods to determine if there are any fixed modes in the system and if they are stable. If the system contains deleterious or unstable fixed modes with respect to $L_f C_f$ we can try and eliminate them by introducing communication between agents.

In this paper we prove that there exists a communication topology with a combination of transmitter-receiver gains for the network of agents and the estimator gain L_f which guarantee that there are no fixed modes in the collected estimation error dynamics if and only if the pair of matrices A, C for the system (1), (2) is observable. This implies that if the observability condition is satisfied, we are able to assign all the eigenvalues of the collected estimation error dynamics, and hence ensure stability and improve the performance of the closed-loop system with the distributed estimation structure. In addition to that, we propose a simple algebraic method for the design of inter-agent communication topology which eliminates the fixed modes with the minimal number of links in the network and minimal dimension of the communication signals.

III. Communication

To introduce communication in the decentralized distributed estimation structure we allow agents of the formation to exchange the information about their formation state estimates, $\hat{x}_i(k)$. This naturally results in the modification of structural properties of the system. We assume that the information transmitted between two agents' estimators through a single unidirectional communication link can be represented by,

$$t_{ij}(k) \quad = \quad H_{ij}\hat{x}_j(k), \tag{8}$$

where $t_{ij}(k) \in I\!R^{k_{ij}}$ is the signal received by estimator i from estimator j. We assume that the transmitter gain matrix, $H_{ij} \in I\!R^{k_{ij}\times n_x}$, and its dimension specified by k_{ij} are to be specified by a designer. This formulation is motivated by the fact that the dimension of a signal transmitted thorough a single communication link can be limited and the limitation is reflected in k_{ij}. The i^{th} estimator can use the received signals and update its state estimate according to the following model,

$$\begin{aligned}\hat{x}_i(k+1) =&(A-B_uK)\hat{x}_i(k) + L_i(y_i(k) - C_i\hat{x}_i(k)) + \\ &\sum_j F_{ij}(t_{ij}(k) - H_{ij}\hat{x}_i(k)),\end{aligned} \tag{9}$$

where F_{ij} is the receiver gain matrix which corresponds to H_{ij} and the sum is taken over all signals received by i^{th} agent.

In order to be able to analyze the structural properties of the collected estimation error dynamics with the introduced communication, we define new input and output matrices describing the communication topology and transmitter-receiver gains. First, we introduce the output matrix $\mathbb{C}_{ij} \in I\!R^{n_x\times Nn_x}$, $i \neq j$, which corresponds to a communication link from agent j to agent i and is defined as follows. It consists of N $n_x \times n_x$ blocks with zero blocks everywhere except for $I \in I\!R^{n_x\times n_x}$ in the j^{th} block column and $-I \in I\!R^{n_x\times n_x}$ in the i^{th} block column. So, for example, $\mathbb{C}_{24} \equiv [0\ \ -I\ 0\ I\ 0\ ...\ 0]$. Note that $\mathbb{C}_{ii}$ are not defined and hence there are $N(N-1)$ matrices $\mathbb{C}_{ij}$ in total. We also introduce an output matrix used to describe information transmitted by agent j to all other $N-1$ agents,

$$\mathbb{C}_j \equiv \left[\mathbb{C}'_{1j}\ \mathbb{C}'_{2j}\ ...\ \mathbb{C}'_{Nj}\right]' \in I\!R^{(N-1)n_x\times Nn_x}. \tag{10}$$

And the collected output matrix describing the complete set of possible communication links from all agents of the formation, $\mathbb{C} \equiv [\mathbb{C}'_1\ \mathbb{C}'_2\ ...\ \mathbb{C}'_N]' \in I\!R^{N(N-1)n_x\times Nn_x}$. In the similar way we define input matrices $\mathbb{B}_{ij} \in I\!R^{Nn_x\times n_x}$, $i \neq j$, which consist of N $n_x \times n_x$ blocks with zero blocks everywhere except for $I \in I\!R^{n_x\times n_x}$ in the i^{th} block row. Note that $\mathbb{B}_{ii}$ are also not defined and, for example, $\mathbb{B}_{2j} \equiv [0\ I\ 0\ ...\ 0]$ for any $j = 1, 3, 4, ..., N$. We also define the input matrix which corresponds to outgoing links of agent j, $\mathbb{B}_j \equiv [\mathbb{B}_{1j}\ \mathbb{B}_{2j}\ ...\ \mathbb{B}_{Nj}] \in I\!R^{Nn_x\times(N-1)n_x}$. And the collected input matrix, $\mathbb{B} \equiv [\mathbb{B}_1\ \mathbb{B}_2\ ...\ \mathbb{B}_N] \in I\!R^{Nn_x\times N(N-1)n_x}$ corresponding to the complete set of possible received signals. Finally, we introduce the collected receiver gain matrix, $F_f = \mathrm{diag}(F_1,\ F_2,\ ...\ F_N) \in I\!R^{N(N-1)n_x\times\sum_{i=1}^N\sum_{j=1}^N k_{ij}}$, where $F_j = \mathrm{diag}(F_{1j},\ F_{2j},\ ...,\ F_{Nj}) \in I\!R^{(N-1)n_x\times\sum_{i=1}^N k_{ij}}$ contains the receiver gains of the agents receiving signals from the j^{th} agent, and $F_{ij} \in I\!R^{n_x\times k_{ij}}$, $i \neq j$. And the collected transmitter gain matrix, $H_f = \mathrm{diag}(H_1,\ H_2,\ ...,\ H_N) \in I\!R^{\sum_{i=1}^N\sum_{j=1}^N k_{ij}\times N(N-1)n_x}$, where $H_j = \mathrm{diag}(H_{1j},\ H_{2j},\ ...,\ H_{Nj}) \in I\!R^{\sum_{i=1}^N k_{ij}\times(N-1)n_x}$, contains transmitter gains of the j^{th} agent, and $H_{ij} \in I\!R^{k_{ij}\times n_x}$, $i \neq j$. The introduced notation allows specification of all possible communication interconnections in the formation with different transmitter-receiver gains, H_{ij}, F_{ij}, assigned for each individual link. If there is no communication link between agent j and agent i, then $F_{ij}H_{ij} = 0^{n_x\times n_x}$.

With the introduced notation, we can rewrite the equation for the collected estimation error dynamics of the formation of agents with the communication between estimators as,

$$\begin{aligned}e(k+1) =&(A_M - L_fC_f - \mathbb{B}F_fH_f\mathbb{C})e(k) \\ =&\left(A_M - [I\ \mathbb{B}]\begin{bmatrix} L_f & 0 \\ 0 & F_fH_f \end{bmatrix}\begin{bmatrix} C_f \\ \mathbb{C} \end{bmatrix}\right)e(k).\end{aligned} \tag{11}$$

As we can see from equation (11), communication between agents introduces a new output feedback term, $\mathbb{B}F_fH_f\mathbb{C}$, in the collected estimation error dynamics. Although this term is potentially a full matrix, if all agents of the formation communicate with each other then there are no zero blocks in $\mathbb{B}F_fH_f\mathbb{C}$, it has structural constraints imposed by blocks of the output matrix $\mathbb{C}$ and it is not obvious that we can freely assign all Nn_x eigenvalues of the collected estimation error dynamics (11), even with a complete communication.

So far we have defined fixed modes for the system (7) as the eigenvalues of the system matrix A_M invariant with respect to the decentralized output feedback L_fC_f. Similarly we can define fixed modes of the system (11) as the eigenvalues of the matrix A_M invariant with respect to the combined output feedback introduced by local agents' measurements, L_fC_f, and the inter-agent communication, $\mathbb{B}F_fH_f\mathbb{C}$. It is clear that the presence of a fixed mode in the system (11) is equivalent to existence of $\lambda \in eig(A_M)$ such that for any L_f, H_f, and F_f, there exists $v \neq 0$ and,

$$(\lambda I - (A_M - L_fC_f - \mathbb{B}F_fH_f\mathbb{C}))v = 0.$$

To prove that there are no fixed modes in the system (11), we, first, have to establish an important observability condition summarized in the following lemma.

Lemma 1: *The pair A, C is observable if and only if the pair A_M, $\begin{bmatrix} C_f \\ \mathbb{C} \end{bmatrix}$ is observable.*

Lemma 1 can be proved with straight forward matrix manipulations and we omit it here due to space limitations. Now we can establish the main result guaranteeing that with enough communication between agents we can assign all eigenvalues of the estimation error dynamics (11).

Theorem 1: *There exists a communication topology and observer L_f, transmitter H_f, and receiver F_f gain matrices which assign all Nn_x poles of the collected estimation error dynamics (11) if and only if the pair A, C is observable.*

Proof. First, we prove the sufficiency part of the theorem. As stated in Lemma 1, observability of the pair A, C is equivalent to observability of the pair A_M, $\begin{bmatrix} C_f \\ \mathbb{C} \end{bmatrix}$. This result bound together with an observation that the pair A_M, $[I\ \mathbb{B}]$ is always controllable since $rank([I\ \mathbb{B}]) = Nn_x$, allows us to use Theorem 4.1, [1] established by Anderson and Clements for guaranteeing assignability of the eigenvalues. To proceed with the proof, we partition the input matrix $[I\ \mathbb{B}]$ into block columns, $\bar{B}_i$, $i \in S \equiv \{1, ..., N + N(N+1)\}$, where each block corresponds to an agent's measurements or a communication link between two agents. With the introduced partition $B_S \equiv [I\ \mathbb{B}] = [\bar{B}_1\ ...\ \bar{B}_N\ \bar{B}_{N+1}\ ...\ \bar{B}_{N+N(N+1)}]$, where there are $N + N(N-1)$ blocks in total, the first N blocks correspond to agents' measurements. Last $N(N-1)$ blocks correspond to inter-agent communication and, respectively, $\bar{B}_{N+1} \equiv \mathbb{B}_{21}$, $\bar{B}_{N+2} \equiv \mathbb{B}_{31}$, ..., $\bar{B}_{N+N(N+1)} \equiv \mathbb{B}_{(N-1)N}$. We introduce a similar partition for the block rows of the output matrix $C_S \equiv \begin{bmatrix} C_f \\ \mathbb{C} \end{bmatrix} = \left[\bar{C}_1'\ ...\ \bar{C}_N'\ \bar{C}_{N+1}'\ ...\ \bar{C}_{N+N(N-1)}'\right]'$, where $\bar{C}_{N+1} \equiv \mathbb{C}_{21}$, $\bar{C}_{N+2} \equiv \mathbb{C}_{31}$, ..., $\bar{C}_{N+N(N+1)} \equiv \mathbb{C}_{(N-1)N}$.

Restating the Theorem 4.1 from [1] for the system (11) and writing it in terms of the introduced variables we arrive at the following.

For the observable pair A_M, $\begin{bmatrix} C_f \\ \mathbb{C} \end{bmatrix}$ and the controllable pair A_M, $[I\ \mathbb{B}]$, $rank(\lambda I - (A_M - L_f C_f - \mathbb{B} F_f H_f \mathbb{C})) < Nn_x$ for all L_f and $F_f H_f$ and some fixed complex number λ *if and only if* there exists a partition of the set S into two disjoint subsets $S_1 = \{i_1, ..., i_k\}$ and $S_2 = \{i_{k+1}, ..., i_{N+N(N-1)}\}$, $S = S_1 \bigcup S_2$, $S_1 \bigcap S_2 = \emptyset$ such that $rank(M_{S_1,S_2}) < Nn_x$, where enumeration of the elements of the set S is arbitrary and

$$M_{S_1,S_2} = \begin{bmatrix} \lambda I - A_M & B_{S_1} \\ C_{S_2} & 0 \end{bmatrix}.$$

By definition, a complex number, λ, is a fixed mode of the system (11) with respect to the output feedback $L_f C_f$, $\mathbb{B} F_f H_f \mathbb{C}$ if $rank(\lambda I - (A_M - L_f C_f - \mathbb{B} F_f H_f \mathbb{C})) < Nn_x$ for all L_f and $F_f H_f$. Hence, based on the result stated above, in order to prove that the system (11) with an observable pair A, C has no fixed modes with respect to $L_f C_f$, $\mathbb{B} F_f H_f \mathbb{C}$, we need to show that there does not exists a partition of the set S into two disjoint subsets S_1, S_2 such that $rank(M_{S_1,S_2}) < Nn_x$.

We show that for any disjoint S_1 and S_2, $rank(M_{S_1,S_2}) = Nn_x$ independently of matrices $\lambda I - A_M$ and C_f. First, we notice that last $N(N-1)$ blocks, $\bar{C}_{N+i}$, $i = 1, ..., N(N-1)$ of the output matrix C_S compose the matrix $\mathbb{C}$ and are grouped into N blocks $\mathbb{C}_j$, $j = 1, ..., N$. Each block $\mathbb{C}_j$ has rank $(N-1)n_x$. If $S_1 \neq \emptyset$ and $rank(C_{S_2}) = (N-1)n_x$ then $rank(M_{S_1,S_2}) = Nn_x$. To see this observe that if $S_1 \neq \emptyset$ then B_{S_1} has at least rank n_x. We can reorder the columns of M_{S_1,S_2} and form a square submatrix from the reordered matrix such that the resulting submatrix has a block upper triangular structure. The diagonal blocks of the resulting block upper triangular matrix will consist of a nonzero block of B_{S_1} of rank n_x and nonzero blocks of C_{S_2} of rank $(N-1)n_x$ and hence the resulting matrix is of rank Nn_x. In order to reduce the rank of C_{S_2} we have to eliminate at least one block row $\bar{C}_i$, and hence corresponding matrices $\mathbb{C}_{ij}$, from each $\mathbb{C}_j$ and corresponding $\mathbb{C}$, but due to the structure of $\mathbb{C}$ and $\mathbb{B}$ such elimination results in appearance of a matrix of rank $2n_x$ in B_{S_1}. The diagonal blocks of the new block upper triangular submatrix will have ranks equal to $2n_x$ and $(N-2)n_x$ and hence total rank of the new submatrix is again Nn_x. We can proceed with this argument to the limiting case when there is only one or no block rows left from each $\mathbb{C}_j$ in C_{S_2}. For these cases $rank(B_{S_1}) = Nn_x$. In a case when $S_1 = \emptyset$ we return to the observability condition. The above argument shows that there does not exists disjoint sets S_1 and S_2 such that $rank(M_{S_1,S_2}) < Nn_x$. This proves the sufficiency part of the theorem.

We prove the necessity by converse. Assume that A, C is unobservable, hence for some eigenvalue of A, λ, there exists $v \neq 0$, such that, $\begin{bmatrix} A - \lambda I \\ C \end{bmatrix} v = 0$. Now we consider,

$$(\lambda I - (A_M - L_f C_f - \mathbb{B} F_f H_f \mathbb{C})) 1_N \otimes v,$$

and observe that $L_f C_f 1_N \otimes v = 0$ since $C_i v = 0$ for any $i = 1, ..., N$ and $\mathbb{B} F_f H_f \mathbb{C}\, 1_N \otimes v = 0$, because $\mathbb{C}\, 1_N \otimes v = 0$ for any v. We arrive at,

$$(\lambda I - A_M) 1_N \otimes v = 1_N \otimes (\lambda I - A) v = 0.$$

This implies that λ and $1_N \otimes v$ are an eigenpair of the system matrix $(A_M - L_f C_f - \mathbb{B} F_f H_f \mathbb{C})$ independent of a choice of the output feedback gain matrices L_f, $F_f H_f$ and hence λ is a fixed mode of the system (11). This proves that if there are no fixed modes in the system (11) then the pair A, C is observable. ∎

Theorem 1 proves that with the sufficient communication we can assign all eigenvalues of the collected estimation error dynamics (11). Now we describe algorithm which allows elimination of all fixed modes of the system (7) by introducing a minimal number of communication links between agents.

IV. Elimination algorithm

To eliminate the fixed modes of the collected estimation error dynamics by introducing communication between agents, we treat the transmitter and the receiver gains, H_f, F_f, as well as the dimensions of communication signals, $k_{ij} \leq n_x$, as the design variables. Initially we assume that there is no communication between agents. Later, following the procedure, we will be introducing communication links by selecting certain columns and rows from the input and output matrices, $\mathbb{B}$, $\mathbb{C}$, to form input and output matrices describing the introduced communication. It is important to notice that while in the proof of the Theorem 1 we operated with the blocks $\mathbb{C}_{ij}$ and hence the links with the dimensions equal to n_x, it is not necessary to use a full dimensional link to eliminate a fixed mode. By operating with individual columns and rows of $\mathbb{B}$ and $\mathbb{C}$ in the elimination procedure below, we not only define the communication topology, but also assign the dimensions, k_{ij}, of the communication links. The elimination procedure follows naturally from the proof of the Theorem 1, and consists of four steps.

First, we characterize the set of the fixed modes of the system (7). There are several methods present in the literature [1], [10], [2] which allow calculation of fixed modes of the system with decentralized output feedback. At this point we assume that there are m_f fixed modes, λ_ℓ, $\ell = 1, ..., m_f$, in the system (7) with respect to the output feedback $L_f C_f$.

Second, we define the index set $S^m \equiv \{1, ..., N\}$ and use it to enumerate the first N blocks, $\bar{B}_i$, $\bar{C}_i$, of the input matrix, $B_S = [I \ \mathbb{B}]$, and the output matrix, $C_S = \begin{bmatrix} C_f \\ \mathbb{C} \end{bmatrix}$. Hence S^m assigns indexes to the blocks of B_S and C_S corresponding to local agents' measurements. Now for each fixed mode λ_ℓ we find all matrices $M(\lambda_\ell)_{S_1^m, S_2^m}$,

$$M(\lambda_\ell)_{S_1^m, S_2^m} = \begin{bmatrix} \lambda_\ell I - A_M & B_{S_1^m} \\ C_{S_2^m} & 0 \end{bmatrix},$$

where S_1^m, S_2^m are all disjoint subsets of S^m. For each λ_ℓ there are $2^N - 2$ such matrices. For further calculations we need only matrices $M(\lambda_\ell)_{S_1^m, S_2^m}$ which have rank less than Nn_x.

At the third step, we use matrices $M(\lambda_\ell)_{S_1^m, S_2^m}$ defined in the previous step, with $rank(M(\lambda_\ell)_{S_1^m, S_2^m}) < Nn_x$ to find communication links eliminating the fixed modes. In order to find these links we augment the matrices $M(\lambda_\ell)_{S_1^m, S_2^m}$ with the columns of $\mathbb{B}$ and the corresponding rows of $\mathbb{C}$ to check for potential communication. Those combinations of columns and rows which raise the rank of the augmented matrix to Nn_x define communication links eliminating the fixed modes. For bookkeeping, we define set $\bar{S} \equiv \{1, ..., N(N-1)n_x\}$ and use it to enumerate the last $N(N-1)n_x$ columns of the input matrix B_S and the last $N(N-1)n_x$ rows of the output matrix C_S. $\bar{S}$ enumerates the columns of B_S and the rows of C_S present due to potential communication between agents and represented in B_S and C_S through $\mathbb{B}$ and $\mathbb{C}$.

For each fixed mode, λ_ℓ, we find all subsets of $\bar{S}$, $\bar{S}^i(\lambda_\ell)$, $i = 1, ..., m_\ell$ such that each subset $\bar{S}^i(\lambda_\ell)$ contains a minimum number of elements of $\bar{S}$ and the matrix $M(\lambda_\ell)_{S_1^m, S_2^m}$ has rank equal to Nn_x, when augmented with any disjoint combination of rows of $\mathbb{B}$ and columns of $\mathbb{C}$ from $\bar{S}^i(\lambda_\ell)$. Note that if the rank deficiency of $M(\lambda_\ell)_{S_1^m, S_2^m}$ is $r_\ell \equiv Nn_x - rank(M(\lambda_\ell)_{S_1^m, S_2^m})$ then each set $\bar{S}^i(\lambda_\ell)$ contains only r_ℓ elements for any $i = 1, ..., m_\ell$, because we need only r_ℓ linearly independent columns of $\mathbb{B}$ and/or rows of $\mathbb{C}$ to raise the rank to Nn_x. Each set $\bar{S}^i(\lambda_\ell)$ uniquely defines links in the communication topology and the dimensions of signals eliminating fixed mode at λ_ℓ. For each λ_ℓ a set $\bar{S}^i(\lambda_\ell)$ always exists due to Theorem 1.

At the last step we form a joint communication input matrix $\bar{B}$ and a joint communication output matrix $\bar{C}$ by picking the columns of $\mathbb{B}$ and rows of $\mathbb{C}$ which correspond to a joint set $\bar{S}^c$. To find $\bar{S}^c$, we check if there exists a set $\bar{S}^i(\lambda_\ell)$, $i = 1, ..., m_\ell$, $\ell = 1, ..., m_f$ which eliminates all fixed modes. If such $\bar{S}^i(\lambda_\ell)$ exists then a set of links necessary to eliminate one of the fixed modes is also sufficient to eliminate all others and we define $\bar{S}^c \equiv \bar{S}^i(\lambda_\ell)$. If such a set does not exist, then we define $\bar{S}^c$ such that it eliminates all fixed modes and $\bar{S}^c \equiv \bigcup^\ell \bar{S}^i(\lambda_\ell)$, where the union is taken over all ℓ and the subsets $\bar{S}^i(\lambda_\ell)$ for each ℓ are chosen to minimize the number of elements in $\bar{S}^c$. The resulting input and output communication matrices, $\bar{B}$, $\bar{C}$, uniquely define the communication topology which guarantees assignability of all eigenvalues of the collected estimation error dynamics with the minimum number of links between the agents and the minimum dimension of each communication signal. In order to minimize the number of links in the topology we should use a different rule for defining $\bar{S}^c$. Instead of minimizing the number of elements when forming $\bar{S}^c$, we should try and keep the number of links corresponding to sets $\bar{S}^i(\lambda_\ell)$ at the minimum.

Note that optimality of the above algorithm is achieved at the expense of the exhaustive search over all possible combinations of elements of the set $\bar{S}$ at the third step and the search over all sets $\bar{S}^i(\lambda_\ell)$ at the last step of the algorithm.

V. EXAMPLE

To illustrate the application of the ideas described in the previous sections we consider a simple example. Consider a system with two agents, $N = 2$, and the collective formation dynamics described by,

$$x(k+1) = \begin{bmatrix} 0.25 & 0 \\ 1 & 1.5 \end{bmatrix} x(k) + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1(k) \\ u_2(k) \end{bmatrix},$$
$$y_1(k) = [1 \ 1]x(k), \ y_2(k) = [1 \ 0]x(k). \tag{12}$$

The pair $A = \begin{bmatrix} 0.25 & 0 \\ 1 & 1.5 \end{bmatrix}$, $C = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ is clearly observable. We assume that the stabilizing state feedback for the system is given by $K = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ with agents' control inputs defined through the projection matrices $\Pi_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $\Pi_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. The closed-loop system dynamics are then given by $A_{clp} = A - B_u K = \begin{bmatrix} 0.25 & 0 \\ 1 & 0.5 \end{bmatrix}$ and A_{clp} has eigenvalues at 0.25 and 0.5.

The collected estimation error dynamics with $L_f = 0$ is given by matrix,

$$A_M = I_N \otimes A_{clp} + M_N = \begin{bmatrix} 0.25 & 0 & 0 & 0 \\ 1 & 0.5 & 0 & 1 \\ 0 & 0 & 0.25 & 0 \\ 0 & 0 & 1 & 1.5 \end{bmatrix}.$$

It has two eigenvalues at 0.25, one at 0.5 and one unstable eigenvalue at 1.5. Now we would like to characterize the set of fixed modes of the collected estimation error dynamics and apply the elimination algorithm if necessary.

First, we would like to check if the collected estimation error dynamics has fixed modes with respect to the distributed output feedback introduced by L_fC_f. For that purpose we define the set $S^m = \{1,2\}$, with the first element corresponding to agent 1 and $[C_1 \;\; 0_{1\times 2}]$, $\begin{bmatrix} I_{2\times 2} \\ 0_{2\times 2} \end{bmatrix}$ output-input matrices, and the second element corresponding to agent 2 and the matrices $[0_{1\times 2} \;\; C_2]$, $\begin{bmatrix} 0_{2\times 2} \\ I_{2\times 2} \end{bmatrix}$. We calculate the matrices $M(0.25)_{1,2}$, $M(0.25)_{2,1}$, $M(0.5)_{1,2}$, $M(0.5)_{2,1}$, $M(1.5)_{1,2}$, and $M(1.5)_{2,1}$. We find that $rank(M(\lambda_\ell)_{S_1^m,S_2^m}) \geq Nn_x$ for all $\lambda = 0.25, 0.5, 1.5$ and S_1^m, S_2^m except $M(1.5)_{1,2}$. For this matrix, $rank(M(1.5)_{1,2}) = 3 < Nn_x = 4$, hence $z = 1.5$ is a fixed mode of the collected estimation error dynamics with respect to the decentralized output feedback L_fC_f. $z = 1.5$ is the only fixed mode of the system.

The original system (12) is observable and by Theorem 1 there exists a communication topology and a combination of gains which eliminate the fixed mode, hence we proceed to the second step of the elimination algorithm. We have already calculated in the previous step matrices $M(\lambda_\ell)_{S_1^m,S_2^m}$ and defined ones which have rank deficiency. To find the communication topology which eliminates the fixed mode at $z = 1.5$ we need only to consider augmenting the matrix $M(1.5)_{1,2}$.

Now we proceed to the third step of the algorithm and construct the communication input and output matrices for the system. We define $\mathbb{C}_{21} = [-I_2 \; I_2]$, $\mathbb{C}_{12} = [I_2 \; -I_2]$, and $\mathbb{B}_{21} = [0_2 \; I_2]'$, $\mathbb{B}_{12} = [I_2 \; 0_2]'$. The collected output matrix which corresponds to the communication is $\mathbb{C} = [\mathbb{C}_1' \; \mathbb{C}_2']' = [\mathbb{C}_{21}' \; \mathbb{C}_{12}']'$ and the collected input matrix is $\mathbb{B} = [\mathbb{B}_1 \; \mathbb{B}_2] = [\mathbb{B}_{21} \; \mathbb{B}_{12}]$. Since $rank(M(1.5)_{1,2}) = 3$ and the rank deficiency, $r = Nn_x - 3 = 1$, we should be able to eliminate the fixed mode at $z = 1.5$ with only one communication link of a dimension one. This corresponds to using a single column and a single row from the communication input and output matrices $\mathbb{B}$, $\mathbb{C}$ respectively. After enumerating columns and corresponding rows of $\mathbb{B}$ and $\mathbb{C}$ we check potential sets $\bar{S}^i(1.5)$ with only one element in each set, $\{1\}$, $\{2\}$, $\{3\}$, or $\{4\}$. In each pair of the corresponding disjoints subsets $\bar{S}_1^i(1.5)$ or $\bar{S}_2^i(1.5)$ only one of the subsets is not empty since there is only one element in each potential $\bar{S}^i(1.5)$.

Going through calculations we observe that ranks of the following matrices,

$$\begin{bmatrix} M(1.5)_{1,2} & \begin{bmatrix} \mathbb{B}_{\{1\}} \\ 0 \end{bmatrix} \end{bmatrix}, \quad \begin{bmatrix} M(1.5)_{1,2} \\ \begin{bmatrix} \mathbb{C}_{\{1\}} & 0_{1\times 2} \end{bmatrix} \end{bmatrix},$$
$$\begin{bmatrix} M(1.5)_{1,2} & \begin{bmatrix} \mathbb{B}_{\{2\}} \\ 0 \end{bmatrix} \end{bmatrix}, \quad \begin{bmatrix} M(1.5)_{1,2} \\ \begin{bmatrix} \mathbb{C}_{\{2\}} & 0_{1\times 2} \end{bmatrix} \end{bmatrix},$$

are equal to 4, while ranks of,

$$\begin{bmatrix} M(1.5)_{1,2} & \begin{bmatrix} \mathbb{B}_{\{3\}} \\ 0 \end{bmatrix} \end{bmatrix}, \quad \begin{bmatrix} M(1.5)_{1,2} & \begin{bmatrix} \mathbb{B}_{\{4\}} \\ 0 \end{bmatrix} \end{bmatrix},$$

are equal to 3. The first two columns of $\mathbb{B}$ and the corresponding first two rows of $\mathbb{C}$ describing communication link from agent 1 to agent 2 raise the rank of the above matrix to Nn_x and hence eliminate the fixed mode. We define $\bar{S}^1(1.5) = \{1\}$, $\bar{S}^2(1.5) = \{2\}$. Communication from agent 2 to agent 1 is not able to eliminate the unstable fixed mode, because $\mathbb{B}_{\{3\}}$, $\mathbb{B}_{\{4\}}$, and hence $\mathbb{B}_{\{3,4\}}$ do not change the rank of the above matrices. The introduction of the communication link from agent 1 to agent 2 with nonzero transmitter and receiver gains $H_{21} = [h_1 \; h_2]$, $F_{21} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$ allows us to assign all eigenvalues of the complete closed-loop system.

The last step of the elimination algorithm is redundant since the above system had only one fixed mode.

VI. Conclusion

In this paper we showed that if the formation is observable from the collective output, then there exists a communication topology and a combination of observer, transmitter and receiver gains which allow us to assign all the poles of the decentralized distributed estimator. We also proposed a method for the design of a minimal communication topology which eliminates the fixed modes. Although after introducing the appropriate communication into the estimation structure we can assign all the poles, it is not clear if we can assign all poles in arbitrary locations. A further investigation of structural properties of the system is necessary to answer that question. Another interesting direction of future research is investigation of pole assignability in the decentralized distributed estimator with uncertain communication topology where communication links are subject to failures.

References

[1] D.B.O. Anderson and D.J. Clements. Algebraic characterization of fixed modes in decentralized control. *Automatica*, 17:703–712, 1981.

[2] E.J. Davison and U. Ozguner. Characterization of decentralized fixed modes for interconnected systems. *Automatica*, 19(2):169–182, 1983.

[3] J. Fax and R. Murray. Information flow and cooperative control of vehicle formations. *Transactions on Automatic Control*, 49(9):1465–1476, 2004.

[4] P. Ferguson and J. How. Decentralized estimation algorithms for formation flying spacecraft. In *Proceedings of the AIAA Guidance, Navigation and Control Conference*, pages 1–12, Austin, Texas, 2003.

[5] K. Kang, J. Yan, and R. Bitmead. Communication design for coordinated control with a non-standard information structure. In *Proceedings of the IEEE Conference on Decision and Control*, pages 7078–7083, Seville, Spain, 2005.

[6] R.S. Smith and F.Y. Hadaegh. Closed-loop dynamics of cooperative vehicle formations with parallel estimators and communication. *IEEE Transactions on Automatic Control*, 52(8):1404–1414, 2007.

[7] M.V. Subbotin and R.S. Smith. Design of a distributed decentralized estimator for a formation with fixed communication topology. In *Proceedings of the American Control Conference*, pages 5025–5030, 2007.

[8] M.V. Subbotin and R.S. Smith. Distributed estimator design for a formation with markovian communication topology. In *Proceedings of the American Control Conference*, pages 5046–5051, 2007.

[9] M. Tarokh. Fixed modes in multivariable systems using constrained controllers. *Automatica*, 21(4):495–497, 1985.

[10] M. Tarokh and M. Jamshidi. Elimination of decentralized fixed modes with minimum number of interconnection gains. *Large Scale Systems*, 11:207–215, 1986.

[11] S. Tatikonda and S. Mitter. Control under communication constraits. *IEEE Transactions on Automatic Control*, 49:1056–1068, 2004.