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A Linear Reaction Technique for Dynamic Asset Allocation in the Presence of Transaction Costs

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Abstract— **Institutional investors manage their strategic mix of asset classes over time to achieve favorable returns in spite of uncertainties. A fundamental issue in this context is to maintain risk under control while achieving the desired return targets. When the asset mix is to be re-balanced many times over the investment horizon, the decision maker faces a rather difficult constrained dynamic optimization problem that should take into account conditional decisions based on future market behavior. This problem is usually solved approximately using scenario-based stochastic programming: a technique that suffers from serious problems of numerical complexity due to the intrinsic combinatorial nature of scenario trees.**

In this paper, we present a novel and computationally efficient approach to constrained discrete-time dynamic asset allocation over multiple periods. This technique is able to control portfolio expectation and variance at both final and intermediate stages of the decision horizon, and may account for proportional transaction costs and intertemporal dependence of the return process. A key feature of the proposed method is the introduction of a linearly-parameterized class of feedback reaction functions, which permits to obtain explicit analytic expressions for the portfolio statistics over time. These expressions are proved to be convex in the decision parameters, hence the multi-stage problem is formulated and solved by means of efficient tools for quadratic or second-order-cone convex programming.

Keywords: **Control of financial risk, strategic asset allocation, multi-stage decision problems, dynamic optimization, convex optimization, portfolio optimization, transaction costs.**

I. INTRODUCTION

The classical Markowitz mean-variance portfolio model [17], [18] assumes a single time period for the determination of optimal asset allocation. For an investor with a long-term investment horizon the question arises whether the optimal multi-period portfolio has any relationship to optimal holdings in a single-period setting. In his seminal works [19], [20], Merton proposes a continuous-time stochastic dynamic programming approach to the multi-period allocation problem. Despite the fact that this model is quite complicated and unsuitable to efficient practical implementation, Merton manages to show that if the objective is maximization of a utility function and if the assets expected returns and covariances are constant over the decision horizon, then the asset allocations are also constant and equal to the one-period solutions for a mean-variance investor, that is, there is no intertemporal hedging demand in this case.

However, Merton's formulation only considers maximization of the terminal wealth (i.e. it ignores the *path* and risk taken to arrive at the terminal wealth) and does not take into account cost of transactions or the possibility of imposing constraints on the portfolio composition at the final and intermediate stages. When these realistic features are added to the model, the myopic one-period approach is no longer equivalent to the optimal multi-period one, hence an investment advantage may be gained by employing a decision model that fully exploits the intertemporal structure of the problem. Moreover, multi-period problems in presence of transaction costs and constraints are in general not tractable analytically by means of the optimal control techniques employed for instance in [5], [6], [10], [19], [20].

The mainstream computational model for dealing with multi-period decision problems in presence of uncertainty is currently provided by multi-stage stochastic programming, see e.g. [1], [2], [3], [14], [24], [26] and the many references therein. However, while stochastic programming provides a conceptually sound framework for posing multi-stage decision problems, from the computational side it results to be impervious to exact and efficient numerical solution, [25]. The key difficulty in the stochastic programming formulation comes from the fact that the stage decisions are actually conditional decision rules, or "policies" that define which action should be taken in response to past outcomes. To model the conditional nature of the problem in some "tractable" way, a discretization of the decision space is typically introduced by constructing a "scenario tree," and this scenario tree may grow exponentially if an accurate and representative discretization is needed, see e.g. [12]. On the other hand, if branching is kept low in the scenario tree, the resulting discretization cannot be guaranteed to be a reliable representation of reality. These computational difficulties are witnessed by the fact that most multi-period problems discussed in the literature deal with few securities over only two or three periods.

In the specific context of financial allocation, a classical stochastic programming method based on Benders decomposition is proposed in [11], and techniques for construction of scenario trees are discussed for instance in [21], [23], [28]. Scenario-based stochastic programming models for portfolio optimization have also been recently proposed in [15], [22]. A survey with theoretical analysis of multi-period models based on scenario trees is provided in [27].

In this paper we propose a different route to multi-stage allocation, which prescinds from the use of decision trees or sample paths, and which leads to explicit convex programming models that can be solved globally and efficiently. We achieve these goals by considering a restricted class of decision policies that are affine functions of the past return innovations. Within this setting, we obtain exact and explicit

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expressions for portfolio expectations and volatilities at all stages of the decision horizon, and these expressions result to be affine or convex quadratic functions of the decision parameters. This approach thus leads to computationally tractable optimization problems (convex quadratic or second order cone programming problems) whose solutions provide suboptimal policies, since affinely parameterized recourse functions constitute only a subclass of the set of possible recourse relations. This sub-optimality is however largely counterweighted by numerical tractability of the ensuing optimization programs, which enables application of multi-stage techniques to real-world investment endeavors. The idea of using affine recourse policies has been first proposed in [9], where a problem limited to two stages and with no transaction costs was considered; a generalization to number of periods larger than two was then proposed in [7]. An extended version of the present paper is available in [8].

a) Notation: If $x \in \mathbb{R}^{n,1}$, then $\mathrm{diag}(x)$ denotes a diagonal matrix having the entries of x on the diagonal. $A^\top$ denotes the transpose of matrix A. The operator $\odot$ denotes the Hadamard (entry-wise) product of conformably sized matrices. I_n, 0_n denote the identity matrix and the zero matrix in $\mathbb{R}^{n,n}$, respectively. For a random vector x, $\mathrm{E}\{x\}$ denotes the expected value of x, and $\mathrm{var}\{x\} \doteq \mathrm{E}\left\{(x-\mathrm{E}\{x\})(x-\mathrm{E}\{x\})^\top\right\}$ its covariance matrix. We shall denote with an over bar the expectation of random quantities and with a tilde the centered quantities, i.e. the quantities with the expectation subtracted, that is $\bar{x} \doteq \mathrm{E}\{x\}$, $\tilde{x} \doteq x - \bar{x}$.

II. Setup and Preliminaries

Consider an investment problem involving n assets or asset classes $\{a_1, \ldots, a_n\}$, which may include cash, over a decision horizon of T periods, where the k-th period starts at time $k-1$ and ends at time k. We denote with $x_i(k)$ the Euro value of the portion of the investor's total wealth invested in security a_i at time k. The *portfolio* at time k is the vector

$$x(k) \doteq \left[\begin{array}{ccc} x_1(k) & \cdots & x_n(k) \end{array}\right]^\top,$$

being $x(0)$ the initial portfolio. The investor's total wealth at time k is

$$w(k) \doteq \sum_{i=1}^{n} x(k) = \mathbf{1}^\top x(k),$$

where $\mathbf{1}$ denotes a vector of ones. At the end of each period, the investor has the opportunity of adjusting his/her investment, by rebalancing the portfolio composition. Specifically, we denote with $x^+(k)$ the portfolio composition just after the adjustment $u(k)$ occurred at time k:

$$x^+(k) \doteq x(k) + u(k),$$

where $u(k) \doteq [u_1(k) \ \cdots \ u_n(k)]^\top$ is the vector of portfolio adjustments. A value of $u_i(k) > 0$ indicates that the portfolio content in asset a_i is increased by $u_i(k)$ Euros (by buying this asset), whereas $u_i(k) < 0$ indicates that the portfolio content in asset a_i is decreased by $u_i(k)$ Euros (by selling this asset). If $p_i(k)$ denotes the market value of a_i at time k, then the (simple) *return* of investment in security a_i over the period of time $[k-1, k]$ is $r_i(k) \doteq (p_i(k) - p_i(k-1))/p_i(k-1) = p_i(k)/p_i(k-1) - 1$, and the one-period *gain* of the same investment is

$$g_i(k) \doteq \frac{p_i(k)}{p_i(k-1)} = r_i(k) + 1.$$

We denote with $g(k)$ the vector collecting the asset gains, and with $G(k) = \mathrm{diag}(g(k))$ the corresponding diagonal matrix. The portfolio composition then evolves in time according to the recursive equations

$$\begin{aligned} x^+(k) &= x(k) + u(k) &\quad (1)\\ x(k+1) &= G(k+1)x^+(k), \quad k = 0, 1 \ldots, T-1. &\quad (2) \end{aligned}$$

Here, we take a standard stochastic view of the market and consider the asset gains $g(k)$, $k = 1, \ldots$, to follow a possibly non-stationary discrete-time stochastic process with finite and possibly time-varying means and covariances. Equation (2) thus specifies a stochastic discrete-time system which describes the time evolution of the portfolio.

A. Problem statement

The purpose of our decision model is to determine investment adjustment *policies* (in our context, a policy is a function expressing $u(k)$ in terms of the previous market gains $g(t)$, $t = 1, \ldots, k$) so to attain a desired target expected return at the end of the investment horizon, while maintaining the associated risk and transaction costs under control. The expected value of cumulative gross return of the investment over the whole horizon is $\mathcal{P} \doteq \mathrm{E}\{w(T)\}/w(0)$. The *risk* associated with the investment strategy is here measured by a weighted sum of wealth variances (volatilities) at all the decision stages:

$$\mathcal{R} \doteq \sum_{k=1}^{T} v_k \mathrm{var}\{w(k)\}, \quad (3)$$

where $v_k \geq 0$ are given weights. For instance, $v_k = 1/T$, $k = 1, \ldots, T$, sets $\mathcal{R}$ to measure the average wealth variance over the decision horizon, whereas $v_k = 0$, $k = 1, \ldots, T-1$, $v_T = 1$, sets $\mathcal{R}$ to measure only end-of-horizon variance.

Let $c_i \geq 0$ denote the proportional transaction cost coefficient for trading in asset a_i. The cost due to all transactions at time k is given by $\sum_{i=1}^{n} c_i |u_i(k)| = \|c \odot u(k)\|_1$, where $\|\cdot\|_1$ denotes the ℓ_1 vector norm and $c^\top = [c_1 \ \cdots \ c_n]$ is the vector of unit transaction costs. The total expected transaction cost over the investment horizon is hence

$$\mathcal{C} \doteq \mathrm{E}\left\{\sum_{k=0}^{T-1} \|c \odot u(k)\|_1\right\}. \quad (4)$$

We consider the portfolio to be self-financing, that is $\sum_{i=1}^{n} u_i(k) = 0$, except that for transaction costs, which are covered by newly injected cash.

Usually, constraints need be enforced on portfolios. These constraints include for instance portfolio composition conditions (minimum and maximum exposure in individual assets or in groups of assets), or no-shortselling constraints. In this paper, we include generic linear constraints in the model by imposing that the expected value of the updated portfolios

$x^+(k)$ lie within a given polytope $\mathcal{X}(k)$. The multi-period asset allocation problem (MAP) we are interested in can now be formally stated as follows:

$$\tilde{J}^* \doteq \min_{u(k)\in\mathcal{U}} \mathcal{R} + \gamma\mathcal{C} \quad (5)$$
$$\text{subject to:} \quad \mathrm{E}\{x^+(k)\} \in \mathcal{X}(k),\ k = 0,\ldots,T-1$$
$$\mathbf{1}^\top u(k) = 0,\ k = 0,\ldots,T-1$$
$$\mathrm{E}\{w(T)\} \geq \psi \cdot w(0),$$

where $\mathcal{U}$ is the class of strictly causal functions of $g(1),\ldots,g(T)$, $\gamma \geq 0$ is a relative objective weight parameter, and ψ is the given target final return.

We notice that considering a general set $\mathcal{U}$ of causal recourse functions in problem (5) makes this problem extremely hard to solve in practice. Indeed, even the computation of the objective may not be tractable for a general set $\mathcal{U}$ of causal functions. On the other extreme, a conventional approximation providing a lower bound on the objective of (5) is to restrict $\mathcal{U}$ to the class of "open-loop" strategies, that is, control functions that are actually independent of the returns. This latter approach, however, might be too coarse and fails to capture the dynamic nature of the decision problem at hand, that is the fact that only the first decision $u(0)$ need actually be determined at time $k = 0$ (the so-called "here-and-now" variable), whereas the decision maker can wait and see what progressively happens to the market before actually deciding the subsequent adjustments $u(1),\ldots,u(T-1)$.

In the following section we consider a tractable approximation to the above problem that uses a restricted class of recourse functions (namely, affine recourse functions). Then, in Section III-B we derive upper and lower bounds for the expectation of the transaction cost term $\mathcal{C}$. The combination of affine recourse strategy and upper-bound approximation for the transaction cost term leads to a problem formulation that is efficiently computable by means of convex programming.

III. Affine Recourse and Portfolio Dynamics

In our approach we postulate that the control action $u(k)$ is an affine, strictly causal function of the returns' innovations:

$$u(0) = \bar{u}(0),$$
$$u(k) = \bar{u}(k) + \sum_{\tau=1}^{k} \Theta_\tau(k)\tilde{g}(\tau),\ k = 1,\ldots,T-1, \quad (6)$$

where $\bar{u}(k) \in \mathbb{R}^n$, $k = 0,\ldots,T-1$, are the nominal portfolio adjustments, with $u(0) = \bar{u}(0)$ representing the "here-and-now" decision variable, and $\Theta_\tau(k) \in \mathbb{R}^{n,n}$, $k = 1,\ldots,n$, $\tau = 1,\ldots,k$, are additional decision variables representing *market reaction* matrices. We rewrite (6) in more compact notation as

$$\begin{aligned} u(0) &= \bar{u}(0), \\ u(k) &= \bar{u}(k) + \mathbf{\Theta}(k)[\mathbf{g}(k) - \bar{\mathbf{g}}(k)], \end{aligned} \quad (7)$$

with $\mathbf{\Theta}(k) \doteq [\Theta_1(k) \cdots \Theta_k(k)] \in \mathbb{R}^{n,kn}$, $\mathbf{g}(k) \doteq [g^\top(1) \cdots g^\top(k)]^\top \in \mathbb{R}^{nk}$.

Remark 1 (Interpretation of the recourse policy): Besides rendering the multi-stage decision model efficiently solvable (as it will be shown shortly), the affine recourse policy (7) also bears a sound interpretation, as explained next. The initial portfolio adjustment $u(0) = \bar{u}(0)$ is the "here-and-now" variable, describing the first "move" to be done by the investor. The successive adjustments $u(k)$, $k = 1,\ldots,T-1$, are "wait-and-see" variables whose value is assumed to be decomposed in the sum of a nominal decision $\bar{u}(k)$ and a recourse term. The nominal adjustment $\bar{u}(k)$ represents the move we would make at time k, if the market during periods preceding k performed as expected. Since the market will never perform exactly as expected, we correct the nominal decision $\bar{u}(k)$ with a term proportional to all past market deviations from expectation. The coefficients of the correction are collected in the market reaction matrices $\Theta_\tau(k)$. In particular, element $[\Theta_\tau(k)]_{ij}$ in row i and column j of matrix $\Theta_\tau(k)$ represents the sensitivity of the control action in the i-th security, $u_i(k)$, with respect to deviations from expectation of the return of the j-th security in the τ-th period. $\diamondsuit$

A. Dynamics of portfolio statistics

Applying the control policy (7) to the portfolio recursion (1), (2), we obtain the following (stochastic) recursions for the controlled portfolio

$$\begin{aligned} x^+(k) &= x(k) + \bar{u}(k) + \mathbf{\Theta}(k)[\mathbf{g}(k) - \bar{\mathbf{g}}(k)] \quad (8) \\ x(k+1) &= G(k+1)x^+(k), \quad k = 0, 1\ldots,T-1, (9) \end{aligned}$$

with $\mathbf{\Theta}(0) = 0$. Our objective in this section is to obtain analytic expressions for the portfolio expectations and covariances. To this end, from repeated application of (9) we obtain the expression for the portfolio composition at a generic instant k:

$$x(k) = \Phi(1,k)x(0) + \mathbf{\Phi}(k)\nu(k) + \sum_{i=1}^{k-1}\sum_{\tau=1}^{i}\Phi(i+1,k)\Theta_\tau(i)\tilde{g}(\tau) \quad (10)$$

where

$$\nu(k) \doteq \begin{bmatrix} \bar{u}(0) \\ \bar{u}(1) \\ \vdots \\ \bar{u}(k-1) \end{bmatrix},\ \mathbf{\Phi}(k) \doteq \begin{bmatrix} \Phi(1,k) & \cdots & \Phi(k,k) \end{bmatrix},$$

being $\Phi(i,k)$, $i \leq k$, a diagonal matrix of compound gains from beginning of period i to end of period k:

$$\Phi(i,k) \doteq G(k)G(k-1)\cdots G(i) \doteq \mathrm{diag}(\varphi(i,k)).$$

Let now $[\Theta_\tau(i)]_c$ denote the c-th column of matrix $\Theta_\tau(i)$, $\theta(k) \in \mathbb{R}^{n^2 k(k-1)/2}$ represent the vector containing all columns $[\Theta_\tau(i)]_c$ ordered lexicographically according to the triple (i,τ,c), for $i = 1,\ldots,k-1$, $\tau = 1,\ldots,i$, $c = 1,\ldots,n$, and let $\mathbf{\Psi}(k) \in \mathbb{R}^{n,nk(k-1)/2}$ be the corresponding matrix formed by horizontal juxtaposition of the diagonal $\Phi(i+1,k)\tilde{g}_c(\tau)$ blocks, in the same lexicographic order. With this notation, $x(k)$ in (10) can be written in the form

$$x(k) = \Phi(1,k)x(0) + \mathbf{\Phi}(k)\nu(k) + \mathbf{\Psi}(k)\theta(k),$$

where $\Phi(1,k)$, $\mathbf{\Phi}(k)$, $\mathbf{\Psi}(k)$ contain all terms depending on stochastic gains, while decision variables are contained in $\nu(k)$, $\theta(k)$. This formalism is useful for expressing the portfolio expectation and wealth variance, since we have

$$\bar{x}(k) = \mathrm{E}\{x(k)\} = \bar{\Phi}(1,k)x(0) + \bar{\mathbf{\Phi}}(k)\nu(k) + \bar{\mathbf{\Psi}}(k)\theta(k), \quad (11)$$

$$\begin{aligned} \mathrm{var}\{w(k)\} &= \mathrm{E}\left\{\tilde{x}^\top(k)\mathbf{1}\mathbf{1}^\top\tilde{x}(k)\right\} \quad (12) \\ &= \begin{bmatrix} x(0) \\ \nu(k) \\ \theta(k) \end{bmatrix}^\top Q(k) \begin{bmatrix} x(0) \\ \nu(k) \\ \theta(k) \end{bmatrix}, \end{aligned}$$

were we defined $\tilde{\phi}(k) \doteq \tilde{\mathbf{\Phi}}^\top(k)\mathbf{1}$, $\tilde{\psi}(k) \doteq \tilde{\mathbf{\Phi}}^\top(k)\mathbf{1}$,

$$Q(k) \doteq \mathrm{E}\left\{ \begin{bmatrix} \tilde{\varphi}(1,k) \\ \tilde{\phi}(k) \\ \tilde{\psi}(k) \end{bmatrix} \begin{bmatrix} \tilde{\varphi}(1,k) \\ \tilde{\phi}(k) \\ \tilde{\psi}(k) \end{bmatrix}^\top \right\}.$$

The previous algebraic derivations actually prove the following statement.

Lemma 1: Consider the portfolio dynamic equations under affine policy (8), (9), and let $G(t) = \mathrm{diag}(g(t))$, $t = 1,2,\ldots$, be a generic, possibly dependent, stochastic return process. Then, the portfolio expectation $\mathrm{E}\{x(k)\}$ is an affine function of the policy parameters $\bar{u}(t)$, $t = 0,1,\ldots,k-1$, and the total wealth variance $\mathrm{var}\{w(k)\}$ is a convex quadratic function of these parameters and of $\mathbf{\Theta}(t)$, $t = 1,\ldots,k-1$.

Notice that although (11), (12) show that the expected portfolio is affine in the decision variables and that the wealth variance is convex and quadratic in the variables, these function are not easy to obtain exactly in practice, since they depend on parameters $\bar{\Phi}(1,k)$, $\bar{\mathbf{\Phi}}(k)$, $\bar{\mathbf{\Psi}}(k)$, $Q(k)$ that are difficult to evaluate for generic and serially correlated return processes. We shall discuss in Section IV-A an approximated technique for computing the parameters $\bar{\Phi}(1,k)$, $\bar{\mathbf{\Phi}}(k)$, $\bar{\mathbf{\Psi}}(k)$, $Q(k)$ in full generality. Also, a key result given in Section V shows how to compute portfolio expectations and covariances *exactly*, under a standard efficient market hypothesis.

B. Bounds on expected transaction cost

The result in the previous lemma implies that, under affine policy, the risk term $\mathcal{R}$ in (3) can be explicitly expressed as a convex quadratic function of the model decision variables $\bar{u}(k)$ and $\mathbf{\Theta}(k)$, and that the final expected wealth $\mathrm{E}\{w(T)\}$ is explicitly given as an affine function of the $\bar{u}(k)$'s. In order to obtain a full convex approximation to problem (5), however, we still need to elaborate on the expected transaction cost term $\mathcal{C}$ in (4), which cannot be expressed in analytic form. In this section, we develop computable upper and lower bounds for the expected transaction cost under the affine recourse policy. First notice that using the recourse rule (7) in (4), we have that

$$\mathcal{C} = \sum_{k=0}^{T-1} \mathrm{E}\left\{\|c \odot (\bar{u}(k) + \mathbf{\Theta}(k)\tilde{\mathbf{g}}(k))\|_1\right\}. \quad (13)$$

The following lemma holds.

Lemma 2 (Upper and lower bounds on $\mathcal{C}$): The expected transaction cost term in (13) is upper and lower bounded as $\underline{\mathcal{C}} \le \mathcal{C} \le \bar{\mathcal{C}}$, where

$$\underline{\mathcal{C}} \doteq \sum_{k=0}^{T-1} \|c \odot \bar{u}(k)\|_1 \quad (14)$$

$$\bar{\mathcal{C}} \doteq \sum_{k=0}^{T-1}\sum_{i=1}^{n} c_i \|\xi_i(k)\|_2, \quad (15)$$

being $\xi_i(k) \doteq \begin{bmatrix} \bar{u}_i(k) & \mathbf{\Theta}_i^\top(k)\Upsilon(k) \end{bmatrix}$, where $\mathbf{\Theta}_i^\top(k)$ denotes the i-th row of $\mathbf{\Theta}(k)$, and $\Upsilon(k)$ is a full-rank factor such that $\Upsilon(k)\Upsilon^\top(k) = \Sigma(k) \doteq \mathrm{var}\{x(k)\}$.

Proof. We first prove the lower bound $\underline{\mathcal{C}} \le \mathcal{C}$. To this end, we recall that the Jensen's inequality (see, e.g., [4]) states that for any random variable X and convex function $f(\cdot)$ it holds that $f(\mathrm{E}\{X\}) \le \mathrm{E}\{f(X)\}$. Considering in particular $f(\cdot) = |\cdot|$, we have that $|\mathrm{E}\{X\}| \le \mathrm{E}\{|X|\}$, thus

$$\begin{aligned} \mathcal{C} &= \sum_{k=0}^{T-1}\sum_{i=1}^{n} c_i \mathrm{E}\left\{|\bar{u}_i(k) + \mathbf{\Theta}_i^\top(k)\tilde{\mathbf{g}}(k)|\right\} \\ &\ge \sum_{k=0}^{T-1}\sum_{i=1}^{n} c_i \left|\mathrm{E}\left\{\bar{u}_i(k) + \mathbf{\Theta}_i^\top(k)\tilde{\mathbf{g}}(k)\right\}\right| \\ &= \sum_{k=0}^{T-1}\sum_{i=1}^{n} c_i |\bar{u}_i(k)| = \sum_{k=0}^{T-1} \|c \odot \bar{u}(k)\|_1 = \underline{\mathcal{C}}. \end{aligned}$$

To prove the upper bound $\mathcal{C} \le \bar{\mathcal{C}}$, we consider again Jensen's inequality with $f(\cdot) = (\cdot)^2$ applied to the random variable $|X|$, which gives $\mathrm{E}\{|X|\} \le \sqrt{\mathrm{E}\{X^2\}}$, hence

$$\begin{aligned} \mathcal{C} &= \sum_{k=0}^{T-1}\sum_{i=1}^{n} c_i \mathrm{E}\left\{|\bar{u}_i(k) + \mathbf{\Theta}_i^\top(k)\tilde{\mathbf{g}}(k)|\right\} \\ &\le \sum_{k=0}^{T-1}\sum_{i=1}^{n} c_i \sqrt{\mathrm{E}\left\{(\bar{u}_i(k) + \mathbf{\Theta}_i^\top(k)\tilde{\mathbf{g}}(k))^2\right\}} \\ &= \sum_{k=0}^{T-1}\sum_{i=1}^{n} c_i \|\xi_i(k)\|_2 = \bar{\mathcal{C}}. \qquad \square \end{aligned}$$

Note that both the upper and lower bounds for the expected transaction cost are convex functions of the decision variables. In particular, $\underline{\mathcal{C}}$ is the sum of absolute values of decision variables, whereas $\bar{\mathcal{C}}$ is the sum of Euclidean norms of linear functions of the decision variables.

IV. Convex MAP Formulations

The multi-period allocation problem stated in (5), under the affine restriction (7) on the allowed recourse functions, is written as

$$\begin{aligned} J^* \doteq \min_{\bar{u}(k)\in\mathbb{R}^n, \mathbf{\Theta}(k)\in\mathbb{R}^{n,nk}} \ & \mathcal{R} + \gamma\mathcal{C}, \quad \text{subject to:} \quad (16) \\ & \mathrm{E}\left\{x^+(k)\right\} \in \mathcal{X}(k),\ k = 0,\ldots,T-1, \\ & \mathbf{1}^\top\bar{u}(k) = 0,\ \mathbf{1}^\top\mathbf{\Theta}(k) = 0, \quad k = 0,\ldots,T-1, \\ & \mathbf{\Theta}(0) = 0_n,\ \mathrm{E}\{w(T)\} \ge \psi \cdot w(0). \end{aligned}$$

Since the affine recourse class is included in the generic causal class $\mathcal{U}$ considered in (5), it clearly holds that $\tilde{J}^* \le$

J^*. Notice that, due to the presence of transaction cost term $\mathcal{C}$, the problem is not yet formulated as a convex program. However, using the convex relaxations developed in the previous section, we can formulate two convex programs that describe upper and lower bounds on J^*, as formalized in the following proposition.

Proposition 1 (Convex formulation of MAP): Consider the multi-period asset allocation problem with affine recourse in (16), having optimal value J^*. We have that

$$J^*_{\rm lb} \le J^* \le J^*_{\rm ub},$$

where $J^*_{\rm lb}$, $J^*_{\rm ub}$ are respectively the optimal values of the following convex optimization problems:

$$J^*_{\rm lb} \doteq \min_{\bar{u}(k)\in\mathbb{R}^n, \mathbf{\Theta}(k)\in\mathbb{R}^{n,nk}} \mathcal{R} + \gamma\underline{\mathcal{C}}, \quad \text{subject to:} \tag{17}$$
$$\mathrm{E}\left\{x^+(k)\right\} \in \mathcal{X}(k),\ k = 0,\ldots,T-1,$$
$$\mathbf{1}^\top \bar{u}(k) = 0,\ \mathbf{1}^\top \mathbf{\Theta}(k) = 0, \quad k = 0,\ldots,T-1,$$
$$\mathbf{\Theta}(0) = 0_n, \quad \mathrm{E}\left\{w(T)\right\} \ge \psi \cdot w(0),$$

$$J^*_{\rm ub} \doteq \min_{\bar{u}(k)\in\mathbb{R}^n, \mathbf{\Theta}(k)\in\mathbb{R}^{n,nk}} \mathcal{R} + \gamma\bar{\mathcal{C}}, \quad \text{subject to:} \tag{18}$$
$$\mathrm{E}\left\{x^+(k)\right\} \in \mathcal{X}(k),\ k = 0,\ldots,T-1,$$
$$\mathbf{1}^\top \bar{u}(k) = 0,\ \mathbf{1}^\top \mathbf{\Theta}(k) = 0, \quad k = 0,\ldots,T-1,$$
$$\mathbf{\Theta}(0) = 0_n, \quad \mathrm{E}\left\{w(T)\right\} \ge \psi \cdot w(0),$$

and where $\underline{\mathcal{C}}$, $\bar{\mathcal{C}}$ are given in (14) and (15), respectively.

A. Sampling approximations

We now go back to expressions (11), (12) for the expected portfolios and wealth variance. Since the parameters $\bar{\Phi}(1,k)$, $\bar{\mathbf{\Phi}}(k)$, $\bar{\mathbf{\Psi}}(k)$, $Q(k)$ appearing in these expressions are not in general expressible in closed form (see Section V for an important exception), we here propose a standard sampling technique to approximate them. Indeed, we notice that, if a stochastic dynamic model is given for the return process $g(k)$, the conditional expectations in (11) and the conditional covariance matrix $Q(k)$ can be efficiently approximated by their empirical counterparts. That is, performing N stochastic simulations of the return series $g(t)$, $t = 1,\ldots,T$, we may build empirical expectations

$$\hat{\Phi}(1,k) = \tfrac{1}{N}\textstyle\sum_{s=1}^{n} \Phi_s(1,k),\ \hat{\mathbf{\Phi}}(k) = \tfrac{1}{N}\textstyle\sum_{s=1}^{n} \mathbf{\Phi}_s(k),$$
$$\hat{\mathbf{\Psi}}(k) = \tfrac{1}{N}\textstyle\sum_{s=1}^{n} \mathbf{\Psi}_s(k),$$
$$\hat{Q}(k) = \tfrac{1}{N-1}\textstyle\sum_{s=1}^{n} \begin{bmatrix} \tilde{\varphi}_s(1,k) \\ \tilde{\phi}_s(k) \\ \tilde{\psi}_s(k) \end{bmatrix} \begin{bmatrix} \tilde{\varphi}_s(1,k) \\ \tilde{\phi}_s(k) \\ \tilde{\psi}_s(k) \end{bmatrix}^\top,$$

where the subscript s denotes the value of the subscripted quantity obtained in the s-th simulation. By the law of large numbers, the above quantities tend to the actual expectations with probability one as N goes to infinity. In practice, they may represent good approximations of the actual expectations, for sufficiently large N. Notice that, although simulations are used in order to estimate these model parameters, the quantities to be estimated do not depend on the decision variables and can be estimated a-priori, that is before the actual optimization is run. Therefore, this approach does not require the construction of exponentially growing scenario trees. For example, using this proposed sampling approach, the multi-stage problems (17), (18) are directly approximated by substituting the sampled quantities in the expressions for the portfolio expectations and covariances (11), (12).

We observe that this approximate formulation is to be used when a full stochastic model for return dynamics is available, and that this setting allows for consideration of completely general inter-temporal statistical dependence in the return process. In the next section we show that under the widely accepted hypothesis of efficient markets we can actually obtain *exact* expressions for the portfolio expectations and covariances and hence exact and explicit convex formulations of the multi-period allocation problem.

V. The Efficient Market Case

In this section we introduce a standard "efficient market" hypothesis (EMH, see [13], [16]) on the return process, that is we assume that gains over different period are statistically independent.

Assumption 1 (Independent returns): Gain $g_i(k_1)$ is statistically independent of $g_j(k_2)$, $\forall\, i,j$ and $\forall\, k_2 \neq k_1$. ⋆

As we shall see, acceptance of this hypothesis permits analytic recursive expressions for the portfolio expectations and covariances, thus avoiding to resort to the sampling approximation of Section IV-A. Also, we shall verify that these expressions only depend on the first two conditional moments of the gain vectors $g(k)$, $k = 1,2,\ldots$:

$$\begin{aligned} \bar{g}(k) &\doteq \mathrm{E}\left\{g(k)\right\}, \quad k = 1,2,\ldots \\ \Sigma(k) &\doteq \mathrm{var}\left\{g(k)\right\} = M(k) - \bar{g}(k)\bar{g}^\top(k),\ k = 1,2,\ldots, \end{aligned}$$

with $M(k) \doteq \mathrm{E}\left\{g(k)g^\top(k)\right\}, \quad k = 1,2,\ldots,$

where the expectations in the previous equations are conditional to past history up to decision time $k = 0$.

The following key lemma states explicitly the dynamic equations for expectation and covariances of the controlled portfolio, under the independence hypothesis in Assumption 1. The proof of this lemma is lengthy and it is therefore skipped here for space reasons.

Lemma 3: (Dynamics of portfolio expectation and covariance) Let Assumption 1 be satisfied, and consider the affinely controlled portfolio equations (8), (9). Then, the controlled portfolio expectation $\bar{x}(k) = \mathrm{E}\left\{x(k)\right\}$, for $k = 0,\ldots,T-1$, obeys to the recursion

$$\bar{x}(k+1) = \bar{G}(k+1)\bar{x}^+(k), \quad \text{with } \bar{x}^+(k) = \bar{x}(k) + \bar{u}(k),$$

while the portfolio covariance $\Gamma(k) \doteq \mathrm{E}\left\{\tilde{x}(k)\tilde{x}^\top(k)\right\}$ follows the recursion

$$\begin{aligned} \Gamma(k+1) = \bar{x}^+(k)\bar{x}^{+\top}(k) \odot \Sigma(k+1) + \Big(\Gamma(k) + \Omega(k)\mathbf{\Theta}^\top(k) \\ + \mathbf{\Theta}(k)\Omega^\top(k) + \mathbf{\Theta}(k)D(k)\mathbf{\Theta}^\top(k)\Big) \odot M(k+1), \end{aligned}$$

initialized with $\Gamma(1) = \bar{x}^+(0)\bar{x}^{+\top}(0)\odot\Sigma(1)$, where $D(k) \doteq \mathrm{diag}(\Sigma(1),\ldots,\Sigma(k))$, and $\Omega(k) \doteq \mathrm{E}\left\{\tilde{x}(k)\tilde{\mathbf{g}}^\top(k)\right\} \in \mathbb{R}^{n,kn}$

is given for $k = 1, \ldots, T-1$ by a parallel recursion

$$\Omega(k) = \Big[\bar{G}(k)\Big(\Omega(k-1) + \mathbf{\Theta}(k-1)D(k-1)\Big)\Big| \\ \mathrm{diag}(\bar{x}^{+}(k-1))\Sigma(k)\Big],$$

initialized with $\Omega(1) = \mathrm{diag}(\bar{x}^{+}(0))\Sigma(1)$. For the total wealth variance $\mathrm{var}\{w(k)\}$, it holds that

$$\mathrm{var}\{w(k+1)\} = \mathbf{1}^\top\Gamma(k+1)\mathbf{1} = \\ \bar{x}^{+\top}(k)\Sigma(k+1)\bar{x}^{+}(k) + \\ \mathrm{Tr}\Big(\Gamma(k) + \mathbf{\Theta}(k)D(k)\mathbf{\Theta}^\top(k) + 2\mathbf{\Theta}(k)\Omega^\top(k)\Big)M(k+1)$$

$\star$

Remark 2: The expressions of $\bar{x}(k)$ and $\mathrm{var}\{w(k)\}$ resulting from the recursions in Lemma 3 can be used in the problem formulations of Section IV to obtain exact and explicit convex formalizations of the MAP problem. These explicit recursions clearly produce a great gain in terms of pre-processing time and absence of approximation errors with respect to the sampling approach discussed in Section IV-A. Also, application of these recursions only requires knowledge of the first and second moments of the return vectors, which are indeed the quantities that are usually available from return time series analysis. These exact recursions should therefore be used in place of the sampling approximations, whenever the efficient market hypothesis can reasonably be assumed to hold (that is, most of the times in practical asset allocation applications). $\Diamond$

VI. Numerical Examples

Examples could not be included here for space reasons. However, the interested reader may find some numerical tests in an extended version [8] of this paper (available on line). I thank Dr. Alberto Cattaneo at ERSEL, Torino, for providing some of the data used in the tests.

VII. Conclusions

In this paper we presented a mean-variance computational model for dynamic asset allocation over multiple discrete-time periods. The main features of the model are (a) the flexibility of dealing with inter-period and end-of-horizon constraints on portfolio expectations and covariances, (b) the inclusion of proportional transaction costs, (c) the generality of stochastic return models that can be used with the method (for instance, non-stationary and possibly time-correlated return processes), and (c) the efficiency in numerical solution, which derives by the finite-dimensional convex representation of the problem. The method completely avoids scenario trees and stochastic approximations. Also, in the efficient market case, the proposed approach provides explicit analytic recursions for the statistics of the controlled portfolio.

The mentioned strong points are achieved at the expense of a degree of sub-optimality of the method, since an affine restriction on the class of reaction functions is imposed a priori. It should however be remarked that a method for efficient and exact computation of the actual optimal solution of the considered problem is to date unavailable. Also, approximate techniques such as stochastic programming currently appear to be able to deal only with problems with few assets and very few periods. It is thus expected that the proposed technique could be employed with success in practical medium-sized problems involving a number of assets and periods of the order of tenths.

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