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A Lagrangian Approach to Constrained Potential Games: Theory and Examples

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Abstract— **In this paper, we use a Lagrangian approach to solve for Nash equilibrium in a continuous non-cooperative game with coupled constraints. We discuss the necessary and the sufficient conditions to characterize the equilibrium of the constrained games. In addition, we discuss the existence and uniqueness of the equilibrium. We focus on the class of potential games and point out a relation between potential games and centralized optimization. Based on these results, we illustrate the Lagrangian approach with symmetric quadratic games and briefly discuss the notion of game duality. In addition, we discuss two engineering potential game examples from network rate control and wireless power control, for which the Lagrangian approach simplifies the solution process.**

I. INTRODUCTION

Classical Lagrangian multiplier theory is well-known to solve standard nonlinear mathematical programming problems, in which an objective function is destined to be optimized subject to certain given constraints [1]. From a system's perspective, such type of optimization problems can be viewed as a centralized approach in that there is one single agent who has the complete system information, searching for an optimal solution. Modern systems are growing in more complexity and are composed of different sub-units forming different hierarchies, for example, optical networks, biological systems, and the Internet. A central type of classical optimization is not practical in terms of information delivery, system scaling and etc. Recently, game-theoretical approach has been considered as an alternative, especially in the application of large-scale network systems, multi-agent systems, market-based systems, etc., [2]. Its distributed feature and inherent assumption of self-optimizing rationality make it a suitable framework to model systems with interactive self-interest components, or agents.

In many current literature, the application of games are mostly limited to problems without coupled constraints, i.e., users' sets of strategies are independent of one another. However, in most engineering scenarios, such restrictions lead to impractical solutions, as most of the existing systems are constrained in a coupled and interative fashion. For example, capacity constraint is one of the notable constraints in a resource allocation problem. Therefore, it is essential to establish a theoretical basis to study non-cooperative games (NG) with coupled constraints.

Attempts for such extension were made in [3] and [4], where the payoff functions of the players are modified to include penalty terms that reduces the payoff when the coupled constraints are violated. This approach circumvents the problem of a constrained game by turning it into an unconstrained game and thus solved in a known way. Since the penalty functions are commonly chosen to be a reciprocal function of the constraints or a logarithmic function of multi-variables, it becomes easily intractable when multiple constraints are involved and utilities become nonlinear and strongly coupled. Though such approach employs the current results on non-cooperative games, for example in [5], it is still challenging to perform analysis on the model. On the other hand, such approach is strongly problem dependent. A different problem may need a different construction of modified payoff functions.

Due to such technical inconvenience, a lot of current applications resort to numerical analysis, such as in [6], [7] and [8]. They follow a semi-analytical approach in which the Nash equilibrium is obtained following relaxation algorithms are derived based on a best response function from maximizing a certain function, e.g. Nikaido-Isoda function in [6] and game potential function in [7]. Some of the approaches are restricted to games with certain properties and are not widely applicable.

In [9], an extension is made on the duality theory to the game-theoretical framework, but the theory is developed by augmenting cost functions into a single game cost function as a two-argument function. Such extension builds upon a component-wise necessary condition for a fixed point solution as indicated in the proof of Lemma 2 in [9]. This result is not further elaborated but used to develop an involved discussion of duality and hierarchical decomposition of constrained games in [9]. We find that the argument made in Lemma 2 is particularly central for continuing the work by Rosen in [10]. In [10], the work is focus on characterizing the uniqueness of a N-person concave game using the condition of diagonally strict concavity. The discussion of a Lagrangian approach in [9] and [10] inspire us to investigate further on using this approach to solve constrained games. We state necessary and sufficient conditions that characterize the NE with a game Lagrangian. With the same techniques used in [10], we are able to state conditions for existence and uniqueness of Nash equilibrium.

Based on these results, we are particularly interested in a class of potential games, recently proposed in [11]. Using a Lagrangian approach, we can arrive at a simple characterization via centralized optimization. This discovery potentially simplifies the techniques used in [12] and supports the approach adopted in [7] for a constrained wireless power control. In addition, the well-known result in [11] on uniqueness and existence of a pure Nash equilibrium of potential games can further be extended to the case of such

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games with coupled constraints.

It should be pointed out that a critical assumption made throughout this paper is that constraints are common information to all the players. This assumption is reasonable since in most engineering situations constraints are open to all users, for example, capacity constraints in power control games. However, for the situation in which part of the constraints are not open information, we need to make a modification using information sets on the Lagrangian so that the constraints are user dependent. We will briefly elaborate this in the directions of the future work.

The contribution of this paper is to 1) review and rigorously state a Lagrangian approach for solving games with coupled constraints; 2) extend results of potential games to the constrained case and establish a connection with classical mathematical programming with potential game theory; 3) briefly discuss the concept of duality in quadratic games that has a computational advantage with lower dimension; 4) point out its practical applications to solve the problem in engineering such as network rate control problem and power control problem in wireless networks. Most notably, the price of anarchy of rate control game described in [12] appears to be 1, independent of constraints. Moreover, for the strongly coupled wireless power control game, we use a parametric approach to design a game for a certain efficiency (price of anarchy) criterion.

This paper is organized as follows. In section II-A, we first review some notations of unconstrained non-cooperative game and concepts from convex games and potential games. In section II-B, we discuss the necessary and the sufficient conditions of constrained Nash games and the issue of existence and uniqueness. In section II-C, we apply these results to potential games and build a connection between a centralized optimization with a potential game problem. In section III-A, we use symmetric quadratic games as an example to illustrate the theory and briefly discuss the concept of game duality. In section III-B, we study potential games arising from network rate control game and wireless power control game. We conclude and point out future direction of research in section IV.

II. Lagrangian Approach to Constrained Games

In this section, we extend the classical Lagrangian multiplier theory for a standard optimization problem to a non-cooperative game with coupled constraints. We assume that the coupled constraints are common information so that every player is subject to the same set of constraints. Applying optimization theory to constrained games has been seen in [10] and more recently in [9]. In [10], a differential form of the necessary and sufficient Kuhn-Tucker (KKT) conditions is used to discuss the uniqueness and existence of Nash equilibrium for a constrained game. However, it was not evident that Lagrangian theory extends to games as well. On the other hand, in [10], an extension of duality theory is made to solve a general class of constrained games. The idea centers around a hierarchical decomposition of the original constrained problem into two unconstrained sub-problems. Lemma 2, upon which the extension is built, is essentially a component-wise Lagrangian approach that coincides with [10]. We rigorously review and connect these two work together and state the following results on the necessary and the sufficient conditions as well as the existence and uniqueness.

A. Preliminaries and Definitions

Let's consider a non-cooperative game (NG) defined by a triplet $\Xi = \langle \mathcal{N}, (A_i), J_i \rangle$, $i \in \mathcal{N}$, where $A_i = [u_{i,\min}, u_{i,\max}]$ is the continuous strategy set, $J_i : \Omega = \prod_i^N A_i \to \mathcal{R}$ is the cost function and $\mathcal{N} = \{1, 2, \cdots, N\}$ is the index set of players. Each player behaves according to its best response function $BR_i(\mathbf{u}_{-i})$ to minimize its cost, without knowing other player's strategy or behavior.

Definition 2.1: Consider an N-player game, in which each player minimizes the cost functions $J_i : \Omega \to \mathcal{R}$. A vector $\mathbf{u}^* = [u_i^*]$ or $\mathbf{u}^* = (\mathbf{u}_{-i}^*, u_i^*) \in \Omega$ is called a Nash equilibrium (NE) of this game if $u_i^* \in BR_i(\mathbf{u}_{-i}^*), \forall i \in \mathcal{N}$, or equivalently, $J_i(u_i^*, \mathbf{u}_{-i}^*) \le J_i(u_i', \mathbf{u}_{-i}^*), \forall u_i' \in A_i, \forall i \in \mathcal{N}$

Definition 2.2: A NG Ξ is a constrained NG if Ξ is subject to coupled inequality constraints $g_i(\mathbf{u}) \le 0, i = \{1, \cdots, M\}$ or $\mathbf{g}(\mathbf{u}) \le 0$ in vector form, with $\mathbf{g}(\mathbf{u}) = [g_1(\mathbf{u}), \cdots, g_M(\mathbf{u})]^T$. We denote such constrained game as a quartet $\Xi_{\mathbf{g}} = \langle N, A_i, J_i, \overline{\Omega} \rangle$. Let $\overline{\Omega} = \{\mathbf{u} \mid \mathbf{g}(\mathbf{u}) \le 0\}$. A point $\mathbf{u}$ is feasible if $\mathbf{u}^* \in \overline{\Omega} \cap \Omega$. $\mathbf{u}^*$ is an NE solution to $\Xi_{\mathbf{g}}$ if

$$J_i(u_i^*, \mathbf{u}_{-i}^*) \le J_i(u_i', \mathbf{u}_{-i}^*), \forall u_i' \in \overline{\Omega}_i(\mathbf{u}_{-i}^*), \forall i \in \mathcal{N},$$

where $\overline{\Omega}_i(\mathbf{u}_{-i}^*)$ is the projection set defined as

$$\overline{\Omega}_i(\mathbf{u}_{-i}^*) = \{u_i \in A_i \mid (u_i', \mathbf{u}_{-i}^*) \in \overline{\Omega} \cap \Omega\}.$$

Definition 2.3: ([11]) A NG Ξ is a potential game (PG) if there exists a function $\Phi(\mathbf{u}) : \Omega \to \mathcal{R}$ such that for all $i \in \mathcal{N}$ and $(u_i, \mathbf{u}_{-i})$ and $(u_i', \mathbf{u}_{-i}) \in \Omega$:

$$J_i(u_i, \mathbf{u}_{-i}) - J_i(u_i', \mathbf{u}_{-i}) = \Phi(u_i, \mathbf{u}_{-i}) - \Phi(u_i', \mathbf{u}_{-i}).$$

If the cost function J_i is continuously differentiable, then a NG is a PG if there exists a potential function $\Phi(\mathbf{u}) : \Omega \to \mathcal{R}$, such that

$$\frac{\partial J_i}{\partial u_i} = \frac{\partial \Phi}{\partial u_i}, \forall i \in \mathcal{N}.$$

Definition 2.4: A potential game Ξ is called a convex potential game (CPG) if $\Phi(\mathbf{u})$ and $\mathbf{g}(\mathbf{u})$ are convex in $\mathbf{u} \in \Omega \cap \overline{\Omega}$. We call a CPG a strictly convex potential game (SCPG) if $\Phi(\mathbf{u})$ is strictly convex.

The definition of CPG allows us to categorize an important class of tractable problems for which we have classical analysis tools for solving NE.

B. A Lagrangian Approach to Non-cooperative Games with Coupled Constraints

In this section, we study a general Lagrangian framework for continuous non-cooperative games with coupled constraints. In particular, we state the necessary and the sufficient conditions for constraints games and then use the strictly diagonally convex condition as in [10] to characterize the existence and uniqueness of the Nash equilibrium.

Theorem 2.1: ([13]) A NG Ξ has a Nash equilibrium if for all $i \in \mathcal{N}$,

- (A1) the set A_i is nonempty compact convex subset of $\mathcal{R}$.
- (A2) J_i is continuous and quasi-convex on A_i.

Theorem 2.2: Consider a NG $\Xi_{\mathbf{g}}$ with constraints given by $\mathbf{g}(\mathbf{u}) \leq 0$. Let $\mathcal{A}(\mathbf{u}) = \{r \mid g_r(\mathbf{u}) = 0\}$ denote the set of active constraints and $\overline{\mathcal{A}} = \mathcal{N}\backslash\mathcal{A}$ the set of non-active constraints. Assume

- (C1) $J_i, i \in \mathcal{N}$ is continuously differentiable.
- (C2) $g_r, r \in \mathcal{A}$ is continuously differentiable.
- (C3) $A_i, i \in \mathcal{N}$ is convex and compact.

(a) (Necessity) Assume that, in addition, $J_i, i \in \mathcal{N}$ is also quasi-convex, and $g_r, r \in \mathcal{A}$ are also convex. Let $\mathbf{u}^*$ be an NE solution. Then there exist unique $\nu^*, \nu_r^* \geq 0, r \in \mathcal{A}$ such that

$$\frac{\partial \mathcal{L}_i}{\partial u_i}(u_i^*, \mathbf{u}_{-i}^*, \nu^*) = 0, \tag{1}$$

and $\nu_r^* = 0, \forall r \in \overline{\mathcal{A}}$, i.e.,

$$\nu_r^* g_r(\mathbf{u}) = 0, r \in \mathcal{N} \tag{2}$$

where $\mathcal{L}_i, \forall i \in \mathcal{N}$ is the game Lagrangian defined by

$$\mathcal{L}_i(u_i, \mathbf{u}_{-i}, \nu) = J_i(u_i, \mathbf{u}_{-i}) + \nu^T \mathbf{g}(u_i, \mathbf{u}_{-i}).$$

(b) (Sufficiency) Let $\mathbf{u}^*$ be a feasible point, i.e., $\mathbf{u}^* \in \Omega \cap \overline{\Omega}$ and ν^* be the game Lagrangian multipliers such that $\nu_r^* g_r(\mathbf{u}) = 0, r \in \mathcal{N}$. If $\mathbf{u}^*$ minimizes the game Lagrangian $\mathcal{L}_i, \forall i$, then $\mathbf{u}^*$ is an NE to the constrained Nash game.

Proof: Omitted due to length ■

Remark 2.1: We observe that the sufficiency doesn't require assumptions of convexity in the utility functions and constraints. In addition, the necessity part of Theorem 2.2 only assumes quasi-convexity of J_i for the existence of NE. It relaxes the assumption of convexity in [9] in Lemma 2.

Existence of NE of a game is usually shown using Kakutani fixed point theorem [13], whereas, the uniqueness of NE can be quite involved. For a two-person game described in Proposition 4.1 in [5] may need a stringent condition for uniqueness. In the following, we adopt the definition of strict diagonal convexity from [10] and show that uniqueness of NE under such conditions.

Definition 2.5: A map $\mathbf{J}(\mathbf{u}) : \Omega \in \mathcal{R}^N \to \mathcal{R}^N$ is strictly diagonally convex (SDC) [10] if

$$(\mathbf{u}^1 - \mathbf{u}^2)^T \left(D_{\mathbf{u}}\mathbf{J}(\mathbf{u}^1) - D_{\mathbf{u}}\mathbf{J}(\mathbf{u}^2)\right) > 0,$$

for every $\mathbf{u}^1, \mathbf{u}^2 \in \Omega$, where $\mathbf{J} = [J_1(\mathbf{u}), \cdots, J_N(\mathbf{u})]^T$ and $D_{\mathbf{u}}\mathbf{J}(\mathbf{u}) = [\frac{dJ_1}{du_1}(\mathbf{u}), \frac{dJ_2}{du_2}(\mathbf{u}), \cdots, \frac{dJ_N}{du_N}(\mathbf{u})]^T$.

Similarly, $J_i, i \in \mathcal{N}$ is diagonally convex if we replace the strict inequality with inequality. It is also easy to verify that for a potential game with strict convex potential function $\Phi(\mathbf{u})$, the cost functions J_i form a diagonally strict convex map.We will use this fact later in Corollary 2.8 to show the uniqueness of NE in strictly convex potential games. Moreover, when a game Ξ is linear [14], i.e., $D_{\mathbf{u}}\mathbf{J}(\mathbf{u}) = \mathbf{C}\mathbf{u} - \mathbf{d}$, for some $\mathbf{C} \in \mathcal{R}^{N\times N}$ and $\mathbf{b} \in \mathcal{R}^N$, then J_i yields a diagonally strictly convex map if $\mathbf{C}$ is positive definite. To see this, we let $\mathbf{u}^1$ and $\mathbf{u}^2$ be two feasible points and, for $i = 1, 2$, we have $D_{\mathbf{u}}\mathbf{J}(\mathbf{u}^i) = \mathbf{C}\mathbf{u}^i - \mathbf{d}$. Take the difference between the two expressions and we arrive at $(\mathbf{u}^1 - \mathbf{u}^2)^T\mathbf{C}(\mathbf{u}^1 - \mathbf{u}^2) > 0$ for all $\mathbf{u}^1, \mathbf{u}^2 \in \Omega$ if $\mathbf{C}$ is positive definite.

Theorem 2.3: (Existence and Uniqueness of NE in Constrained Nash Game $\Xi_{\mathbf{g}}$) Consider a NG $\Xi_{\mathbf{g}}$ in Theorem 2.2 and the conditions (C1)-(C2) on J_i, g_r, and A_i are satisfied. In addition, J_i is quasi-convex and g_r is convex.

- **(Existence)** There exists an NE to $\Xi_{\mathbf{g}}$.
- **(Uniqueness)** There exists a unique NE to $\Xi_{\mathbf{g}}$ provided that J_i is strictly diagonally convex.

Proof: Omitted due to length. ■

Remark 2.2: From the uniqueness proof, we can also state that NE is unique when J_i is diagonally convex (not strictly) but $\mathbf{g}(\mathbf{u})$ is strictly convex (instead of just being convex). In this case, we have $W_1 \leq 0$ but $W_2 < 0$. Applying the same argument, we can arrive at the conclusion of uniqueness of NE.

C. Constrained Potential Games

Following the last section, we are equipped to study potential games subject to coupled constraints. With the definition of the potential games with continuous strategy set in section II-A, we are able to simplify the Lagrangian characterization for this special type of games. From Definition 2.3, we have $\frac{\partial J_i}{\partial u_i} = \frac{\partial \Phi}{\partial u_i}, \forall i \in \mathcal{N}$, and subsequently, the necessary condition (1) of Theorem 2.2 is reduced to

$$\frac{\partial \mathcal{L}_\Phi}{\partial u_i}(u_i, \mathbf{u}_{-i}, \nu) = 0, \tag{3}$$

or in terms of the constraints and potential function,

$$\frac{\partial \Phi}{\partial u_i}(u_i, \mathbf{u}_{-i}) + \nu^T \frac{\partial \mathbf{g}}{\partial u_i}(u_i, \mathbf{u}_{-i}) = 0, \tag{4}$$

where $\mathcal{L}_\Phi = \Phi(\mathbf{u}) + \nu^T\mathbf{g}(\mathbf{u})$, and $\nu^T\mathbf{g}(\mathbf{u}) = 0$.

Theorem 2.4: Suppose there exists a potential function $\Phi(\mathbf{u}) : \Omega \to \mathcal{R}$ of the NG Ξ. The solution to the optimization problem (PPG) is a NE to the constrained Nash game $\Xi_{\mathbf{g}}$.

$$\begin{array}{ll} \text{(PPG)} & \min \Phi(\mathbf{u}) \\ \text{subject to} & \mathbf{g}(\mathbf{u}) \leq 0, \mathbf{u} \in \Omega. \end{array}$$

Proof: Let $\mathbf{u}'$ be the optimal solution to (PPG). Since $\mathbf{u}'$ minimizes $\mathcal{L}_\Phi$, $\mathbf{u}'$ minimizes $\mathcal{L}_i, \forall i$. Therefore, $\mathbf{u}'$ is a NE to the constrained game from the sufficiency of Theorem 2.2. ■

It should be noted that under general cases, not every equilibrium from the constrained game solves (PPG) because (3) only gives a necessary condition. It is essential to use other criteria to choose from the candidate solutions the one that minimizes the Lagrangian $\mathcal{L}_i(\mathbf{u})$, or $\Phi(\mathbf{u})$.

Corollary 2.5: Every potential game such that $\Phi(\mathbf{u})$ is continuous and $\Omega \cap \overline{\Omega}$ is nonempty and compact has a NE.

Proof: Using Weierstrass' Theorem, we can conclude that there always exists an optimal solution $\mathbf{u}^{\text{opt}}$ to (PPG).

From Theorem 2.4, $\mathbf{u}^{\text{opt}}$ is a corresponding NE to the constrained noncooperative game. Therefore, the result follows. ∎

Corollary 2.5 indicates that quasi-convexity or convexity are not necessary to ensure the existence of NE, but rather just the continuity and compactness of the feasible set. Under the case where no coupled constraints are present, the result of Corollary 2.5 actually coincides with Lemma 4.3 from [11], stating that '*Every continuous potential game with compact strategy sets possesses a pure-strategy equilibrium point.*' Corollary 2.5 extends this result to a potential game with coupled constraints.

Corollary 2.6: Assume that $\Phi(\mathbf{u})$ is continuous and differentiable on $\Omega \cap \overline{\Omega}$. If a potential game is convex, then solving the constrained Nash potential game is equivalent to solving (PPG).

Proof: From Theorem 2.2, the necessary condition for an NE is given by

$$\frac{\partial \mathcal{L}_\Phi}{\partial u_i} = 0. \tag{5}$$

Since $\Phi(\mathbf{u})$ and $\mathbf{g}(\mathbf{u})$ are convex in $\mathbf{u} \in \Omega \cap \overline{\Omega}$ from Definition 2.4, (5) also becomes a sufficient condition for NE. Therefore, the problem of solving constrained Nash game is equivalent to solving the convex program (PPG). ∎

Theorem 2.7: (Saddle-point Characterization of CPG) Consider a convex potential game (CPG) defined in Definition 2.4. Assume the constraints satisfy Slater's condition, i.e., there exists a vector $\mathbf{u}' \in \Omega \cap \overline{\Omega}$ such that $g_r(\mathbf{u}') < 0$, for every $r \in \mathcal{M}$. Then $\mathbf{u}^*$ is an NE if an only if there exists $\nu^* \in \mathcal{R}^M$, $\nu^* \geq 0$ such that

$$\mathcal{L}_\Phi(\mathbf{u}^*, \nu) \leq \mathcal{L}_\Phi(\mathbf{u}^*, \nu^*) \leq \mathcal{L}_\Phi(\mathbf{u}, \nu^*),$$

for every $\mathbf{u} \in \Omega \cap \overline{\Omega}$ and $\nu \in \mathcal{R}^M_+$.

Proof: From Corollary 2.6, we note that a CPG has a corresponding convex program. The proof of Theorem 2.7 immediately follows from the saddle-point characterization of optimality for a convex program, as appeared in [1] and [15]. ∎

The saddle point theorem does not assume differentiability or continuity and has its implication in duality of the convex potential games, i.e., we can find the optimal solution (1) either fixing ν and minimizing with respect to $\mathbf{u}$, or (2) fixing $\mathbf{u}$ and maximizing the Lagrangian with respect to ν. It can be shown easily that under the Slater's condition, the two approaches give rise to the same optimal value of $\mathcal{L}_\Phi$, i.e. the property of strong duality. We will use this result for quadratic games in section III-A to discuss the notion of game duality.

Lastly, we use Theorem 2.3 to prove the the uniqueness of the NE to convex potential game (CPG).

Corollary 2.8: Every strictly convex potential game (SCPG) admits a unique equilibrium

Proof: Since $\Phi(\mathbf{u})$ is strictly convex in $\mathbf{u}$, thus, it is equivalent that

$$(\nabla\Phi(\mathbf{u}^1) - \nabla\Phi(\mathbf{u}^2)) \cdot (\mathbf{u}^1 - \mathbf{u}^2) > 0.$$

Therefore, $J_i, \forall i \in \mathcal{N}$ is strictly diagonally convex. Directly following Theorem 2.3, we obtain the result. ∎

We can also prove Corollary 2.8 from a different perspective, using the result in section II-C. Since $\Phi(\mathbf{u})$ is strictly convex, (PPG) has a unique solution. According to Theorem 2.6, this solution is a NE to the potential game. Due to the game equivalence with optimization problem (PPG) from Corollary 2.6, we can conclude that the solution is also unique for the game. Following the Remarks 2.2, we can also modify Corollary 2.8 and state that convex potential games (not strictly convex) have unique equilibrium if $\mathbf{g}(\mathbf{u})$ is strictly convex.

III. Examples and Applications

In this section, we first demonstrate the use of the Lagrangian approach for quadratic games subject to linear constraints. We also use examples from network rate control and wireless power control to show the simplification of finding Nash equilibrium using results in section II.

A. Symmetric Quadratic Games

Quadratic game is an important class of linear games which are analytically tractable [5]. The case without constraints has been discussed in [5]. We first assume that the linear best response functions are given by $BR_i(\mathbf{u}) = \mathbf{c}_i^T\mathbf{u} + d_i$. Let $\mathbf{C} \in \mathcal{R}^{N\times N}$ be a matrix with its i-th column given by $\mathbf{c}_i^T$ and $\mathbf{d} = \mathcal{R}^N = [d_i]$. We consider a symmetric case in which game possesses potential function is a quadratic form, i.e., $\mathbf{C}$ has the symmetry such that $C_{ij} = C_{ji}$. We denote quadratic games with such property symmetric quadratic games (SQG). We can introduce a general potential function $\Phi_S(\mathbf{u})$ for SQG.

$$\Phi_S(\mathbf{u}) = \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} C_{ij}u_iu_j - \sum_{i=1}^{N} d_iu_i = \frac{1}{2}\mathbf{u}^T\mathbf{C}\mathbf{u} - \mathbf{d}^T\mathbf{u}. \tag{6}$$

For an SQG without constraints, with the assumption that $\mathbf{C}$ is positive definite, it is obvious to verify that the unconstrained Nash equilibrium is a solution to $\min_{\mathbf{u}\in\mathcal{R}^+} \Phi_S(\mathbf{u})$. Therefore, the Nash solution is uniquely determined by $\mathbf{u}^* = \mathbf{C}^{-1}\mathbf{d}$.

Let's now consider an SQG with linear constraints $\mathbf{g}(\mathbf{u}) = \mathbf{B}\mathbf{u} - \mathbf{v} \leq 0$. We again assume that $\mathbf{C}$ is positive definite. From Corollary 2.6, we can conclude that a Nash equilibrum is a solution from (QCPG) and, from Theorem 2.7, we are ready to use the duality theory to solve and derive an iterative algorithm for the equilibrium.

$$\begin{array}{ll} \text{(QCPG)} & \min_{\mathbf{u}} \Phi_S(\mathbf{u}) \\ \text{subject to} & \mathbf{B}\mathbf{u} \leq \mathbf{v}, \mathbf{u} \geq 0. \end{array}$$

The problem (QCPG) is in the form of a standard quadratic programming problem (QCPG). The Lagrangian is given by

$$\mathcal{L}(\mathbf{u}, \nu) = \frac{1}{2}\mathbf{u}^T\mathbf{C}\mathbf{u} - \mathbf{d}^T\mathbf{u} + \nu^T(\mathbf{B}\mathbf{u} - \mathbf{v}).$$

Since $\mathcal{L}$ is convex, a necessary and sufficient condition for a minimum $\mathbf{u}^*$ is to satisfy

$$\mathbf{C}\mathbf{u} + \mathbf{B}^T\nu - \mathbf{d} = 0.$$

that is, the Nash solution $\mathbf{u}^*$ is given by

$$\mathbf{u}^* = -\mathbf{C}^{-1}(\mathbf{B}^T\nu - \mathbf{d}). \tag{7}$$

It is easy to observe that (7) coincides with the result in Theorem 2.2 obtained in [14]. We can easily convert (QCPG) into an unconstrained dual problem (DQPG) given by

$$\begin{array}{ll}\text{(DQPG)} & \min D_S(\nu) \\ \text{subject to} & \nu \geq 0.\end{array}$$

where $D_S = \frac{1}{2}\nu^T\mathbf{D}\nu + \nu^T\mathbf{h} - \frac{1}{2}\mathbf{d}^T\mathbf{C}^{-1}\mathbf{d}$, and $\mathbf{D} = -\mathbf{B}\mathbf{C}^{-1}\mathbf{B}^T, \mathbf{h} = -\mathbf{v} + \mathbf{B}\mathbf{C}^{-1}\mathbf{d}$.

At this stage, we can introduce the concept of duality in quadratic games. We denote $\Xi_d = \langle M, B_i, U_i, \overline{\Omega}_d\rangle$ as the dual game with respect to $\Xi_{\mathbf{g}} = \langle N, A_i, J_i, \overline{\Omega}\rangle$. Let $B_i = [\nu_{\min}, \nu_{\max}]$ and

$$U_i = \sum_{j\neq i} D_{ij}\nu_j\nu_i + h_i\nu_i + \frac{1}{2}D_{ii}\nu_i^2.$$

In dual game Ξ_d, each individual maximize his utility U_i with respect to u_i rather than minimizing his cost function as in $\Xi_{\mathbf{g}}$. It is obvious that the dual game of a quadratic game is also a quadratic game and that the NE of $\Xi_{\mathbf{g}}$ and the NE of Ξ_d are related by (43). The dual game Ξ_d does not have coupled constraints, i.e., $\overline{\Omega}_d = \mathcal{R}^M$. In addition, the dual game Ξ_d has less dimension when $M < N$.

B. Engineering Applications

1) Network Rate Control: In this section, we discuss a network rate allocation problem inspired by recent effort to alleviate congestion in the Internet [16], [17]. Consider a network with L links and N users. Let $\mathcal{N} = \{1, 2, \cdots, N\}$ denote the set of users, $\mathcal{L} = \{1, 2, \cdots, L\}$ the set of links and $x_i \in A_i \subset \mathcal{R}$ the rate allocated to i-th user, where $A_i = [x_{i,\min}, x_{i,\max}]$. Each user $i \in \mathcal{N}$ has a utility function $U_i(x_i) : \mathcal{R} \to \mathcal{R}$ and each link has a capacity $c_l, l \in \mathcal{R}$. Let $\mathbf{A} = [A_{li}] \in \mathcal{R}^{L\times N}$ be the matrix describing the user-link relation of the network, i.e.,

$$A_{ij} = \begin{cases} 1, & \text{if user } i \text{ uses link } l; \\ 0, & \text{otherwise.}\end{cases}$$

The network system is subject to the capacity constraint of $\sum_{i\in\mathcal{N}} A_{li}x_i \leq c_l$, or in the matrix form $\mathbf{A}\mathbf{x} \leq \mathbf{c}$, i.e., $\mathbf{c} = [c_l]$, a vector of link constraints. In addition, we let $C_i(x_i) : \mathcal{R} \to \mathcal{R}$ be the pricing term for each user, and thus the payoff function of each user as $F_i(x_i) = U_i(x_i) - C_i(x_i)$ for $i \in \mathcal{N}$. Therefore, using the game-theoretical framework in [18] and [12], we have a game $\Xi_R = \langle \mathcal{N}, (A_i), F_i\rangle$ subject to the capacity constraint $\mathbf{A}\mathbf{x} \leq \mathbf{c}$. The approach used in [18] is to embed the constraints as penalty terms into the payoff function so that the payoff decreases when the constraint is violated. This method is analytically cumbersome, especially when we have multiple constraints. We observe that the game Ξ_R is a potential game due to the fact that each user has his own utility decoupled from other users. The potential function $\Phi_R(\mathbf{x})$ for this game is simply the sum of the payoff functions, i.e.,

$$\Phi_R(\mathbf{x}) = \sum_{i\in\mathcal{N}} F_i(x_i). \tag{8}$$

Therefore, a Nash solution $\mathbf{x}^*$ to the constrained game can be found by the optimal solution $\mathbf{x}^{\text{opt}}$ to the optimization problem (PPGF).

$$\begin{array}{ll}\text{(PPGF)} & \max_{\mathbf{x}} \Phi_R(\mathbf{x}) \\ \text{subject to} & \mathbf{A}\mathbf{x} \leq \mathbf{c}, x_i \in A_i.\end{array}$$

With the assumption of concavity of $\Phi_R(\mathbf{x})$, we can establish the equivalence between (PPGF) and Ξ_R. It should be noted that the relation between (PPGF) and Ξ gives rise to a centralized optimization scheme proposed in [19]. Therefore, we can state that the Nash equilibrium to the network rate control problem coincides with the centralized optimal solution. In addition, with the assumption of a logarithmic utility function, the Nash solution also agrees with the Nash bargaining solution described in [20].

From the perspective of efficiency, the price of anarchy ρ, defined in [21] as the ratio between game optimal and the social optimal, is always 100% in this case, i.e.,

$$\rho = \frac{\Phi_R(\mathbf{x}^*)}{\Phi_R(\mathbf{x}^{\text{opt}})} = 1. \tag{9}$$

The inclusion of coupled constraints into the game Ξ_R actually doesn't affect the game efficiency at all.

2) Wireless Power Control Game: In this section, we discuss a power control potential game described in [7]. In a typical CDMA wireless systems, mobile users respond to the time-varying nature of the channel by regulating their transmitter powers. The major goal is to optimize signal-to-interference ratio and minimize the interference level. In [22], a game theoretical framework is used for uplink power control problems in CDMA networks. We extend the problem by considering a capacity constrained power control game.

Let's consider a single-cell CDMA communication system with total bandwidth W Hz and unspread bandwidth B Hz, supporting N users. Let $h_i, i = 1, \cdots, N$ denote normalized the slowly-varying channel gain with respect to thermal noise power. The signal to noise ratio (SNR) of $i-th$ user at the receiver is given by

$$\text{SNR}_i(\mathbf{p}) = \frac{W}{B}\frac{|h_i|^2p_i}{1 + \sum_{j\neq i}|h_j|^2p_j},$$

where $\mathbf{p} = [p_1, p_2, \cdots, p_N], p_i$ is the transmit power of the $i-th$ user.

Consider a well-know strategic noncooperative wireless power control game $\Xi_W = \langle \mathcal{N}, (A_i), P_i\rangle$, where $A_i = [p_{\min,i}, p_{\max,i}]$ and $P_i(\mathbf{p}) = \ln\left(1 + \frac{|h_i|^2p_i}{1+\sum_{j\neq i}|h_j|^2p_j} - r_ip_i\right)$. Described in [7], this game has a potential function given by

$$\Phi_W(\mathbf{p}) = \ln\left(1 + \sum_{i=1}^N |h_i|^2p_i\right) - \sum_{i=1}^N r_ip_i.$$

Without coupled constraints, for example, capacity constraints $\sum_{i=1}^{N} p_i \leq C_0$, the Nash equilibrium has been studied in [22]. With coupled constraints, using the Lagrangian theory of constrained games, we can arrive at solving the optimization problem (WPG), coinciding with the result obtained in [7] from a different analysis.

$$\begin{array}{ll} \text{(WPG)} & \max_{\mathbf{p}} \Phi_W(\mathbf{p}) \\ \text{subject to} & \mathbf{1}^T\mathbf{p} \leq C_0, \mathbf{p} \geq 0. \end{array}$$

The minimization problem (WPG) provides us a basis to compare NE with classical centralized optimization problem (CCP) as in [23], [24]. It circumvents the difficulty of relating classical Lagrangian method for solving optimal solutions to game-theoretical solutions and provides two optimization problems in the similar structure. We expect this will facilitate , at least numerically, the investigation on efficiency of Nash equilibrium in the network.

$$\begin{array}{ll} \text{(CCP)} & \max_{\mathbf{p}} \sum_{i=1}^{N} P_i(\mathbf{p}) \\ \text{subject to} & \mathbf{1}^T\mathbf{p} \leq C_0, \mathbf{p} \geq 0. \end{array}$$

For example, suppose we use the same definition of price of anarchy in (9) for the wireless power control game and we desire the resulting Nash equilibrium to reach certain efficiency criterion, i.e, the price of anarchy $\rho \geq \theta$, where $\theta \in \mathcal{R}$ is a feasible target. We then have a single design optimization problem (DPA) as follows.

$$\begin{array}{ll} \text{(DPA)} & \max_{\mathbf{p}} \Phi_W(\mathbf{p}) \\ \text{subject to} & \mathbf{1}^T\mathbf{p} \leq C_0, \mathbf{p} \geq 0, \\ & \theta\overline{S} - \sum_{i=1}^{N} P_i(\mathbf{p}) \leq 0, \end{array}$$

where $\overline{S}$ is the optimal value of (CCP). By imposing an extra constraint, we limit the feasible set of NE and the (DPA) framework allows us to find an NE that can satisfy a given efficiency criterion. We can also optimize (DPA) with respect to $\mathbf{p}$ and θ and form a parametric optimization problem (DPA,θ). Under a certain Slater's condition, we are also able to show that (DPA) is also a stable parametric programming in θ, [25].

IV. Conclusion and Future Works

This paper deals with constrained Nash games with continuous sets of strategies. We use a Lagrangian approach to obtain necessary and sufficient conditions to characterize the Nash equilibrium under coupled constraints. The paper also discusses existence and uniqueness of the Nash equilibrium under certain conditions on the cost functions and constraints. We extend these fundamental results to potential games and discover an extension of existence of Nash equilibrium in potential games with the presence of coupled constraints. In addition, we illustrate the application of Lagrangian approach on quadratic games, wireless power control games and rate control games.

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