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Passivity-based tracking control of multiconstraint complementarity Lagrangian systems

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Abstract— **In this study one considers the tracking control problem of a class of nonsmooth fully actuated Lagrangian systems subject to frictionless unilateral constraints. A passivity-based switching controller that guarantees some stability properties of the closed-loop system is designed. A particular attention is paid to transition (impacting) and detachment phases of motion. This work extends previous works on the topic as it considers multiconstraint n-degree-of-freedom systems.**

I. Introduction

This paper focuses on the problem of tracking control of complementarity Lagrangian systems [12] subject to frictionless unilateral constraints whose dynamics may be expressed as:

$$\begin{cases} M(X)\ddot{X} + C(X,\dot{X})\dot{X} + G(X) = U + \nabla F(X)\lambda_X \\ 0 \le \lambda_X \perp F(X) \ge 0, \\ \text{Collision rule} \end{cases} \tag{1}$$

where $X \in \mathbb{R}^n$ is the vector of generalized coordinates, $M(X) = M^T(X) \in \mathbb{R}^{n\times n}$ is the positive definite inertia matrix, $F(X) \in \mathbb{R}^m$ represents the distance to the constraints, $C(X,\dot{X})$ is the matrix containing Coriolis and centripetal forces, $G(X)$ contains conservative forces, $\lambda_X \in \mathbb{R}^m$ is the vector of the Lagrangian multipliers associated to the constraints and $U \in \mathbb{R}^n$ is the vector of generalized torque inputs. For the sake of completeness we precise that ∇ denotes the Euclidean gradient $\nabla F(X) = (\nabla F_1(X), \ldots, \nabla F_m(X)) \in \mathbb{R}^{n\times m}$ where $\nabla F_i(X) \in \mathbb{R}^n$ represents the vector of partial derivatives of $F_i(\cdot)$ with respect to the components of X. We assume that the functions $F_i(\cdot)$ are continuously differentiable and that $\nabla F_i(X) \neq 0$ for X with $F_i(X) = 0$. It is worth to precise here that for a given function $f(\cdot)$ its derivative with respect to the time t will be denoted by $\dot{f}(\cdot)$. For any function $f(\cdot)$ the limit to the right at the instant t will be denoted by $f(t^+)$ and the limit to the left will be denoted by $f(t^-)$. A simple jump of the function $f(\cdot)$ at the moment $t = t_\ell$ is denoted $\sigma_f(t_\ell) = f(t_\ell^+) - f(t_\ell^-)$.

The admissible domain associated to the system (1) is the closed set Φ where the system can evolve and it is described as follows:

$$\Phi = \{X \mid F(X) \ge 0\} = \bigcap_{1\le i\le m} \Phi_i,$$

where $\Phi_i = \{X \mid F_i(X) \ge 0\}$ considering that a vector is non-negative if and only if all its components are non-negative. In order to have a well-posed problem with a physical meaning we consider that Φ contains at least a closed ball of positive radius.

Definition 1: A singularity of the boundary $\partial\Phi$ of Φ is the intersection of two or more codimension-one surfaces $\Sigma_i = \{X \mid F_i(X) = 0\}$.

The presence of $\partial\Phi$ can induce some impacts that must be included in the dynamics of the system. It is obvious that $m > 1$ allows both simple impacts (when one constraint is involved) and multiple impacts (when singularities or surfaces of codimension larger than 1 are involved). The collision (or restitution) rule in (1), is a relation between the post-impact velocity and the pre-impact velocity. Among the various models of collision rules, Moreau's rule is an extension of Newton's law which is energetically consistent [8] and is numerically tractable [1]. For these reasons throughout this paper the collision rule will be defined by Moreau's relation [12]:

$$\begin{aligned} \dot{X}(t_\ell^+) = (1+e_n) \operatorname*{arg\,min}_{z\in T_\Phi(X(t_\ell))} \frac{1}{2}[z - \dot{X}(t_\ell^-)]^T \\ \times M(X(t_\ell))[z - \dot{X}(t_\ell^-)] - e_n\dot{X}(t_\ell^-) \end{aligned} \tag{2}$$

where $\dot{X}(t_\ell^+)$ is the post-impact velocity, $\dot{X}(t_\ell^-)$ is the pre-impact velocity, $e_n \in [0,1]$ is the restitution coefficient and $T_\Phi(X)$ is the tangent cone to Φ at X [12], [14]. Denoting by T the kinetic energy of the system, we can compute the kinetic energy loss at the impact t_ℓ as [9]:

$$T_L(t_\ell) = -\frac{1-e_n}{2(1+e_n)}\sigma_{\dot{X}}^\top(t_\ell)M(X(t_\ell))\sigma_{\dot{X}}(t_\ell) \le 0 \tag{3}$$

The structure of the paper is as follows: in Section 2 one presents some basic concepts and prerequisites necessary for the further developments. Section 3 is devoted to the controller design. In Section 4 one defines the desired (or "exogenous") trajectories entering the dynamics. The desired contact-force that must occur on the phases where the motion is constrained, is explicitly defined in Section 5. Section 6 focuses on the strategy for take-off at the end of constraint phases. The main results related to the closed-loop stability analysis are presented in Section 7. One example and concluding remarks end the paper.

The tracking control problem under consideration was studied in [7] mainly in the 1-dof (degree-of-freedom) case and in [4] in the n-dof case. Both of these papers consider systems with only one unilateral frictionless constraint. Here we not only consider the multiconstraint case but the results

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in Section 7 relax some very hard to verify condition imposed in [4].

The following standard notations will be adopted: $||\cdot||$ is the Euclidean norm, $b_p \in \mathbb{R}^p$ and $b_{n-p} \in \mathbb{R}^{n-p}$ are the vectors formed with the first p and the last $n-p$ components of $b \in \mathbb{R}^n$, respectively. $N_\Phi(X_p = 0)$ is the normal cone $N_\Phi(X)$ to Φ at X (see [12], [14]) when X satisfies $X_p = 0$, $\lambda_{min}(\cdot)$ and $\lambda_{max}(\cdot)$ represent the smallest and the largest eigenvalues, respectively. We also note that an LCP is a system as $0 \leq \lambda \perp A\lambda + b \geq 0$, which has a unique solution for all $b \in \mathbb{R}^n$ if and only if $A \in \mathbb{R}^{n\times n}$ is a P-matrix.

II. Basic concepts

A. Typical task

In the case $m = 1$ (only one unilateral constraint) the system dynamics alternates free-motion phases Ω_{2k} when the constraint is not active($F(X) > 0$), and constraint-motion phases Ω_{2k+1} when the constraint is active ($F(X) = 0$). Between free and constraint phases the dynamical system always passes through a transition phase I_k containing some impacts (more details can be found in [7]).
In the case $m \geq 2$ (multiple constraints) things complicate since the number of typical phases increases due to the singularities of $\partial\Phi$ that must be taken into account. Explicitly, the constraint-motion phases need to be decomposed in sub-phases where some specific constraints are active. As we shall see later the tracking during this phases does not present particular difficulties for different number of active constraints. Thus the goal is to control the system during a generic phase (constraint or not) and the passage between phases when the number of active constraints increases or decreases. We stress out that an impacting transition occurs when the number of active constraints increases but there is no impact (and no change of the dynamics) when the number of active constraints decreases. Therefore, without any loss of generality we study the following typical task:

$$\mathbb{R}^+ = \bigcup_{k\geq 0} \left(\Omega_{2k}^{J_k} \cup I_k^{J_k} \cup \Omega_{2k+1}^{J_k'}\right), \quad J_k, J_{k+1} \subset J_k' \tag{4}$$

where the superscript J_k represents the set of active constraints during the corresponding motion phase. Throughout the paper, the sequence $\Omega_{2k}^{J_k} \cup I_k^{J_k} \cup \Omega_{2k+1}^{J_k'}$ will be referred to as the cycle k of the system's evolution. Furthermore for robustness reasons, during impacting transition phases $I_k^{J_k}$ we impose a closed-loop dynamics (with impacts) that mimics somehow the bouncing-ball dynamics (see e.g. [5]).

B. Stability analysis criteria

The system (1) is a complex nonsmooth and nonlinear dynamical system which involves continuous and discrete time phases. A stability framework for this type of systems has been proposed in [7] and extended in [4]. This is an extension of the Lyapunov second method adapted to closed-loop mechanical systems with unilateral constraints. In the sequel we introduce some basic concepts in order to clarify the framework.
The trajectories playing a role in the dynamics and the design of the controller are:
- $X^{nc}(\cdot)$ – the desired trajectory of the unconstrained system (i.e. the trajectory that the system should track if there were no constraints). We suppose that $F(X^{nc}(t)) < 0$ for some t, otherwise the problem reduces to the tracking control of a system with no constraints.
- $X_d^*(\cdot)$ – the signal entering the control input and playing the role of the desired trajectory during some parts of the motion.
- $X_d(\cdot)$ – the signal entering the Lyapunov function. This function is set on the boundary $\partial\Phi$ after the first impact of each cycle.

These signals may coincide on some time intervals as we shall see later.

Throughout the paper Ω denotes the complement of $I = \bigcup_{k\geq 0} I_k^{J_k}$. The Lebesgue measure of Ω, denoted $\lambda[\Omega]$, is assumed infinite. Consider $x(\cdot)$ the state of the closed-loop system in (1) with some feedback controller $U(X, \dot{X}, X_d^*, \dot{X}_d^*, \ddot{X}_d^*)$.

Definition 2 (Weakly Stable System [7]): The closed loop system is called weakly stable if for each $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that $||x(0)|| \leq \delta(\epsilon) \Rightarrow ||x(t)|| \leq \epsilon$ for all $t \geq 0$, $t \in \Omega$. The system is asymptotically weakly stable if it is weakly stable and $\lim_{t\in\Omega,\, t\to\infty} x(t) = 0$. Finally, the practical weak stability holds if there exists $0 < R < +\infty$ and $t^* < +\infty$ such that $||x(t)|| < R$ for all $t > t^*$, $t \in \Omega$.
Consider $I_k^{J_k} \triangleq [\tau_0^k, t_f^k]$ and $V(\cdot)$ such that there exists strictly increasing functions $\alpha(\cdot)$ and $\beta(\cdot)$ satisfying the conditions: $\alpha(0) = 0$, $\beta(0) = 0$ and $\alpha(||x||) \leq V(x,t) \leq \beta(||x||)$.
In the sequel, we consider that for each cycle the sequence of impact instants t_ℓ^k has an accumulation point t_∞^k. We note that a finite accumulation period (i.e. $t_\infty^k < +\infty$) implies that $e_n < 1$ ($e_n = 1 \Rightarrow t_\infty^k = +\infty$ see [3]).

The following is inspired from [4], and will be used to study the stability of the system (1).

Proposition 1 (Weak Stability): Assume that the task admits the representation (4) and that

a) $\lambda[I_k^{J_k}] < +\infty, \quad \forall k \in \mathbb{N}$,
b) outside the impact accumulation phases $[t_0^k, t_\infty^k]$ one has $\dot{V}(x(t), t) \leq -\gamma V(x(t), t)$ for some constant $\gamma > 0$,
c) $\sum_{\ell\geq 0} \left[V(t_{\ell+1}^{k-}) - V(t_\ell^{k+})\right] \leq K_1 V^{p_1}(\tau_0^k), \quad \forall \ell \geq 0$ for some $p_1 \geq 0$, $K_1 \geq 0$,
d) the system is initialized on Ω_0 such that $V(\tau_0^0) \leq 1$,
e) $\sum_{\ell\geq 0} \sigma_V(t_\ell^k) \leq K_2 V^{p_2}(\tau_0^k) + \xi$ for some $p_2 \geq 0$, $K_2 \geq 0$ and $\xi \geq 0$.

If $p = \min\{p_1, p_2\} < 1$ then $V(\tau_0^k) \leq \delta(\gamma, \xi)$, where $\delta(\gamma, \xi)$ is a function that can be made arbitrarily small by increasing the value of γ. The system is practically weakly stable with $R = \alpha^{-1}(\delta(\gamma, \xi))$.

Remark 1: Since the Lyapunov function is exponentially decreasing on the Ω_k phases, assumption (d) in Proposition 1 means that the system is initialized on Ω_0 sufficiently far from the moment when the trajectory $X^{nc}(\cdot)$ leaves the admissible domain and is therefore not stringent.

The practical stability is very useful because attaining asymptotic stability is not an easy task for the unilaterally constrained systems described by (1) especially when $n \geq 2$ and $M(q)$ is not a diagonal matrix (i.e. there are inertial couplings, which is the general case). *Precisely, the practical weak stability is characterized by an "almost decreasing" Lyapunov function* $V(x(\cdot).\cdot)$ as shown in Figure 4.

III. Controller design

In order to overcome some difficulties that can appear in the controller definition, the dynamical equations (1) will be expressed in the generalized coordinates introduced by McClamroch & Wang [10]. We suppose that the generalized coordinates transformation holds globally in Φ, which may obviously not be the case in general. However, the study of the singularities that might be generated by the coordinates transformation is out of the scope of this paper. Let us consider $D = [I_m \vdots O] \in \mathbb{R}^{m\times n}$, $I_m \in \mathbb{R}^{m\times m}$ the identity matrix. The new coordinates will be $q = Q(X) \in \mathbb{R}^n$, with $q = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$, $q_1 = \begin{bmatrix} q_1^1 \\ \vdots \\ q_1^m \end{bmatrix}$ such that $\Phi = \{q \mid Dq \geq 0\}$.

The controller used here consists of different low-level control laws for each phase of the system. More precisely, the switching controller can be expressed as

$$T(q)U = \begin{cases} U_{nc} & \text{for } t \in \Omega_{2k}^{\emptyset} \\ U_t^J & \text{for } t \in I_k^J \\ U_c^J & \text{for } t \in \Omega_k^J \end{cases} \tag{5}$$

where $T(q) = \begin{pmatrix} T_1(q) \\ T_2(q) \end{pmatrix} \in \mathbb{R}^{n\times n}$ is full-rank under some assumptions (see [10]). The dynamics becomes:

$$\begin{cases} M_{11}(q)\ddot{q}_1 + M_{12}(q)\ddot{q}_2 + C_1(q,\dot{q})\dot{q} + g_1(q) = T_1(q)U + \lambda \\ M_{21}(q)\ddot{q}_1 + M_{22}(q)\ddot{q}_2 + C_2(q,\dot{q})\dot{q} + g_2(q) = T_2(q)U \\ q_1^i \geq 0,\ q_1^i\lambda_i = 0,\ \lambda_i \geq 0,\ 1 \leq i \leq m \\ \text{Collision rule} \end{cases} \tag{6}$$

where the set of complementary relations can be written more compactly as $0 \leq \lambda \perp Dq \geq 0$. In the sequel U_{nc} coincides with the fixed-parameter controller proposed in [13]. First, let us introduce some notations: $\tilde{q} = q - q_d$, $\bar{q} = q - q_d^*$, $s = \dot{\tilde{q}} + \gamma_2\tilde{q}$, $\bar{s} = \dot{\bar{q}} + \gamma_2\bar{q}$, $\dot{q}_r = \dot{q}_d - \gamma_2\tilde{q}$ where $\gamma_2 > 0$ is a scalar gain and q_d, q_d^* represent the desired trajectories defined in the previous section. Using the above notations the controller is given by $T(q)U \triangleq$

$$\begin{cases} U_{nc} = M(q)\ddot{q}_r + C(q,\dot{q})\dot{q}_r + G(q) - \gamma_1 s \\ U_t^J = U_c^J,\ t \leq t_0^k \\ U_t^J = M(q)\ddot{q}_r + C(q,\dot{q})\dot{q}_r + G(q) - \gamma_1\bar{s},\ ,\ t > t_0^k \\ U_c^J = U_{nc} - P_d + K_f(P_q - P_d) \end{cases} \tag{7}$$

where $\gamma_1 > 0$ is a scalar gain, $K_f > 0$, $P_q = D^T\lambda$ and $P_d = D^T\lambda_d$ is the desired contact force during persistently constrained motion. It is clear that during Ω_k^J not all the constraints are active and, therefore, some components of λ and λ_d are zero.

In order to prove the stability of the closed-loop system (6)–(7) we will use the following positive definite function:

$$V(t,s,\tilde{q}) = \frac{1}{2}s^T M(q)s + \gamma_1\gamma_2\tilde{q}^T\tilde{q} \tag{8}$$

IV. Tracking control framework

In this paper we treat the tracking control problem for the closed-loop dynamical system (5)–(7) with the complete desired path a priori taking into account the complementarity conditions and the impacts. In order to define the desired trajectory let us consider the motion of a virtual and unconstrained particle perfectly following a trajectory (represented by $X^{nc}(\cdot)$ on Figure 1) with an orbit that leaves the admissible domain for a given period. Therefore, the orbit of the virtual particle can be split into two parts, one of them belonging to the admissible domain (inner part) and the other one outside the admissible domain (outer part). In the sequel we deal with the tracking control strategy when the desired trajectory is constructed such that:

(i) when no activated constraints, it coincides with the trajectory of the virtual particle,
(ii) when p activated constraints, its orbit coincides with the projection of the outer part of the virtual particle's orbit on the surface of codimension p defined by the activated constraints (X_d between A'' and C in Figure 1),
(iii) the desired detachment moment and the moment when the virtual particle re-enters the admissible domain (w.r.t. $p \leq m$ constraints) are synchronized.

Therefore we have not only to track a desired path but also to impose a desired velocity allowing the motion synchronization on the admissible domain.

The main difficulties here consist of:

- stabilizing the system on $\partial\Phi$ during the transition phases $I_k^{J_k}$ and incorporating the velocity jumps in the overall stability analysis;
- deactivating some constraints at the moment when the unconstrained trajectory re-enters the admissible domain with respect of them;
- maintaining a constraint movement between the moment when the system was stabilized on $\partial\Phi$ and the detachment moment.

A. Design of the desired trajectories

Throughout the paper we consider $I_k^{J_k} = [\tau_0^k, t_f^k]$, where τ_0^k is chosen by the designer as the start of the transition phase $I_k^{J_k}$ and t_f^k is the end of $I_k^{J_k}$. We note that all superscripts $(\cdot)^k$ will refer to the cycle k of the system motion. We also use the following notations:

- t_0^k is the first impact during the cycle k,
- t_∞^k is the accumulation point of the sequence $\{t_\ell^k\}_{\ell\geq 0}$ of the impact instants during the cycle k ($t_f^k \geq t_\infty^k$),
- τ_1^k will be explicitly defined later and represents the instant when the desired signal X_d^* reaches a given value chosen by the designer in order to impose a closed-loop dynamics with impacts during the transition phases,
- t_d^k is the desired detachment instant, therefore the phases $\Omega_{2k+1}^{J_k'}$ can be expressed as $[t_f^k, t_d^k]$.

It is noteworthy that t_0^k, t_∞^k, t_d^k are state-dependent whereas τ_1^k and τ_0^k are exogenous and imposed by the designer. To better understand the definition of these specific instants we present in Figure 1 a simplified representation of the signals introduced in the previous section.

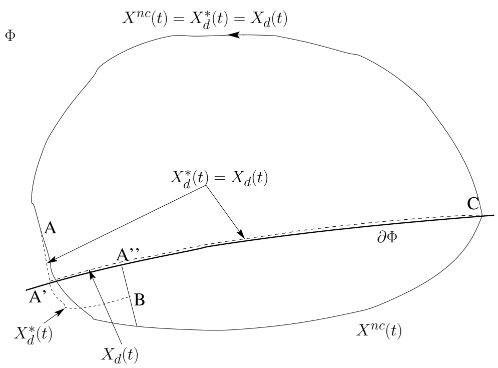

Fig. 1. The closed-loop desired trajectory and control signals

The points A, A', A'' and C in Figure 1 correspond to the moments τ_0^k, t_0^k, t_f^k and t_d^k respectively. On the other hand in Figure 1 we see that starting from A the desired trajectory $X_d(\cdot) = X_d^*(\cdot)$ is deformed compared to $X^{nc}(\cdot)$. In order to reduce this deformation τ_0^k and implicitly the point A must be close to $\partial\Phi$. Taking into account just the constraints $J_k' \setminus J_{k+1}$ we can identify t_d^k with the moment when $X_d(\cdot)$ and $X^{nc}(\cdot)$ rejoin at C.

B. Design of $q_d^*(\cdot)$ and $q_d(\cdot)$ on the phases $I_k^{J_k}$

During the transition phases the system must be stabilized on $\partial\Phi$. Obviously, this does not mean that all the constraints have to be activated. Let us consider that only the first p constraints (eventually reordering the coordinates) define the border of Φ where the system must be stabilized. On $[\tau_0^k, t_0^k)$ we define $q_d^*(\cdot)$ as a twice differentiable signal such that $q_d^*(t)$ approaches a given point of the normal cone $N_\Phi(q_p = 0)$ on $[\tau_0^k, \tau_1^k]$. Precisely, we define $q_d^*(\cdot)$ such as:

$$\begin{aligned} &q_d^*(\tau_0^k) = q^{nc}(\tau_0^k), \quad \dot{q}_d^*(\tau_0^k) = q^{nc}(\tau_0^k) \\ &q_d^*(\tau_0^k + \delta) = q^{nc}(\tau_0^k), \quad \dot{q}_d^*(\tau_0^k + \delta) = 0 \\ &(q_d^i)^*(\tau_1^k) = -\varphi V^{1/3}(\tau_0^k),\ (\dot{q}_d^*)_p(\tau_1^k) = 0,\ i = 1, \ldots, p \end{aligned} \tag{9}$$

where $\varphi > 0$, and $\delta > 0$ is a small constant introduced in order to assure the twice differentiability of q_d^* before the first impact. The rationale behind the choice of $q_d^*(\cdot)$ is on one hand to assure a robust stabilization on $\partial\Phi$, mimicking the bouncing-ball dynamics; on the other hand to enable one to compute suitable upper-bounds that will help using Proposition 1 (hence $V^{1/3}(\cdot)$ terms in (9) with $V(\cdot)$ in (8)).

Remark 2: Two different situations are possible. The first is given by $t_0^k > \tau_1^k$ (see Figure 2) and we shall prove that in this situation all the jumps of the Lyapunov function in (8) are negative. The second situation was pointed out in [4] and is given by $t_0^k < \tau_1^k$. In this situation the first jump at t_0^k in the variation of the Lyapunov function is positive, therefore the system can be only weakly stable.

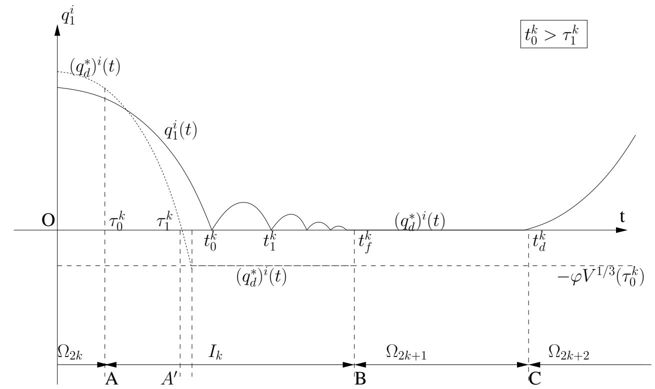

Fig. 2. The design of q_{1d}^* on the transition phases $I_k^{J_k}$

During the transition phases $I_k^{J_k}$ we define $(q_d)_{n-p}(t) = (q_d^*)_{n-p}(t)$. Assuming a finite accumulation period, the impact process can be considered in some way equivalent to a plastic impact. Therefore, $(q_d)_p$ and $(\dot{q}_d)_p$ are set to zero on the right of t_0^k.

V. Design of the desired contact force during constraint phases

For the sake of simplicity we consider the case of the constraint phase Ω_k^J, $J \neq \emptyset$ with $J = \{1, \ldots, p\}$. Obviously a sufficiently large desired contact force P_d assures a constrained movement on Ω_k^J. Nevertheless at the end of the Ω_{2k+1}^J phases a detachment from some surfaces Σ_i has to take place. It is clear that a take-off implies not only a well defined desired trajectory but also some small values of the corresponding contact force components. On the other hand, if the components of the desired force decrease too much a detachment can take place before the end of the Ω_{2k+1}^J phases which can generate other impacts. Therefore we need a lower bound of the desired force which assures the contact during the Ω_k^J phases.

Dropping the time argument, the dynamics of the system on Ω_k^J can be written as

$$\begin{cases} M(q)\ddot{q} + F(q, \dot{q}) = U_c + D_p^T \lambda_p \\ 0 \leq q_p \perp \lambda_p \geq 0 \end{cases} \tag{10}$$

where $F(q, \dot{q}) = C(q, \dot{q})\dot{q} + G(q)$ and $D_p = [I_p \vdots O] \in \mathbb{R}^{p \times n}$. On Ω_k^J the system is permanently constrained which implies $q_p(\cdot) = 0$ and $\dot{q}_p(\cdot) = 0$. In order to assure these conditions it is sufficient to have $\lambda_p > 0$. In the following let us denote $M^{-1}(q) = \begin{pmatrix} [M^{-1}(q)]_{p,p} & [M^{-1}(q)]_{p,n-p} \\ [M^{-1}(q)]_{n-p,p} & [M^{-1}(q)]_{n-p,n-p} \end{pmatrix}$ and $C(q, \dot{q}) = \begin{pmatrix} C(q, \dot{q})_{p,p} & C(q, \dot{q})_{p,n-p} \\ C(q, \dot{q})_{n-p,p} & C(q, \dot{q})_{n-p,n-p} \end{pmatrix}$ where the meaning of each component is obvious.

Proposition 2: On Ω_k^J the constraint motion of the closed-loop system (10),(5),(7) is assured if the desired force is defined by

$$(\lambda_d)_p \triangleq \beta - \frac{\bar{M}_{p,p}(q)}{1+K_f}\left([M^{-1}(q)]_{p,p}C_{p,n-p}(q,\dot{q})+ [M^{-1}(q)]_{p,n-p}C_{n-p,n-p}(q,\dot{q})+\gamma_1[M^{-1}(q)]_{p,n-p}\right)s_{n-p} \quad (11)$$

where $\bar{M}_{p,p}(q)=\left([M^{-1}(q)]_{p,p}\right)^{-1}$ and $\beta\in\mathbb{R}^p$, $\beta>0$.

Proof: For the sake of brevity we give here only the idea of the proof. The result is based on the solution of the LCP derived combining (10) and (7), which is:

$$0\le D_pM^{-1}(q)\big[-F(q,\dot{q})+U_{nc}-(1+K_f)D_p^T(\lambda_d)_p\big] +(1+K_f)D_pM^{-1}(q)D_p^T\lambda_p\perp\lambda_p\ge 0 \quad (12)$$

It is worth to precise that the LCP (12) has a unique solution since $(1+K_f)D_pM^{-1}(q)D_p^T>0$. ■

VI. Strategy for take-off at the end of constraint phases Ω^J_{2k+1}

In this section we are interested in finding the conditions on the control signal U^J_c that assure the take-off at the end of constraint phases Ω^J_{2k+1}. As we have already seen before, the phase Ω^J_{2k+1} can be expressed as the time interval $[t^k_f,t^k_d)$. The dynamics on $[t^k_f,t^k_d)$ is given by (10) and the system is permanently constrained, which implies $q_p(\cdot)=0$ and $\dot{q}_p(\cdot)=0$. Let us also consider that the first r constraints ($r<p$) have to be deactivated. Thus, the detachment takes place at t^k_d if $\ddot{q}_r(t^{k+}_d)>0$ which requires $\lambda_r(t^{k-}_d)=0$. The last $p-r$ constraints remain active which means $\lambda_{p-r}(t^{k-}_d)>0$.

To simplify the notation we drop the time argument in many equations of this section. We denote the LCP matrix as:

$$(1+K_f)D_pM^{-1}(q)D_p^T=\begin{pmatrix} A_1(q) & A_2(q)\\ A_2(q)^T & A_3(q)\end{pmatrix}$$

with $A_1\in\mathbb{R}^{r\times r}$, $A_2\in\mathbb{R}^{r\times(p-r)}$ and $A_3\in\mathbb{R}^{(p-r)\times(p-r)}$.

Proposition 3: For the closed-loop system (10),(5),(7) the decrease of active constraints from p to $r<p$, is possible if at the instant t^k_d

$$\begin{aligned}(\lambda_d)_r &= \left(A_1-A_2A_3^{-1}A_2^T\right)^{-1}\left(b_r-A_2A_3^{-1}b_{p-r}-A_2C_2\right)-C_1\\ (\lambda_d)_{p-r} &= C_2+A_3^{-1}\left(b_{p-r}-A_2^T(\lambda_d)_r\right)\end{aligned} \quad (13)$$

where

$$b_p=b(q,\dot{q},U_{nc})\triangleq D_pM^{-1}(q)[U_{nc}-F(q,\dot{q})]\ge 0$$

and $C_1\in\mathbb{R}^r$, $C_2\in\mathbb{R}^{p-r}$ such that $C_1\ge 0$, $C_2>0$

Proof: The result follows solving the LCP (12). ■

Proposition 4: The closed-loop system (10),(5), (7) is permanently constrained on $[t^k_f,t^k_d)$ and a smooth detachment is guaranteed on $[t^k_d,t^k_d+\epsilon)$ (ϵ is a small positive real number chosen by the designer) if

(i) $(\lambda_d)_p$ is defined on $[t^k_f,t^k_d)$ by (13) where C_1 is replaced by $C_1(t-t^k_d)$.

(ii) On $[t^k_d,t^k_d+\epsilon)$

$$q^*_d(t)=q_d(t)=\begin{pmatrix} q^*_r(t)\\ q^{nc}_{n-r}(t)\end{pmatrix},$$

where $q^*_r(t)$ is a twice differentiable function such that

$$\begin{aligned} q^*_r(t^k_d)=0,\quad & q^*_r(t^k_d+\epsilon)=q^{nc}_r(t^k_d+\epsilon),\\ \dot{q}^*_r(t^k_d)=0,\quad & \dot{q}^*_r(t^k_d+\epsilon)=\dot{q}^{nc}_r(t^k_d+\epsilon)\end{aligned} \quad (14)$$

and $\ddot{q}^*_r(t^k_d)=a>\max\left(0,-A_1(q)\left((\lambda_d)_r(t^{k-}_d)\right)\right)$.

Proof: The uniqueness of solution of the LCP (12) assures us that (11) and (13) agree if $C_1<0$. In other words, replacing C_1 by $C_1(t-t^k_d)$ in (13) we assure a constrained motion on $[t^k_f,t^k_d)$ and the necessary conditions for detachment on $[t^k_d,t^k_d+\epsilon)$. The second item simply says that the detachment is assured by a sufficiently large positive desired acceleration $\ddot{q}^*_r$. Relations (14) assure the twice differentiability of q_d and q^*_d. ■

VII. Closed-loop stability analysis

In the case $\Phi=\mathbb{R}^n$, the function $V(t,s,\tilde{q})$ in (8) can be used to prove the closed-loop stability of the system (6), (7) (see for instance [6]). In the case studied here ($\Phi\subset\mathbb{R}^n$) the analysis becomes more complicated as shown in [7].

To simplify the notation $V(t,s(t),\tilde{q}(t))$ is denoted as $V(t)$. In order to introduce the main result of this paper we make the next assumption, which is verified in practice for dissipative systems.

Assumption 1: The controller U_t in (7) assures that the sequence $\{t^k_\ell\}_{\ell\ge 0}$ of the impact times possesses a finite accumulation point t^k_∞ i.e. $\lim_{\ell\to\infty} t^k_\ell=t^k_\infty<+\infty$, $\forall k$.

Theorem 1: Let Assumption 1 hold, $e_n\in[0,1)$ and $(q^*_d)_p$ defined as in (9). The closed-loop system (5)-(7) initialized on Ω_0 such that $V(\tau^0_0)\le 1$, satisfies the requirements of Proposition 1 and is therefore practically weakly stable with the closed-loop state $x(\cdot)=[s(\cdot),\tilde{q}(\cdot)]$ and $R=\sqrt{Ke^{-\gamma(t^k_f-t^k_\infty)}}$ where $\gamma=2\gamma_1/\lambda_{max}(M(q))$ and $K>0$ is a real constant.

Proof: The proof consist of verifying the conditions b), c) and e) of Proposition 1. The details can be found in [11]. ■

VIII. Illustrative example

The numerical simulations are done with the Moreau's time-stepping algorithm of the Siconos software platform [2]. The choice of a time-stepping algorithm was mainly dictated by the presence of accumulations of impacts which render the use of event-driven methods difficult. The influence of different parameter as $\gamma_1,\gamma_2,e_n,\tau^k_0$ or the time-step, is studied in [11] by simulating the behavior of a planar two-link rigid-joint manipulator in presence of one unilateral constraint.

Let us consider in the sequel a planar two-link rigid-joint manipulator with two constraints. Precisely we impose an admissible domain $\Phi=\{(x,y)\mid y\ge 0,\ 0.7-x\ge 0\}$. Let us also consider an unconstrained desired trajectory X^{nc} whose orbit is given by the circle $\{(x,y)\mid (x-0.7)^2+y^2=0.5\}$ that violates both constraints. In other words, the two-link planar manipulator must track a quarter-circle; stabilize on and then follow the line $\Sigma_1=\{(x,y)\mid y=0\}$; stabilize on the intersection of Σ_1 and $\Sigma_2 = \{(x, y) \mid x = 0.7\}$; detach from Σ_1 and follow Σ_2 until the unconstrained circle re-enters Φ and finally take-off from Σ_2 in order to repeat the previous steps. It is noteworthy that the task presented above is not of type (4) since after a constraint phase (when the end-effector is attached to Σ_1) follows a transition phase instead of a detachment. However, as we have pointed out in Section II the succession of phases is not determinant and the manipulator can accomplish the task under consideration in a weakly stable way.

Let us consider in this case that a cycle is $\Omega_{2k} \cup I_k^1 \cup \Omega_{2k+1}^1 \cup I_k^2 \cup \Omega_{2k+1}^2$ where Ω_{2k} is the free-motion phase and I_k^i, Ω_{2k+1}^i are the impacting transients and the constrained phases associated to the surface Σ_i. The numerical values used for the dynamical model are again $l_1 = l_2 = 0.5m$, $I_1 = I_2 = 1kg.m^2$, $m_1 = m_2 = 1kg$ and the restitution coefficient $e_n = 0.7$. We impose a period of 10 seconds for each cycle and we simulate the dynamics during 6 cycles. Setting the controller gains $\gamma_1 = 15$, $\gamma_2 = 15$ we see in Figure 3 that the desired trajectory is accurately followed. The same conclusion can be deduced looking at the variation of the Lyapunov function plotted in Figure 4.

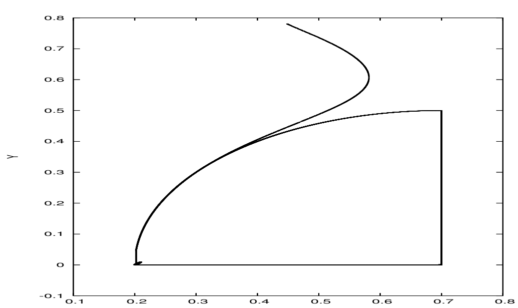

Fig. 3. The trajectory of the system during 6 cycles.

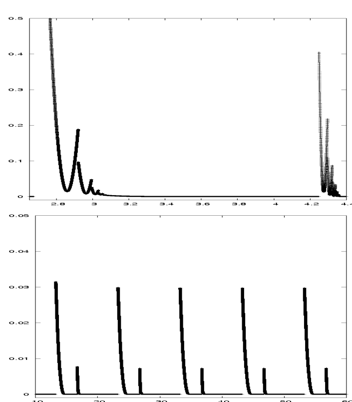

Fig. 4. Up: Lyapunov function-zoom on the transition phases during the first cycle; Down: The variation of the Lyapunov function on the last 5 cycles.

The simulation are done imposing a constant contact-force λ_1 during the motion on the surface Σ_1 (see Figure 5 (left)) and a decreasing contact-force, that allows a smooth detachment, during the motion on Σ_2 (see Figure 5 (right)).

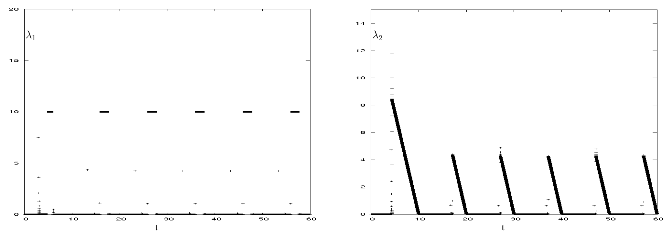

Fig. 5. Left: Variation of the contact force during the motion on Σ_1; Right: Variation of the contact force during the motion on Σ_2

IX. CONCLUSIONS

In this paper we have proposed a methodology to study the tracking control of fully actuated Lagrangian systems subject to multiple frictionless unilateral constraints. The main contribution of the work is twofold: first, it formulates a general framework and second, it provides a complete stability analysis for the class of systems under consideration. It is noteworthy that even in the case of only one frictionless unilateral constraint the paper presents some improvements with respect to the existing works in the literature. Precisely, the stability analysis result is significantly more general than those presented in [4] and [7] and, each element entering the dynamics (desired trajectory, contact force) is explicitly defined. Numerical simulations for an illustrative example, using the SICONOS software platform [2] complete the study.

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