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Structured Semidefinite Representation of Some Convex Sets

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Abstract— **Linear matrix Inequalities (LMIs) have had a major impact on control but formulating a problem as an LMI is an art. Recently there is the beginnings of a theory of which problems are in fact expressible as LMIs. For optimization purposes it can also be useful to have "lifts" which are expressible as LMIs. We show here that this is a much less restrictive condition and give methods for actually constructing lifts and their LMI representation.**

I. INTRODUCTION

Recently, there is a lot of work [Las01], [ND05], [NDS06], [Par00], [ParStu03] in solving global polynomial optimization problems by using sum of squares (SOS) methods or semidefinite programming (SDP) relaxations. The basic idea is to approximate a semialgebraic set S by a collection of convex sets called SDP relaxations each of which has an SDP representation. This leads to the fundamental problem of which sets can be represented with LMIs or projections of LMIs.

A set S is said to have an *LMI representation* or be *LMI representable* if

$$S = \{x \in \mathbb{R}^n : A_0 + \sum_{i=1}^{n} A_i x_i \succeq 0\} \qquad \text{(I.1)}$$

for some symmetric matrices A_i. Here the notation $X \succeq 0\,(\succ 0)$ means the matrix X is positive semidefinite (definite). Obvious necessary conditions for S to be LMI representable are that S must be convex and S must have the form

$$S = \{x \in \mathbb{R}^n : g_1(x) \geq 0, \cdots, g_m(x) \geq 0\} \qquad \text{(I.2)}$$

where $g_i(x)$ are multivariate polynomials; such are called basic closed semialgebraic sets.

It turns out that many convex sets are not LMI representable, see Helton and Vinnikov [HV07]. For instance, the convex set

$$\{x \in \mathbb{R}^2 : 1 - (x_1^4 + x_2^4) \geq 0\}$$

does not admit an LMI representation. However, the set S is the projection onto x-space of the set

$$\hat{S} := \Big\{(x,w) \in \mathbb{R}^2 \times \mathbb{R}^2 : \begin{bmatrix} 1 & x_2 \\ x_2 & w_2 \end{bmatrix} \succeq 0$$
$$\begin{bmatrix} 1+w_1 & w_2 \\ w_2 & 1-w_1 \end{bmatrix} \succeq 0, \begin{bmatrix} 1 & x_1 \\ x_1 & w_1 \end{bmatrix} \succeq 0 \Big\}$$

in $\mathbb{R}^4$ which is representable by an LMI.

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More generally a set $S \subseteq \mathbb{R}^n$ is said to be *semidefinite representable* or *SDP representable* if S can be described as

$$S = \Big\{x \in \mathbb{R}^n : \exists w \in \mathbb{R}^M \text{ s.t. }$$
$$A + \sum_{i=1}^{n} x_i B_i + \sum_{j=1}^{M} w_j C_j \succeq 0\Big\}.$$

Here A, B_i, C_j are symmetric matrices of appropriate dimensions. Conceptually, one can think of S as the projection into $\mathbb{R}^n$ of a set $\hat{S}$ in $\mathbb{R}^{(n+M)}$ having the LMI representation:

$$\hat{S} = \Big\{(x,w) \in \mathbb{R}^{(n+M)} :$$
$$A + \sum_{i=1}^{n} x_i B_i + \sum_{j=1}^{M} w_j C_j \succeq 0\Big\}. \qquad \text{(I.3)}$$

The representation (I.3) is called a *semidefinite representation* or *SDP representation* of the set S. We refer to the w_j as *auxillary variables*.

The key use of an SDP representation is illustrated by optimizing a linear function $\ell^T x$ over S. Note that minimizing $\ell^T x$ over S is equivalent to problem

$$\min_{(x,y)\in\hat{S}} \ell^T x,$$

which is a conventional LMI, so can be attacked by standard toolboxes. Nesterov and Nemirovski ([NN94]) in their book which introduced LMIs and SDP gave collections of examples of SDP representable sets thereby leading to:

Question: *Which convex sets S are the projection of a set $\hat{S}$ having an LMI representation?*

In §4.3.1 of his excellent 2006 survey [Nem06], Nemirovsky commented "this question seems to be completely open". Now much more is known and this paper describes both qualitative theory and SDP constructions.

Recently, Helton and Nie [HN1], [HN2] proved some sufficient conditions that guarantee the convex set S is SDP representable. For instance, one sufficient condition is called the so-called *sos-convexity* or *sos-concavity*. A polynomial $f(x)$ is called sos-convex if its Hessian matrix $\nabla^2 f(x) = W(x)^T W(x)$ for some matrix polynomials ($W(x)$ is not necessarily square). A polynomial $g(x)$ is called sos-concave if $-g(x)$ is sos-convex. Helton and Nie [HN1] proved the following theorem:

Theorem 1.1 ([HN1]): If every $g_i(x)$ is sos-concave, then S is SDP representable.

An explicit construction of one SDP representation of S when every $g_i(x)$ is sos-concave will be given in Section II. For general polynomials $g_i(x)$, the constructed SDP representation in Section II is usually very big. However, when polynomials $g_i(x)$ are sparse, the SDP representation can be reduced to have smaller sizes. This will be addressed in Section III.

There are also some sufficient conditions other than sos-concavity that guarantee the SDP representability. For instance, when the boundary of S is positively curved, then S is SDP representable. This will be discussed in Section IV.

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II. General SDP representation

Suppose S is a convex set given in the form

$$S = \{x \in \mathbb{R}^n : g_1(x) \geq 0, \cdots, g_m(x) \geq 0\}.$$

In this section, we assume every $g_i(x)$ is a sos-concave polynomial. A natural SDP relaxation of S is

$$R = \left\{x : \exists\, y \ s.t.\ g(x,y) \geq 0,\ M_d(x,y) \succeq 0\right\}. \quad \text{(II.4)}$$

Here $g(x,y)$ is a vector valued linear function and $M_d(x,y)$ is a matrix valued linear function defined in what follows. The integer $2d$ is the minimum upper bound of the degrees of $g_i(x)$. The vector $g(x,y)$ is of the form

$$g(x,y) = \tilde{g}_0 + \sum_{i=1}^{n} x_i \tilde{g}_i + \sum_{1<|\alpha|\leq 2d} y_\alpha \tilde{g}_\alpha$$

whose coefficients are such that

$$\begin{bmatrix} g_1(x) \\ \vdots \\ g_m(x) \end{bmatrix} = \tilde{g}_0 + \sum_{i=1}^{n} x_i \tilde{g}_i + \sum_{1<|\alpha|\leq 2d} x^\alpha \tilde{g}_\alpha.$$

The matrix $M_d(x,y)$ is the d-th order *moment matrix* constructed as

$$M_d(x,y) = A_0 + \sum_{i=1}^{n} x_i A_i + \sum_{1<|\alpha|\leq 2N} y_\alpha A_\alpha. \quad \text{(II.5)}$$

Here the symmetric matrices A_α are such that

$$\mathbf{m}_d(x)\mathbf{m}_d(x)^T = A_0 + \sum_{i=1}^{n} x_i A_i + \sum_{1<|\alpha|\leq 2d} x^\alpha A_\alpha.$$

The notation $\mathbf{m}_d(x)$ above denotes the column vector of monomials with degree up to d, i.e.,

$$\mathbf{m}_d(x) = \begin{bmatrix} 1 & x_1 & \cdots & x_1^2 & x_1x_2 & \cdots & x_n^d \end{bmatrix}^T.$$

This construction of SDP relaxations of the set S was proposed by Parrilo [Par06] and Lasserre [Las06]. When every $g_i(x)$ is sos-concave, Helton and Nie [HN1] proved $R = S$. This result lends itself to implementation which we now illustrate with two examples. After that we improve this SDP construction to exploit sparsity structure when it is present in the defining polynomials g_i.

Example 2.1: Consider the set $S = \{x \in \mathbb{R}^n : g(x) \geq 0\}$ where

$$g(x) = 1 - (x_1^4 + x_2^4 - x_1^2 x_2^2).$$

Direct calculation shows

$$-\nabla^2 g(x) = \begin{bmatrix} x_1 & \\ & x_2 \end{bmatrix} \underbrace{\begin{bmatrix} 12 & -4 \\ -4 & 12 \end{bmatrix}}_{\succeq 0} \begin{bmatrix} x_1 & \\ & x_2 \end{bmatrix}.$$

So $g(x)$ is sos-concave. Thus we know S can be represented by R constructed in (II.4), which in this specification becomes

$$1 - y_{40} - y_{04} + y_{22} \geq 0,$$
$$\begin{bmatrix} 1 & x_1 & x_2 & y_{20} & y_{11} & y_{02} \\ x_1 & y_{20} & y_{11} & y_{30} & y_{21} & y_{12} \\ x_2 & y_{11} & y_{03} & y_{21} & y_{12} & y_{03} \\ y_{20} & y_{30} & y_{21} & y_{40} & y_{31} & y_{22} \\ y_{11} & y_{21} & y_{12} & y_{31} & y_{22} & y_{13} \\ y_{02} & y_{12} & y_{03} & y_{22} & y_{13} & y_{04} \end{bmatrix} \succeq 0.$$

The matrix above is the second order moment matrix. In this SDP representation, there are 12 auxiliary variables y_{ij}.

Example 2.2: Consider the set $S = \{x \in \mathbb{R}^n : 1 - p(x) \geq 0\}$ where p is a homogeneous polynomial:

$$p(x) = [x^d]^T B [x^d].$$

Here $d > 0$ is an integer and

$$B = (b_{ij})_{1\leq i,j\leq n} \succeq 0$$

is a symmetric matrix, and $[x^d]$ denotes the vector

$$[x^d] = \begin{bmatrix} x_1^d & x_2^d & \cdots & x_n^d \end{bmatrix}^T.$$

Direct calculation shows

$$\nabla^2 p(x) = \mathrm{diag}([x^{d-1}]) \cdot W \cdot \mathrm{diag}([x^{d-1}])$$

where the symmetric matrix W is defined to be

$$W = d^2 B + (3d^2 - 2d)\mathrm{diag}(B).$$

Since $B \succeq 0$, we also have $W \succeq 0$. So $p(x)$ is sos-convex. Therefore S can be represented as

$$\left\{ x : \exists\, y,\ 1 - \sum_{i,j=1}^{n} b_{ij} y_{d(e_i+e_j)} \geq 0, M_d(x,y) \succeq 0 \right\}.$$

Here e_i denotes the i-th standard unit basis vector of $\mathbb{R}^n$ and $M_d(x,y)$ is the d-th order moment matrix.

III. Sparse SDP representation

The SDP relaxations in [Las06], [HN1], [HN2], [Par06] have not exploited the special structures of polynomials

$$g_1(x), \cdots, g_m(x)$$

such as dependence of each polynomial on only a few variables (termed sparsity). On the other hand, in polynomial optimization the sparsity structure of polynomials can be exploited to improve the computation efficiency of their semidefinite relaxations [KKW05], [Las06spr], [Nie06], [ND06], [Par03], [WKKM06]. In this paper we show that when the defining polynomials for S are sparse, their structures can also be exploited to get a "sparser" SDP representation.

This section gives a structured SDP relaxation and proves a sufficient condition such that this structured SDP relaxation represents S exactly.

Throughout this section, we assume every polynomial $g_k(x)$ is sos-concave. Let

$$\mathcal{H} = \{x \in \mathbb{R}^n : a^T x \geq b\} \supseteq S$$

be a supporting half space and $a^T u = b$ for some $u \in \partial S$. When S has nonempty interior, there exists Lagrange multipliers $\lambda_1 \geq 0, \cdots, \lambda_m \geq 0$ such that

$$a = \sum_{k=1}^{m} \lambda_k \nabla g_k(u),\ \lambda_i g_i(u) = 0,\ i = 1, \cdots, m.$$

Helton and Nie [HN1] showed that the Lagrange function

$$a^T x - b - \sum_{k=1}^{m} \lambda_k g_k(x)$$

is an SOS polynomial when every polynomial $g_k(x)$ is sos-concave.

Assume that there exists a partition $\{I_1, \cdots, I_K\}$ is a partition of the index set $\{1, 2, \cdots, n\}$ such that $I_i \cap I_j = \emptyset$ whenever $i \neq j$, and suppose that for any a, b, λ there is a decomposition such that

$$a^T x - b - \sum_{k=1}^{m} \lambda_k g_k(x) = \phi_\lambda^{(1)}(x_{I_1}) + \cdots + \phi_\lambda^{(K)}(x_{I_K}),$$

where each $\phi_\lambda^{(i)}(x_{I_i})$ is a polynomial in variables x_{I_i}. Here x_{I_i} denotes the subvector of x whose indices are in I_i. In other words, the polynomials

$$\phi_\lambda^{(1)}(x_{I_1}), \cdots, \phi_\lambda^{(K)}(x_{I_K})$$

are uncoupled.

Given a polynomial $p(x)$, denote by supp$(p(x))$ the support of $p(x)$, i.e., the set of exponents of existing monomials of $p(x)$. If $p(x)$ is SOS and has decomposition $p(x) = \sum_i q_i^2(x)$, then it holds

$$\text{supp}(q_i(x)) \subseteq \text{convex hull}\left(\frac{1}{2}\text{supp}(p(x))\right),$$

by Theorem 1 in Reznick [Rez78]. So we define F_i to be the maximum lattice set such that

$$F_i \subseteq \text{convex hull}\left(\frac{1}{2}\text{supp}(\phi_\lambda^{(i)})\right).$$

Now define symmetric matrices M_α^j as follows

$$\begin{aligned}&\mathbf{m}_{F_i}(x_{I_i})\mathbf{m}_{F_i}(x_{I_i})^T\\ =&M_0^{(i)} + \sum_{j\in I_i} x_j M_j^{(i)} + \sum_{1<|\alpha|\le 2N} x^\alpha M_\alpha^{(i)}.\end{aligned}$$

Here $\mathbf{m}_{F_i}(x_{I_i})$ denotes the vector of monomials whose exponents lie in F_i. Then define linear matrices

$$M_{F_i}(x,y) = M_0^{(i)} + \sum_{j\in I_i} x_j M_j^{(i)} + \sum_{1<|\alpha|\le 2d} x^\alpha M_\alpha^{(i)}. \quad \text{(III.6)}$$

Lemma 3.1: Let a, b, λ be the above. Then there are symmetric matrices $W_1, \cdots, W_K \succeq 0$ such that

$$a^T x - b - \sum_{k=1}^m \lambda_k g_k(x) = \sum_{i=1}^K \mathbf{m}_{F_i}(x_{I_i})^T \cdot W_i \cdot \mathbf{m}_{F_i}(x_{I_i}).$$

Proof: By the structure assumption, we have representation

$$\begin{aligned}L_a(x) &:= a^T x - b - \sum_{k=1}^m \lambda_k g_k(x)\\ &= \eta_1(x_{I_1}) + \cdots + \eta_K(x_{I_K})\end{aligned}$$

for some polynomials $\eta_1(x_{I_1}), \cdots, \eta_K(x_{I_K})$. We know $L_a(x)$ is nonnegative polynomial and u is one global minimizer such that $L_a(u) = 0$. Let $u^{(i)}$ denote the subvector of u whose coordinates correspond to the variables x_{I_i}. Then $u^{(i)}$ is one global minimizer of $\eta_i(x_{I_i})$. So we know

$$L_a(x) = \sum_{i=1}^k \left(\eta_i(x_{I_i}) - \eta_i(u^{(i)})\right)$$

is SOS by Section 3 in [HN1]. In the above, fix one index i and set $x^{(j)} = u^{(j)}$ for $j \neq i$, then we can see $\eta_i(x_{I_i}) - \eta_i(u^{(i)})$ must also be SOS in x_{I_i}. Furthermore, by Theorem 1 in Reznick [Rez78], the polynomial $\eta_i(x_{I_i}) - \eta_i(u^{(i)})$ has the representation

$$\eta_i(x_{I_i}) - \eta_i(u^{(i)}) = \mathbf{m}_{F_i}(x_{I_i})^T \cdot W_i \cdot \mathbf{m}_{F_i}(x_{I_i}),$$

for some symmetric matrix $W_i \succeq 0$. Thus the Lemma is proven. ■

Theorem 3.2: Under the above assumptions, the convex set S has the SDP representation

$$\begin{aligned}L = \Big\{x \in \mathbb{R}^n :\, \exists y,\ s.t.\ g(x,y) \ge 0,\\ M_{F_i}(x,y) \succeq 0,\ i = 1,\cdots,K\Big\}.\end{aligned} \quad \text{(III.7)}$$

That is, $S = L$.

Proof: We have seen $S \subseteq L$. If $L \neq S$, then there must exist some point $\hat{x} \in L/S$. By the Convex Set Separation Theorem, there exists one supporting hyperplane of S

$$\mathcal{H} = \{x \in \mathbb{R}^n :\, a^T x \ge b\} \supseteq S$$

such that $a^T u = b$ for some $u \in \partial S$ and $a^T \hat{x} < b$. Consider the linear optimization problem

$$\begin{aligned}b = \min_{x\in\mathbb{R}^n} &\quad a^T x\\ s.t. &\quad g_1(x) \ge 0, \cdots, g_m(x) \ge 0.\end{aligned}$$

Then u is one minimizer for the above. Let $\lambda_1 \ge 0, \cdots, \lambda_m \ge 0$ be the corresponding Lagrange multipliers. Then, by the previous lemma, we have shown

$$\begin{aligned}&a^T x - b - \sum_{i=1}^m \lambda_k g_i(x)\\ =&\sum_{i=1}^K \mathbf{m}_{F_i}(x_{I_i})^T \cdot W_i \cdot \mathbf{m}_{F_i}(x_{I_i})\end{aligned}$$

for some symmetric matrices $W_1, \cdots, W_K \succeq 0$. So we have

$$\begin{aligned}b = \ &\max \quad \gamma \ \text{ s.t.}\\ a^T x - \gamma - \sum_{i=1}^m \lambda_k g_i(x) &= \sum_{i=1}^K \mathbf{m}_{F_i}(x_{I_i})^T \cdot W_i \cdot \mathbf{m}_{F_i}(x_{I_i})\\ \lambda_1, \cdots, \lambda_m &\ge 0, W_1, \cdots, W_K \succeq 0.\end{aligned}$$

The dual of the above SOS program is

$$\min\ a^T x \quad s.t.\ x \in L.$$

Since $\hat{x} \in L$, by weak duality, it holds $b \le a^T \hat{x}$, which contradicts the previous assertion $a^T \hat{x} < b$. ■

Now let us show some examples for the sparse SDP representation constructed in (III.7).

Example 3.3: Consider the convex set

$$S = \{x \in \mathbb{R}_+^n : g(x) := 1 - (x_1^8 + x_1^2 + x_1 x_2 + x_2^2) \ge 0\}.$$

Obviously $g(x)$ is sos-concave. The convex hull of $\left(\frac{1}{2}\text{supp}(g)\right)$ contains only the following integer points:

$$(0,0), (1,0), (2,0), (3,0), (4,0), (0,1).$$

By the sparsity theorem, S can be represented as

$$1 - y_{80} - y_{20} - y_{11} - y_{02} \ge 0,$$
$$\begin{bmatrix} 1 & x_1 & x_2 & y_{20} & y_{30} & y_{40}\\ x_1 & y_{20} & y_{11} & y_{30} & y_{40} & y_{50}\\ x_2 & y_{11} & y_{02} & y_{21} & y_{31} & y_{41}\\ y_{20} & y_{30} & y_{21} & y_{40} & y_{50} & y_{60}\\ y_{30} & y_{40} & y_{31} & y_{50} & y_{60} & y_{70}\\ y_{40} & y_{50} & y_{41} & y_{60} & y_{70} & y_{80} \end{bmatrix} \succeq 0.$$

The matrix above is the sparse moment matrix constructed in (III.6). There are totally 11 auxiliary variables y_{ij}.

Example 3.4: Consider the set $S = \{x \in \mathbb{R}^n : 1 - p(x) \ge 0\}$ where

$$p(x) = \sum_{i=1}^n p_i(x_i), \quad p_i(x_i) = \sum_{k=1}^{2d} \frac{x_i^k}{k!}.$$

Obviously $p(x)$ is sos-convex, because each univariate polynomial $p_i(x_i)$ is convex and hence sos-convex. Thus S can be represented as

$$1 - \sum_{i=1}^n \sum_{k=1}^{2d} \frac{y_k^{(i)}}{k!} \ge 0$$
$$H_1(x_1, y^{(i)}) \succeq 0, \cdots, H_n(x_n, y^{(n)}) \succeq 0$$

where $H_i(x_i, y^{(i)})$ are defined as

$$H_i(x_i, y^{(i)}) = \begin{bmatrix} 1 & x_i & y_2^{(i)} & \cdots & y_d^{(i)}\\ x_i & y_2^{(i)} & y_3^{(i)} & \cdots & y_{d+1}^{(i)}\\ y_2^{(i)} & y_3^{(i)} & y_4^{(i)} & \cdots & y_{d+2}^{(i)}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ y_d^{(i)} & y_{d+1}^{(i)} & y_{d+2}^{(i)} & \cdots & y_{2d}^{(i)} \end{bmatrix}.$$

The symmetric matrices $H_i(x_i, y^{(i)})$ are sparse moment matrices constructed in (III.6). There are totally $2n(d-1)$ auxiliary variables $y_k^{(i)}$.

IV. Positive curvature condition

Section II and Section III show the explicit construction of SDP representation when all the defining polynomials $g_i(x)$ are sos-concave. If some $g_i(x)$ is not sos-concave, these constructions usually do not represent S. However, there are other sufficient conditions that guarantees S is SDP representable, which is called *positive curvature*.

Assume S in (I.2) is convex, compact and has nonempty interior. Denote by ∂S the boundary of S. Let $Z_i = \{x : g_i(x) = 0\}$ and note $\partial S \subset \cup_i Z_i$. We say the defining functions of S are *nondegenerate* provided $\nabla g_i(x) \neq 0$ for all $x \in Z_i \cap \partial S$. The boundary of S is said to have *positive curvature* provided that there exist nondegenerate defining functions g_i for S such that at each $x \in \partial S \cap Z_i$

$$-v^T \nabla^2 g_i(x) v > 0, \quad \forall\, 0 \neq v \in \nabla g_i(x)^{\perp}, \qquad \text{(IV.8)}$$

in other words, the Hessian of g_i compressed to the tangent space (the second fundamental form) is negative definite. A standard fact in geometry is that this does not depend on the choice of $g_i(X)$.

Obviously, necessary conditions for S to be SDP representable are that S must be convex and semialgebraic (describable by a boolean combination of of polynomial equalities or inequalities over the real numbers). The following, Theorem 3.3 of [HN2], goes in the direction of the converse.

Theorem 4.1: Suppose S is a convex compact set with nonempty interior which has nondegenerate defining polynomials $S = \{x \in \mathbb{R}^n : g_1(x) \geq 0, \cdots, g_m(x) \geq 0\}$. If the boundary ∂S is positively curved, then S is SDP representable.

If S is convex with nondegenerate defining functions, then its boundary has nonnegative curvature. Thus the positive curvature assumption is not a huge restriction beyond being strictly convex. The nondegeneracy assumption is another restriction.

Finally comes an example where the defining polynomial is not concave but the boundary has positive curvature.

Example 4.2: Consider the set

$$S = \{x \in \mathbb{R}^n_+ : g(x) := x_1 x_2 \cdots x_n - 1 \geq 0\}.$$

We can easily see that S is convex but the defining polynomial $g(x)$ is not concave. Note that

$$\frac{\nabla g(x)}{g(x)+1} = \begin{bmatrix} \frac{1}{x_1} & \frac{1}{x_2} & \cdots & \frac{1}{x_n} \end{bmatrix}^T$$

$$\frac{\nabla^2 g(x)}{g(x)+1} = \begin{bmatrix} 0 & \frac{1}{x_1 x_2} & \cdots & \frac{1}{x_1 x_n} \\ \frac{1}{x_1 x_2} & 0 & \cdots & \frac{1}{x_2 x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{x_1 x_n} & \cdots & \frac{1}{x_{n-1} x_n} & 0 \end{bmatrix}.$$

We claim that the boundary ∂S has positive curvature, which is justified by the following observation:

$$-\nabla^2 g(x) + \nabla g(x) \nabla g(x)^T$$
$$\succeq (g(x)+1)\,\mathrm{diag}\left(\frac{1}{x_1^2}, \frac{1}{x_2^2}, \cdots, \frac{1}{x_n^2}\right) \succ 0, \ \forall\, x \in \partial S.$$

Since ∂S has positive curvature, Theorem 4.1 guarantees S has an SDP representation whose construction was in Section 5 in [HN1].

V. Concluding remarks

This paper gives an explicit construction, (II.4), of an SDP representation for a convex set S and a sparser one (III.7) when polynomials $g_k(x)$ are sos-concave. There are also some other constructions of SDP relaxations [Las06], [HN1], [HN2] for S, which are also SDP representations of S when $g_k(x)$ are strictly concave on the boundary ∂S of S or when the boundary ∂S has positive curvature.

In theory a hierarchy of SDP relaxations converging to S within finitely many steps can be constructed when the boundary ∂S has positive curvature (weaker than our hypothesis). However, these refined constructions of SDP representations are usually more complicated than (II.4) or (III.7), for example, usually it is difficult to predict which step of their hierarchy of relaxations represents S exactly. In contrast, the size of construction (II.4) or (III.7) is explicit. We refer to [HN1], [HN2] for more details.

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