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Synthesis of complete orthonormal fractional bases

Hüseyin Akçay

Abstract— **In this paper, fractional orthonormal basis functions which generalize the well-known fixed pole rational basis functions are synthesized. For a range of non-integer differentiation orders and under mild restrictions on the choice of the basis poles, the synthesized basis functions are complete in the space of functions which are analytic on the open right-half plane and square-integrable on the imaginary axis. This presents an extension of the completeness results for the fractional Laguerre and Kautz bases to fractional orthonormal bases with prescribed pole locations.**

I. INTRODUCTION

The fractional calculus is a generalization of the traditional calculus that leads to similar concepts and tools but with a much wider applicability. The mathematical concept and formalism of fractional calculus originate from the works of Liouville [1] and Riemann [2]. For almost three hundred years, it has remained an interesting, but abstract, mathematical concept. In recent years, fractional calculus has been taken up by scientists and engineers and applied in an increasing number of fields, namely in the areas of thermal engineering, acoustics, electromagnetism, control, robotics, viscoelasticity, diffusion, turbulence, signal processing, and many others.

There are many linear systems with transfer functions that can be represented as fractional differential systems, *that is*, as functions $G(s)$ which involve fractional powers of the Laplace variable s. For instance, in the field of diffusion, recent work [3] generalized diffusion equations based on non-integer derivatives. In thermal diffusion, it was shown in [4] that in a semi-infinite homogeneous medium the exact solution of the heat equation links thermal flux to a half order derivative of the surface temperature on which the flux is applied. Expressing such a relation by the use of rational models would require a much higher number of parameters. Diffusion phenomena were investigated in semi-infinite planar, spherical and cylindrical media in [5], [6], [7] where it was shown that the involved transfer functions use exponents of s that are multiples of 0.5. In electrochemical diffusion of charges in the electrode and the electrolyte, the most common physical model used in the literature is the Randles model [8] which uses Warburg impedance that involves an integrator of order 0.5. A fractal model for anomalous losses in ferromagnetic materials was used in [9]. In rheology, stress in a viscoelastic material is proportional to a non-integer derivative of deformation [10].

Department of Electrical and Electronics Engineering, Anadolu University, 26470 Eskişehir, Turkey. E-mail: huakcay@anadolu.edu.tr; Tel: + (90) 222 335 0580 -X 6459; Fax: +(90) 222 323 9501. This work was supported in part by the Scientific & Technological Research Council of Turkey under Grant 106E108.

In the area of control, the idea of using fractional systems for modeling ideal loop transfer functions dates back to Bode [11]. Recently, in [12] the advantages of a fractional order controller known as commande robust d'ordre non entier (CRONE) with respect to classical devices were shown. Fractional proportional-integral-derivative controller applications are reported in [13]. System identification with fractional models was initiated in [14]. Recently, in [15] fractional models were used to identify thermal diffusive systems. An overview of system identification methods based on fractional models is presented in [16].

In signal processing, non-integer derivative was used in the synthesis of fractal noise [17]. The works of Mandelbrot on fractals led to a significant impact in several scientific areas. Presently, new themes are the object of active research such as fractional delay filtering [18], fractional splines and wavelets [19], [20], [21]. In a similar line of thought, the concept of fractional Fourier transform [22] can be mentioned. This tool has mostly been applied in the field of optics. But, some applications to filtering, encoding, watermarking, and phase retrieval have appeared in the literature on signal analysis.

Of the greatest interest to the signal processing and control engineering communities is the fact that the fractional systems have both short and long term memories. Some basic properties of fractional systems such as stability [23], [24], observability and controllability [25], the H_2-norm [26], and the H_∞-norm [27] have been investigated over the last ten years.

A fundamental idea in various areas of applied mathematics, control theory, signal processing, and system analysis is that of decomposing (perhaps infinite dimensional) descriptions of linear-time-invariant dynamics in terms of an orthonormal basis. This approach is of greatest utility when accurate system descriptions are achieved with only a small number of basis functions. In recognition of this, there has been much work [28], [29] over the past several decades and, with renewed interest, more recently [30], [31], [32], [33], [34] on the construction, analysis, and application of rational orthonormal bases suitable for providing linear system characterizations.

An important motivation for the consideration of orthonormal parameterizations is for approximation purposes. In this setting, a dominant question must arise as the quality of the approximation. Pertaining to this, one of the most fundamental properties that might be required is *completeness*. Formally, a model set A is complete in a space X if the closure of the linear span of A under the norm on X equals X.

In Laguerre model structures, prior knowledge of the relative stability of a transfer function is encoded in terms of a single basis pole. In the case of systems for which prior knowledge of a resonant mode exists, it is more appropriate to employ two-parameter Kautz bases. The well-known Laguerre and Kautz bases [31] are special cases of the general orthonormal bases [30] where the basis poles are again restricted to a finite set. In [33], [34], model sets spanned by fixed pole orthonormal bases which generalize the Laguerre, two-parameter Kautz, and general orthonormal bases were investigated. These model sets were shown to be complete in $H_2(\Pi)$, the space of functions which are analytic on the open right-half plane denoted by Π and square integrable on the imaginary axis, provided that the chosen basis poles satisfy a mild condition. This generalization enjoys increased flexibility of pole location. As a result, a fewer number of basis functions may be used without sacrificing model accuracy.

Intuitively, one is led to the conclusion that the Laguerre functions can be extended to fractional differentiation orders by simply allowing their differentiation orders to be positive real numbers [35]. However, the classical Laguerre functions are divergent whenever their differentiation orders are non-integer [36]. The first complete fractional orthonormal basis, the so-called fractional Laguerre basis, was synthesized in [37]. This extension from the rational Laguerre basis to a fractional one provides a new class of fixed denominator models for system approximation and identification. A fractional orthogonal Kautz basis, that happens to be complete from the completeness of the fractional Laguerre functions in [37], was synthesized in [38].

The purpose of the current paper is to generalize the results in [37], [38] to fractional bases with infinitely many prescribed poles subject to mild restrictions on the choice of poles. This generalization is not straightforward. The key idea in [32], [33], [34] in showing completeness of the basis functions was to to re-parameterize the chosen model structures into a new one with equivalent fixed poles, but for which the basis functions are orthonormal in $H_2(\Pi)$. Then, it was possible to derive analytic expressions for approximation errors of the rational basis functions in terms of the Blaschke products [39] formed by the basis poles. These analytic expressions yielded necessary and sufficient conditions for the completeness of the basis functions not only in $H_2(\Pi)$, but also in many spaces. It was also possible to express each basis function as a product of a Blaschke product with a first order system.

This approach can not be utilized in the synthesis of fixed pole fractional bases with infinitely many poles since fractional analogs of the Blaschke products can not simply be defined by inserting s^γ in place of s due to the branch cut along the negative real line. The deficiency in defining fractional Blaschke products makes completeness study significantly harder for the fractional rationals because the orthonormality can not be employed either as an implementional tool or as an analysis tool. The use of the conformal mapping technique in [37] is limited only to the synthesis of fractional Laguerre bases.

This paper is organized as follows. In § II, mathematical background on the fractional derivatives and the fractional transfer functions is briefly reviewed. In § III, completeness results for the synthesized fractional basis functions with prescribed poles are presented. Orthonormalization and calculation of impulse-responses of the synthesized basis functions as well as the completeness proofs can be found in [40]. In § IV, a numerical example is used to illustrate the basis synthesis scheme and the impulse responses of the synthesized basis functions are computed. § V outlines future research directions and concludes the paper.

A. Notation

The field of the real and the complex numbers are denoted respectively by $\mathbf{R}$ and $\mathbf{C}$. The set of the positive numbers and its complement in $\mathbf{R}$ are respectively denoted by $\mathbf{R}_+$ and $\mathbf{R}_-$. The real and the imaginary parts of z are denoted respectively by $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$. The upper and the lower open half planes are denoted respectively by Π_2 and Π_4, $\Pi_1 = \Pi$, and Π_3 denotes the open left half plane.

Let $\mathscr{C}_\gamma$ denote the open sector defined by

$$\mathscr{C}_\gamma = \{s \in \mathbf{C} : |\arg(s)| < \pi[1-(\gamma/2)]\} \qquad (0<\gamma<2).$$

Thus, $\mathscr{C}_0 = \mathbf{C} - \mathbf{R}_-$ and $\mathscr{C}_1 = \Pi$. As γ increases, $\mathscr{C}_\gamma$ decreases. Let $\mathbf{D}_\varepsilon(z)$ denote the open disk with center $z \in \mathbf{C}$ and radius $\varepsilon > 0$.

he Hardy spaces of functions $F(s)$ analytic on Π and such that $\|F\|_p < \infty$ $(0 < p \le \infty)$ are denoted by $H_p(\Pi)$ where

$$\|F\|_p = \begin{cases} \left[\frac{1}{2\pi}\sup_{x>0}\int_{-\infty}^{\infty}|F(x+jy)|^p\,dy\right]^{\frac{1}{p}}, & p \ne \infty \\ \sup_{s\in\Pi}|F(s)|, & p=\infty. \end{cases}$$

II. Fractional linear systems

In this section, we will review definitions and results of fractional calculus pertinent to our analysis. The readers are referred to [41] and the references therein for details.

A. Fractional differential equations

The inverse Laplace transform of $F(s)$ denoted by $f(t)$ is defined by

$$f(t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} F(s)e^{st}\,ds \qquad (t>0) \tag{1}$$

where $\sigma \in \mathbf{R}$ is inside the region of convergence. It is related to $F(s)$ by the Laplace transformation:

$$F(s) = \int_0^\infty f(t)e^{-st}\,dt. \tag{2}$$

Note the following relation:

$$\int_0^\infty D_{\mathrm{d}}^\gamma f(t)e^{-st}\,dt = s^\gamma F(s) \qquad \text{for } \mathrm{Re}(s)>0 \tag{3}$$

where $D_{\mathrm{d}}^\gamma f(t)$ denotes the (direct) Grünwald-Letnikov fractional derivative of order γ of $f(t)$ [42].

The multi-valued function s^γ becomes an analytic function in the complement of its branch cut line as soon as a branch cut line, *i.e.* $\mathbf{R}_-$, is specified. This choice is made for a causal system.

Let us define a linear system through a fractional differential equation in the form:

$$\sum_{n=0}^{N} a_n D_{\rm d}^{\gamma_n} y(t) = \sum_{m=0}^{M} b_m D_{\rm d}^{\gamma_m} u(t) \tag{4}$$

where the differentiation orders, γ_n, are all positive, and $a_N, b_M \neq 0$. Applying (2) to (4) and using (3), we obtain the transfer function of the system:

$$F(s) = \frac{\sum_{m=0}^{M} b_m s^{\gamma_m}}{\sum_{n=0}^{N} a_n s^{\gamma_n}}. \tag{5}$$

The transfer function $F(s)$ is said *commensurable* of order $\gamma \in \mathbf{R}_+$ if γ_m, γ_n in (5) are integer multiples of γ, and γ is the largest number with this property. Thus, a commensurable transfer function F is a rational function in s^γ; and assuming $N > M$, by partial fraction expansion it can be decomposed as

$$F(s) = \sum_{k=1}^{L} \sum_{\ell=1}^{n_k} \frac{\alpha_{k,\ell}}{(s^\gamma + \lambda_k)^\ell} \tag{6}$$

for some complex numbers $\alpha_{k,\ell}$, λ_k, and positive integers n_k where $\sum_{k=1}^{L} n_k = N$.

B. Stability of s^γ-rational functions

A system with transfer function $G(s)$ is said to be stable if $G \in H_\infty(\Pi)$. This means that the system defined by (4) maps bounded energy inputs $u(t)$ to bounded energy outputs $y(t)$. In fact, if this happens then (4) maps magnitude bounded inputs to magnitude bounded outputs as well, *that is*, the fractional linear system (4) is bounded-input/bounded-output (BIBO) stable.

The stability of the fractional system defined by (4) with $\gamma_k = k\gamma$ for all k can be checked by checking γ and the arguments of λ_k denoted by $\arg(\lambda_k)$ in the partial fraction expansion of $F(s)$. Matignon [23] showed that the fractional system defined by (4) with $\gamma_k = k\gamma$ for all k is BIBO stable if and only if for all k,

$$0 < \gamma < 2 \qquad \text{and} \qquad |\arg(\lambda_k)| < \pi(1 - \frac{\gamma}{2}). \tag{7}$$

Henceforth, we will restrict γ to the interval $(0,2)$.

C. Fractional Orthonormal Bases

The partial fraction expansion (6) of a fractional linear system (4) with $\gamma_k = k\gamma$ for all k suggests approximating arbitrary functions in $H_2(\Pi)$ by linear combinations of the functions $(s^\gamma + \lambda_k)^{-\ell}$, $1 \leq \ell \leq n_k$; $k \geq 1$. There are several degrees of freedom and constraints in doing so. First of all, the stability constraint (7) has to be taken into account, which can be dealt with easily by suitably selecting the sequence λ_k for a fixed γ satisfying $\gamma \in (0,2)$. Another degree of freedom comes from the choice of the parameters n_k.

For the sequence λ_k, we consider arbitrary choices and multiplicities subject to the argument restrictions in § II-B. Thus, for all k, we assume $(s^\gamma + \lambda_k)^{-1} \in H_\infty(\Pi)$. It remains to satisfy $(s^\gamma + \lambda_k)^{-\ell} \in H_2(\Pi)$ so that their orthonormalized versions span a dense subset of $H_2(\Pi)$. Then, it suffices to let $\ell\gamma > \frac{1}{2}$ to assure $(s^\gamma + \lambda_k)^{-\ell} \in H_2(\Pi)$. Further details will be supplied later. Thus the problem studied is of synthesizing complete fractional orthonormal bases in $H_2(\Pi)$. The completeness problem boils down deriving sufficient conditions in terms of the parameters λ_k and their multiplicities.

After establishing the completeness of the functions: $(s^\gamma + \lambda_k)^{-\ell}$, $1 \leq \ell \leq n_k$; $k \geq 1$ in $H_2(\Pi)$, the next task is to orthonormalize them. This is a non-trivial process due to the branch cut along the negative real axis. Since $s = 0$ is a branch point, the following inner products

$$\begin{gathered} < (s^\gamma + \lambda_{k_1})^{-\ell_1}, (s^\gamma + \lambda_{k_2})^{-\ell_2} >= \\ \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega}{[(j\omega)^\gamma + \lambda_{k_1}]^{\ell_1} \overline{[(j\omega)^\gamma} + \overline{\lambda_{k_2}}]^{\ell_2}} \end{gathered} \tag{8}$$

are to be interpreted in a principal-value integral sense where the value at $s = 0$ is defined by continuation. The inner products (8) can be calculated explicitly by the aid of the residue method. Note that

$$(j\omega)^\gamma = |\omega|^\gamma \begin{cases} e^{j\pi\gamma/2}, & \omega > 0 \\ e^{-j\pi\gamma/2}, & \omega < 0. \end{cases} \tag{9}$$

The final task is to obtain the impulse responses of the fractional orthonormal basis functions, which are complicated expressions due to the branch cut.

D. Motivation for the completeness study

Let us consider the simple fractional rational transfer function $(s^\gamma + \lambda_k)^{-\ell}$, $\lambda_k \in \mathbf{R}$. By taking its inverse Laplace transform, we obtain the impulse response of this system denoted by $\widehat{\phi}_{k\ell}(t)$, $t > 0$ as follows

$$\begin{aligned} \widehat{\phi}_{k\ell}(t) \quad &= \quad \sum_{p=0}^{1 \text{ or } 2} \sum_{q=0}^{\ell-1} c_{pq} t^q e^{\alpha_{pq} t} \\ &+ \frac{1}{\pi} \sum_{i=1}^{\ell} \frac{\ell!}{i!(\ell-i)!} \int_0^\infty \frac{\lambda_k^{\ell-i} \sin(\pi\gamma i) x^{\gamma i} e^{-xt}\, dx}{[x^{2\gamma} + 2\lambda_k x^\gamma \cos(\pi\gamma) + \lambda_k^2]^\ell} \end{aligned} \tag{10}$$

for some complex numbers c_{pq}, α_{pq}. The above expression, as pointed out in [37], has two terms: the first term is the sum of the exponential modes originating from the poles of $(s^\gamma + \lambda_k)^{-\ell}$ and the second term is the combination of an infinite number of exponentials originating from the branch cut. The presence of the first term and the range of p as well as the numbers c_{pq}, α_{pq} depend on the values of γ and λ_k. This term has the character of a linear time-invariant dynamics and quickly dies out since $\mathrm{Re}(\alpha_{pq}) < 0$ for all p and q. The second term is more profound. In fact, from the definition of the gamma function:

$$\Gamma(\beta) = \int_0^\infty e^{-z} z^{\beta-1}\, dz,$$

we have as $t \to \infty$,

$$\widehat{\phi}_{k\ell}(t) \approx \sum_{i=1}^{\ell} \frac{\ell! \sin(\pi\gamma i) \Gamma(\gamma i + 1)}{i!(\ell-i)! \pi \lambda_k^{\ell+i}} t^{-(\gamma i+1)}.$$

Thus, the fractional rationals appear to be more suitable than the rationals in modeling slowly decaying impulse responses.

III. SYNTHESIS OF COMPLETE FRACTIONAL BASES

In this section, we synthesize complete fractional bases in $\mathscr{H}_2(\Pi)$. As basis functions, we propose the following so-called generator functions

$$\phi_{k\ell}(s) = \frac{(s^\gamma+\lambda)^{1-m}}{(s^\gamma+\lambda_k)^\ell}, \qquad 1\le \ell \le n_k;\ k\ge 1 \tag{11}$$

where $\lambda > 0$, $m \ge 2$ is a convergence factor to be fixed later, $\lambda_k \in \Pi$ are given complex numbers, and $\gamma \in (0,1)$ is a fixed number.

When $\gamma = 1$, it was shown in [33] that the generator functions (11) with $m = 1$ are complete in $\mathscr{H}_2(\Pi)$ if and only if the chosen basis poles satisfy the criterion:

$$\sum_{k=1}^{\infty} n_k \frac{\mathrm{Re}(\lambda_k)}{1+|\lambda_k|^2} = \infty. \tag{12}$$

Our first result in this section establishes that under the same criterion, the generator functions (11) are complete in $\mathscr{H}_2(\Pi)$ if $0 < \gamma < 1$ and m satisfies $2\gamma m > 1+\gamma$.

Theorem 3.1: Let $\gamma \in (0,1]$ be a fixed number and m be a positive integer satisfying $2\gamma m > 1+\gamma$. Consider the generator functions (11) defined by a choice of numbers $\lambda > 0$ and $\lambda_k \in \Pi$. Then, (11) are complete in $\mathscr{H}_2(\Pi)$ if (12) holds.

The completeness condition (12) applies to all $0 < \gamma \le 1$. For a fixed $\gamma \in (0,1)$, restricting λ_k to Π for all k results in the generator functions (11) being analytic on $\mathscr{C}_{\max\{0,2-(1/\gamma)\}}$. This restriction might introduce some conservatism on the choice of λ_k. Nevertheless, (12) does not preclude the possibility of λ_k converging to zero slowly. For example, put $\lambda_k = 1/k$ for all k. Then, (12) is satisfied.

We propose the generator functions for the case $\gamma \in (1,2)$ as follows

$$\eta_{k\ell}(s) = \frac{1}{(s^\gamma+\lambda_k)^\ell}, \qquad 1\le \ell \le n_k;\ k \ge 1 \tag{13}$$

where $\lambda_k \in \mathscr{C}_{\bar\gamma}$ for all k and some $\gamma < \bar\gamma < 2$.

Theorem 3.2: Let $\gamma \in (1,2)$ be a fixed number. Consider the generator functions (13) defined by a choice of the complex numbers $\lambda_k \in \mathscr{C}_{\bar\gamma}$ for all k and some $\gamma < \bar\gamma < 2$. Then, (13) are complete in $\mathscr{H}_2(\Pi)$ if

$$\sum_{k=1}^{\infty} n_k \frac{r_k^a \cos(a\theta_k)}{1+r_k^{2a}} = \infty \tag{14}$$

where $\lambda_k = r_k e^{j\theta_k}$ and $a = (2-\bar\gamma)^{-1}$.

The completeness of the generator functions (13) was established by restricting λ_k to $\mathscr{C}_{\bar\gamma}$ for all k. This set is a proper subset of $\mathscr{C}_\gamma$. It is a difficult question to answer whether it is possible to relax this restriction. Nevertheless, as γ approaches 1, $\bar\gamma$ can be forced to approach 1. Then, (14) coincides with (12) demonstrating that the former is consistent with the boundary case $\gamma = 1$.

IV. NUMERICAL EXAMPLE

In this section, we illustrate the basis synthesis scheme by a numerical example. Let

$$\phi_1(s) = \frac{1}{s^{1.5}+e^{j\frac{\pi}{6}}}, \qquad \phi_2(s) = \frac{1}{s^{1.5}+e^{-j\frac{\pi}{6}}}$$

be two given generator functions. We are to construct two orthonormal basis functions from ϕ_1 and ϕ_2 and compute their impulse responses as well. The basis functions with real-valued impulse responses are easily found as

$$\begin{aligned}
\breve{g}_1(s) &= \phi_1(s)+\phi_2(s) = \frac{2s^{1.5}+\sqrt{3}}{s^3+\sqrt{3}s^{1.5}+1},\\
\breve{g}_2(s) &= j[\phi_1(s)-\phi_2(s)] = \frac{1}{s^3+\sqrt{3}s^{1.5}+1}.
\end{aligned}$$

The difficult part is the orthonormalization of $\breve{g}_1$ and $\breve{g}_2$. To this end, first we compute the inner products $\langle \phi_k, \phi_l\rangle$ for $k,l = 1,2$ as

$$\langle \phi_1,\phi_2\rangle = \frac{1}{1.5\pi}\int_0^\infty \frac{x^{-\frac{1}{3}}\,dx}{(x+e^{-j\frac{7\pi}{12}})(x+e^{j\frac{11\pi}{12}})},$$

$$\begin{aligned}
\langle \phi_1,\phi_2\rangle &= \frac{\frac{2\pi j}{1.5\pi}}{1-e^{-j\frac{2\pi}{3}}}\left[\frac{(-e^{-j\frac{7\pi}{12}})^{-\frac{1}{3}}}{e^{j\frac{11\pi}{12}}-e^{-j\frac{7\pi}{12}}} + \frac{(-e^{j\frac{11\pi}{12}})^{-\frac{1}{3}}}{e^{-j\frac{7\pi}{12}}-e^{j\frac{11\pi}{12}}}\right]\\
&= \frac{4}{\sqrt{27}}e^{-j\frac{2\pi}{9}} = 0.5897 - j0.4948,
\end{aligned}$$

$$\begin{aligned}
\|\phi_1\|_1^2 &= \|\phi_2\|_2 = \frac{1}{3\sin(\frac{\pi}{3})}\left[\frac{\sin(\frac{11\pi}{36})}{\sin(\frac{11\pi}{12})}+\frac{\sin(\frac{7\pi}{36})}{\sin(\frac{7\pi}{12})}\right]\\
&= 1.4468
\end{aligned}$$

where the details are omitted. The Gram-Schmidt procedure applied to $\breve{g}_1$ and $\breve{g}_1$ yields two basis functions that are orthogonal to each other:

$$\tilde{\chi}_1 = \breve{g}_1, \qquad \tilde{\chi}_2 = \breve{g}_1 - \frac{\|\breve{g}_1\|_2^2}{\langle \breve{g}_1,\breve{g}_2\rangle}\breve{g}_2.$$

The inner products of $\breve{g}_1$ and $\breve{g}_2$ are computed as

$$\begin{aligned}
\langle \breve{g}_1,\breve{g}_2\rangle &= -2\,\mathrm{Im}\langle \phi_1,\phi_2\rangle = 0.9896,\\
\|\breve{g}_1\|_2^2 &= 2\|\phi_1\|_2^2 + 2\,\mathrm{Re}\langle \phi_1,\phi_2\rangle = 4.0730,\\
\|\breve{g}_2\|_2^2 &= 2\|\phi_1\|_2^2 - 2\,\mathrm{Re}\langle \phi_1,\phi_2\rangle = 1.7142.
\end{aligned}$$

Thus,

$$\begin{aligned}
\|\tilde{\chi}_1\|_2 &= \|\breve{g}_1\|_2 = 2.0182,\\
\|\tilde{\chi}_2\|_2^2 &= -\|\breve{g}_1\|_2^2 + \frac{\|\breve{g}_1\|_2^4}{\langle \breve{g}_1,\breve{g}_2\rangle^2}\|\breve{g}_2\|_2^2 = 24.9653.
\end{aligned}$$

It follows that the following basis functions

$$\chi_1(s) = 0.4955\breve{g}_1, \quad \chi_2(s) = 0.2001\breve{g}_1 - 0.8237\breve{g}_2. \tag{15}$$

are orthonormal and their linear span equals the linear span of ϕ_1 and ϕ_2. More explicitly, χ_1 and χ_2 can be written as

$$\chi_1(s) = \frac{0.9910s^{1.5}+0.8582}{s^3+\sqrt{3}s^{1.5}+1},$$
$$\chi_2(s) = \frac{0.4002s^{1.5}-0.4771}{s^3+\sqrt{3}s^{1.5}+1}.$$

If the impulse responses of $\breve{g}_1$ and $\breve{g}_2$ are known, then the impulse responses of χ_1 and χ_2 denoted by $\hat{\chi}_1$ and $\hat{\chi}_2$, respectively, can be computed from (15) by superposition. The former impulse responses are also computed by superposition from the impulse responses of ϕ_1 and ϕ_2 denoted respectively by h_1 and h_2. Notice that ϕ_1 has two poles in Π_3 at $z_{11}=e^{-j\frac{5\pi}{9}}$, $z_{12}=e^{j\frac{7\pi}{9}}$ and ϕ_2 has two poles at the conjugate points: $z_{21}=e^{j\frac{5\pi}{9}}$, $z_{22}=e^{-j\frac{7\pi}{9}}$. For ϕ_1, the residue term, *i.e.,* the first term in (10) is computed as

$$\sum_{k=1}^{2}\frac{e^{z_{1k}t}}{(s^{1.5}+e^{j\frac{\pi}{6}})'_{s=z_{1k}}} = \frac{1}{1.5}e^{-t\cos(\frac{4\pi}{9})-j[t\sin(\frac{4\pi}{9})-\frac{5\pi}{18}]} + \frac{1}{1.5}e^{-t\cos(\frac{2\pi}{9})+j[t\sin(\frac{2\pi}{9})-\frac{7\pi}{18}]}.$$

A similar computation is made for ϕ_2. Thus,

$$h_1(t) = -\frac{1}{\pi}\int_0^\infty \frac{x^{1.5}}{x^3+e^{j\frac{\pi}{3}}}\,e^{-xt}\,dx+\frac{1}{1.5}\{ e^{-t\cos(\frac{4\pi}{9})-j[t\sin(\frac{4\pi}{9})-\frac{5\pi}{18}]}+e^{-t\cos(\frac{2\pi}{9})+j[t\sin(\frac{2\pi}{9})-\frac{7\pi}{18}]}\},$$
$$h_2(t) = -\frac{1}{\pi}\int_0^\infty \frac{x^{1.5}}{x^3+e^{-j\frac{\pi}{3}}}\,e^{-xt}\,dx+\frac{1}{1.5}\{ e^{-t\cos(\frac{2\pi}{9})-j[t\sin(\frac{2\pi}{9})-\frac{7\pi}{18}]}+e^{-t\cos(\frac{4\pi}{9})+j[t\sin(\frac{4\pi}{9})-\frac{5\pi}{18}]}\}.$$

Hence,

$$\begin{aligned}\hat{\chi}_1(t) &= 0.6607\,e^{-0.1736t}\cos(0.9848t-0.8727)\\ &\quad +0.6607\,e^{-0.7660t}\cos(0.6428t-1.2217)\\ &\quad -\int_0^\infty \frac{0.3154x^{4.5}+0.1577x^{1.5}}{x^6+x^3+1}\,e^{-xt}\,dx,\\ \hat{\chi}_2(t) &= -1.1302\,e^{-0.1736t}\sin(0.9848t-1.1110)\\ &\quad +1.1302\,e^{-0.7660t}\sin(0.6428t-0.9834)\\ &\quad -\int_0^\infty \frac{0.1274x^{4.5}-0.3904x^{1.5}}{x^6+x^3+1}\,e^{-xt}\,dx.\end{aligned}$$

The periodic modes in both responses are linear combinations of two damped sinusoids, which quickly die off. Fig 1 shows the impulse responses of χ_1 and χ_2. As expected from the initial value theorem, both responses start at zero. Note that

$$\phi_k(0+) = \lim_{s\to\infty} s\phi_k(s) = 0, \qquad k=1,2.$$

V. Conclusions

In this paper, fractional orthonormal basis functions with prescribed poles were synthesized. These basis functions were shown to be complete in $\mathscr{H}_2(\Pi)$ under mild restrictions on the choice of the basis poles. This result enables one to approximate systems in $\mathscr{H}_2(\Pi)$, in particular the systems with both short and long memories, by convergent Fourier series of the fractional orthormal basis functions of this paper.

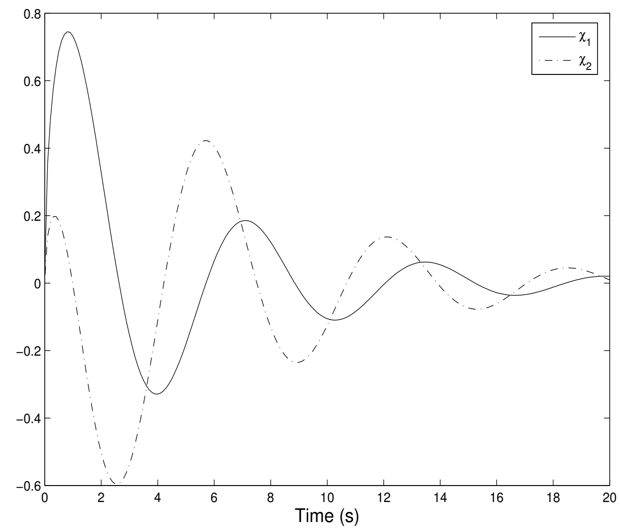

Fig. 1. The impulse responses of the synthesized basis functions.

The work initiated in this paper can be continued in several directions. First, completeness properties of the synthesized bases in different spaces, for example the spaces in which the rational orthonormal bases have been shown to be complete, should be investigated. The convergence and the approximation properties of the Fourier series formed by the fractional orthonormal basis functions over some known subsets of these spaces need to be explored. It is worth to study the completeness problem for fractional incommensurable rationals with prescribed poles. Fast and reliable numerical methods are needed to evaluate the impulse-responses of the synthesized basis functions. Then, it will be possible to quickly calculate time responses of the synthesized basis functions to arbitrary inputs.

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