Proceedings of the
47th IEEE Conference on Decision and Control
Cancun, Mexico, Dec. 9-11, 2008

LMI Conditions for H_∞ Static Output Feedback Control of Discrete-time Systems

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***Abstract*— This paper is concerned with the problems of H_∞ static output feedback (SOF) control synthesis for discrete-time systems. New linear matrix inequality (LMI) characterizations are derived, which enable one SOF controller by using parameter-dependent Lyapunov function. The relationship between the proposed methods include and the existing ones are clarified, which shows that our new results include those results as special cases. Numerical examples are included for illustration.**

I. Introduction

Static output feedback (SOF) control synthesis problem has been a longstanding one in the control community and has been investigated extensively (see [9]). It is generally cast as a biaffine matrix inequality (BMI) one which is known to be NP-hard and cumbersome in numerical.

During the last decades, many researchers have been tried their best to obtain some sufficient conditions in terms of linear matrix inequalities (LMIs) for SOF controller design at the expense of necessity. Sufficient LMI conditions are presented in [12] [13] [14] by forcing the Lyapunov variable to have a special structure and in [15] by inserting a linear matrix equality constraint on a Lyapunov functions. Special congruence transformations are adopted in [16] [17] respectively for continuous-time and discrete-time systems to exploit more degrees of in Lyapunov functions. An linear parameter dependent (LPD) method is proposed in [18] [19]. To remove the restriction on the Lyapunov variable, a LMI based method with a block diagonal structure slack variable has been proposed in recent papers, such as [1] for H_∞ SOF stabilization and [2] [3] for SOF stabilization. Since all the above mentioned LMI-based conditions are only of sufficiency, thus, how to reduce the conservativeness becomes the most important issue in this research direction.

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In this paper, new sufficient conditions for H_∞ output feedback stabilization of linear discrete-time systems with/without polytopic uncertainties are proposed. Those design conditions are composed by two parallel sets of LMIs, i.e. then there may exist some cases that one set of LMIs is feasible while the other set of LMIs is infeasible and vise versa. However, all the LMI conditions presented in a uniform framework based on Finsler's lemma. Therefore, one can clearly see the distinctness and relationship between them. Moveover, The connections between the proposed methods include and the existing ones [1] [2] [3] are pointed out, it shows that the results of [1] [2] [3] can be recovered form our proposed methods by specializing the key slack variables. Then, the proposed certainly lead to less conservative results.

II. Preliminaries

Consider a nominal linear time-invariant discrete-time system described by:

$$\Sigma_0: \quad \begin{aligned} x(k+1) &= Ax(k) + B_1\omega(k) + B_2u(k) \\ z(k) &= C_1x(k) + D_{11}\omega(k) + D_{12}u(k) \\ y(k) &= C_2x(t) \end{aligned} \tag{1}$$

and an uncertain linear discrete-time system described by

$$\Sigma_\Delta: \quad \begin{aligned} x(k+1) &= \mathscr{A}x(k) + \mathscr{B}_1\omega(k) + \mathscr{B}_2u(k) \\ z(k) &= \mathscr{C}_1x(k) + \mathscr{D}_{11}\omega(k) + \mathscr{D}_{12}u(k) \\ y(k) &= \mathscr{C}_2x(t) \end{aligned} \tag{2}$$

where $x(k) \in \mathbb{R}^n$ is the state, $y(k) \in \mathbb{R}^q$ is the measured output, $z(k) \in \mathbb{R}^p$ is the regulated output, $w(k) \in \mathbb{R}^p$ is the exogenous input, and $u(k)$ is the control input The matrices $\mathscr{A}, \mathscr{B}_1, \mathscr{B}_2, \mathscr{C}_1, \mathscr{D}_{12}, \mathscr{C}_2$ of uncertain system Σ_Δ (2) belong to the following uncertainty polytope:

$$\mathcal{P} = \left\{ \left[\begin{array}{c|c|c} \mathscr{A} & \mathscr{B}_1 & \mathscr{B}_2 \\ \hline \mathscr{C}_1 & \mathscr{D}_{11} & \mathscr{D}_{12} \\ \hline \mathscr{C}_2 & \mathscr{D}_{21} & \end{array}\right] = \sum_{i=1}^{N} \alpha_i \left[\begin{array}{c|c|c} A_i & B_{1i} & B_{2i} \\ \hline C_{1i} & D_{11i} & D_{12i} \\ \hline C_{2i} & & \end{array}\right] \right\} \tag{3}$$

where $\alpha_i > 0$, $\sum_{i=1}^{N} \alpha_i = 1$

The following lemmas play an essential role in the later development.

Lemma 1. *(Finsler' Lemma)*
Letting that $\xi \in \mathbb{R}^N$, $\mathcal{P} = \mathcal{P}^T \in \mathbb{R}^{N\times N}$, and $\mathcal{H} \in \mathbb{R}^{M\times N}$

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such that $rank(\mathcal{H}) = R < N$, then the following statements are equivalent:
(a) $\xi^T \mathcal{P} \xi < 0$, *for all* $\xi \neq 0$, $\mathcal{H}\xi = 0$;
(b) $\exists \mathcal{M} \in \mathbb{R}^{N\times M}$ *such that* $\mathcal{P} + He(\mathcal{M}\mathcal{H}) < 0$.

Lemma 2. *(Geromel and Korogui [4])*
If the symmetric matrices $V_{ij} \in R^{n\times n}$ are such that

$$\begin{aligned} &V_{ij} + V_{ji} \geq 0, && 1 \leq j < i \leq N \\ &\sum_{i=1}^{N} (V_{ij} + V_{ji}) \leq 0, && j = 1, ..., N \end{aligned} \tag{4}$$

then the following inequlity

$$\sum_{i=1}^{N} \sum_{j=1}^{N} \alpha_i \alpha_j V_{ij} \leq 0 \qquad \forall \alpha \in \Lambda \tag{5}$$

holds, where Λ is the simplex

$$\Lambda := \left\{ \alpha \in R^N : \alpha_i \geq 0, \; \sum_{i=1}^{N} \alpha_i = 1 \right\} \tag{6}$$

Lemma 3. *(Bounded Real Lemma).*
The unforced nominal system

$$\begin{aligned} x(k+1) &= Ax(k) + B\omega(k) \\ z(k) &= Cx(k) + D\omega(k) \end{aligned} \tag{7}$$

is said to be stable with $\|H(z)\|_\infty < \gamma$ if one of the following three conditions holds, where $H(z)$ is the transfer function of (7), i.e. $H(z) = C(zI - A)^{-1}B + D$.

(3.a) *There exist matrices $P = P^T > 0$ such that*

$$\begin{bmatrix} A^TPA - P + C^TC & A^TPB + C^TD \\ B^TPA + D^TC & B^TPB + D^TD - \gamma^2 I \end{bmatrix} < 0 \tag{8}$$

(3.b) *There exist matrix P, G such that*

$$\begin{bmatrix} P - G - G^T & 0 & G^TA^T & G^TC^T \\ 0 & -I & B^T & D^T \\ AG & B & -P & 0 \\ CG & D & 0 & -\gamma^2 I \end{bmatrix} < 0 \tag{9}$$

(3.c) *There exist matrix $P = P^T > 0, G, F$ such that*

$$\begin{bmatrix} P - G - G^T & 0 & G^TA^T - F & G^TC^T \\ 0 & -I & B^T & D^T \\ AG - F^T & B & -P + AF + F^TA^T & F^TC^T \\ CG & D & CF & -\gamma^2 I \end{bmatrix} < 0 \tag{10}$$

Proof. We will proof the condition (3.c) first here. Notice that the inequality (10) can be rewritten as

$$\mathcal{P} + He(\mathcal{M}\mathcal{H}) < 0 \tag{11}$$

where

$$\mathcal{P} = \begin{bmatrix} P & 0 & 0 & 0 \\ 0 & -P & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & -\gamma^2 I \end{bmatrix}$$
$$\mathcal{M} = \begin{bmatrix} G^T & 0 \\ F^T & 0 \\ 0 & I \\ 0 & 0 \end{bmatrix} \tag{12}$$
$$\mathcal{H} = \begin{bmatrix} -I & 0 \\ A & B \\ 0 & -I \\ C & D \end{bmatrix}^T$$

Let us consider the following system

$$\begin{aligned} x'(k+1) &= A^T x'(k) + C^T \omega'(k) \\ z'(k) &= B^T x'(k) + D^T \omega'(k) \end{aligned} \tag{13}$$

which is the dual version of system (7). Form the well-known duality of linear systems, we have

$$\mathcal{H}\xi = 0 \tag{14}$$

where

$$\xi = \begin{bmatrix} x'(k+1) \\ x'(k) \\ z'(k) \\ \omega'(k) \end{bmatrix}$$

Now, one can conclude that the following inequality

$$\begin{bmatrix} * \\ * \\ * \\ * \end{bmatrix}^T \begin{bmatrix} P & 0 & 0 & 0 \\ 0 & -P & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & -\gamma^2 I \end{bmatrix} \begin{bmatrix} x'(k+1) \\ x'(k) \\ z'(k) \\ \omega'(k) \end{bmatrix} < 0 \tag{15}$$

holds by applying the Finsler Lemma. It can be easily verified that the inequality (15) coincide with the time domain interpretation of bounded realness. i.e. the system (7) is stable with $\|H(z)\|_\infty < \gamma$.
Similar with the above statements, one can proof the condition (3.b) just by letting the matrix $\mathcal{M}$ of (12) as

$$\mathcal{M} = \begin{bmatrix} G^T & 0 \\ 0 & 0 \\ 0 & I \\ 0 & 0 \end{bmatrix}$$

Likewise, one can fulfill the proof of condition (3.a) by letting

$$\mathcal{M} = \begin{bmatrix} P & 0 \\ 0 & 0 \\ 0 & I \\ 0 & 0 \end{bmatrix}$$

Remark 1.
a) Let us look into the three matrices $\mathcal{P}, \mathcal{M}, \mathcal{H}$ in the equality (12), the matrix $\mathcal{P}$ represents the desired performance (i.e. bounded realness), the matrix $\mathcal{H}$ is composed by system matrices, and $\mathcal{M}$ is a multiplier which decoupling the

system matrices and the Lyapunov matrix P, and we call the matrices G, F in $\mathcal{M}$ auxiliary variables throughout of this paper.
b) Introducing auxiliary variables for system analysis or synthesis is known as the parameter dependent Lyapunov function (PDLF) method. In [20], Oliveira et al derived a relaxed LMI stability condition for discrete-time system by introducing one auxiliary variable (i.e. G). In [7], Peaucelle et al proposed further less conservative robust stability conditions for continuous-time and discrete-time systems by introducing two auxiliary matrix variables (i.e. G and F). The PDLF method has been extended and applied on many system analysis and design problems. For example, in [8], Duan et al given an improved robust filter design by proposing a proper structure of the auxiliary variables. Here, the two auxiliary variables G and F were introduced just by employing the Finsler lemma.

The objective of this paper is to design a static output feedback controller

$$u(k) = Ky(k) \tag{16}$$

for the systems (1) and (2) such that the resulting closed-loop systems (17) and (18):

$$\begin{aligned} \Sigma_0^c: \; x(k+1) &= Ax(k) + B_1\omega(k) + B_2 u(k) \\ z(k) &= (C_1 + D_{12}KC_2)x(k) + D_{11}\omega(k) \end{aligned} \tag{17}$$

and

$$\begin{aligned} \Sigma_\Delta^c: \; x(k+1) &= \mathscr{A}x(k) + \mathscr{B}_1\omega(k) + \mathscr{B}_2 u(k) \\ z(k) &= (\mathscr{C}_1 + \mathscr{D}_{12}K\mathscr{C}_2)x(k) + \mathscr{D}_{11}\omega(k) \end{aligned} \tag{18}$$

are (robust) stable with $\|H(z)\|_\infty < \gamma$.

III. Main Results

A. H_∞ static output feedback control for nominal systems

Without loss of generality, assume that the output matrix C_2 of the nominal system is of full row rank, then there exist nonsingular transformation matrix T such that

$$C_2 T = \begin{bmatrix} I & 0 \end{bmatrix} \tag{19}$$

Remark 2. *For any given C_2, the corresponding T generally are not unique. A special T can be obtained by following formula,*

$$T = \begin{bmatrix} C_2^T(C_2C_2^T)^{-1} & C_2^\perp \end{bmatrix} \tag{20}$$

where $C_2 C_2^\perp = 0$

Theorem 1. *If there exist a scalar λ, symmetric positive matrix P, and matrices G, F, Y with the following structure*

$$G = \begin{bmatrix} G^{11} & 0 \\ G^{21} & G^{22} \end{bmatrix} F = \begin{bmatrix} \lambda G^{11} & 0 \\ F^{21} & F^{22} \end{bmatrix} Y = \begin{bmatrix} Y_1 & 0 \end{bmatrix} \tag{21}$$

satisfying one of the following LMI conditions

$$T1.1: \quad \Phi = \begin{bmatrix} \Phi_{11} & * & * & * \\ 0 & -I & * & * \\ \Phi_{31} & B_1 & \Phi_{33} & * \\ \Phi_{41} & D_{11} & \Phi_{34} & -\gamma^2 I \end{bmatrix} < 0 \tag{22}$$

where

$$\begin{aligned} \Phi_{11} &= P - TG - G^T T^T \\ \Phi_{31} &= ATG + B_2 Y - F^T T^T \\ \Phi_{41} &= C_1 TG + D_{12} Y \\ \Phi_{33} &= -P + He(ATF + \lambda B_2 Y) \\ \Phi_{34} &= C_1 TF + \lambda D_{12} Y \end{aligned}$$

$$T1.2: \quad \Psi = \begin{bmatrix} \Psi_{11} & * & * & * \\ 0 & -I & * & * \\ \Psi_{31} & B_1 & \Psi_{33} & * \\ \Psi_{41} & D_{11} & \Psi_{34} & -\gamma^2 I \end{bmatrix} < 0 \tag{23}$$

where

$$\begin{aligned} \Psi_{11} &= P - TGT^T - TG^T T^T \\ \Psi_{31} &= ATGT^T + B_2 YT^T - TF^T T^T \\ \Psi_{41} &= C_1 TGT^T + D_{12} YT^T \\ \Psi_{33} &= -P + He(ATFT^T + \lambda B_2 YT^T) \\ \Psi_{34} &= C_1 TF + \lambda D_{12} Y \end{aligned}$$

then the static output feedback controller (16) with $K = Y_1 G_{11}^{-1}$ renders the the closed-loop system (17) stable with $\|H(z)\|_\infty < \gamma$

Proof. From the structure of G, Y and (3), (10), we can obtain

$$\begin{aligned} Y &= \begin{bmatrix} KG_{11} & 0 \end{bmatrix} = \begin{bmatrix} K & 0 \end{bmatrix} \begin{bmatrix} G_{11} & 0 \\ G_{21} & G_{22} \end{bmatrix} \\ &= K \begin{bmatrix} I & 0 \end{bmatrix} T^{-1} TG = KC_2 TG \end{aligned} \tag{24}$$

by matrix substitution, one can derive that the LMI condition (22) is equivalent to

$$\mathcal{P} + He(\mathcal{M}\mathcal{H}) < 0 \tag{25}$$

where

$$\begin{aligned} \mathcal{P} &= \begin{bmatrix} P & 0 & 0 & 0 \\ 0 & -P & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & -\gamma^2 I \end{bmatrix} \\ \mathcal{M} &= \begin{bmatrix} G^T T^T & 0 \\ F^T T^T & 0 \\ 0 & I \\ 0 & 0 \end{bmatrix} \\ \mathcal{H} &= \begin{bmatrix} -I & 0 \\ A + B_2 KC_2 & B_1 \\ 0 & -I \\ C_1 + D_{12}KC_2 & D_{11} \end{bmatrix}^T \end{aligned} \tag{26}$$

Similar with the proof of Lemma 3, one can obtain that the static output feedback controller $K = Y_1 G_{11}^{-1}$ renders the closed-loop system (17) stable with $\|H(z)\|_\infty < \gamma$. Moreover, form (22), we can deduce that the matrix G is positive-definite (not necessarily symmetric) which implies that the matrix G, and implicitly G_{11} is invertible. Then

$G_{11}K = Y_1$ admits the solution of the controller gain $K = G_{11}^{-1}Y_1$.

One can proof the other part of this theorem (i.e. condition *T1.2*) in same manner, the only difference lies in letting $\mathcal{M}$ in (41) as

$$\mathcal{M} = \begin{bmatrix} TG^TT^T & 0 \\ TF^TT^T & 0 \\ 0 & I \\ 0 & 0 \end{bmatrix} \tag{27}$$

Thus, the proof is complete. □

Remark 3.
a): since λ is a scalar, then the corresponding LMI optimization problem (22) and (23)are convex for an given λ. One can obtain the final optimized results by using line searches
b): Theorem 1 presents sufficient conditions for designing H_∞ static output feedback controllers for discrete-time system (1). The conditions (22) and (23) are parallelled with each other instead of equivalent, then there may exist some cases that the condition (22) is feasible while the condition (23) is infeasible and vise versa, using the two conditions (22) and (23) together will increasing the possibility of solvability for an given H_∞ static output feedback control problem.

B. Relationships with existing results

There are some relevant existing results that designing static output feedback controller via LMI approach. Lee presents H_∞ LMI design condition in [1]. Moreover, Oliveira [2] and Dong [3] proposed LMI conditions for static output feedback stabilization, which can be directly extended to H_∞ static output feedback controller design. The above mentioned LMI methods are not presented here for the reasons of space, please see [1] [2] [3] for detail. In the sequel, we will show that the LMI methods [1] [2] [3] are special cases of our result (i.e. Theorem 1).

a): How to recover the result of [1]
Letting matrix $\mathcal{M}$ in (41) as

$$\mathcal{M} = \begin{bmatrix} TG^TT^T & 0 \\ TF^TT^T & 0 \\ 0 & I \\ 0 & 0 \end{bmatrix} \text{ with } G = \begin{bmatrix} G^{11} & 0 \\ 0 & G^{22} \end{bmatrix}$$

b): How to recover the result of [2]
Letting matrix $\mathcal{M}$ in (41) as

$$\mathcal{M} = \begin{bmatrix} TG^TT^T & 0 \\ TF^TT^T & 0 \\ 0 & I \\ 0 & 0 \end{bmatrix} \text{ with } G = \begin{bmatrix} G^{11} & 0 \\ G^{21} & G^{22} \end{bmatrix}$$

c): How to recover the result of [3]
Letting matrix $\mathcal{M}$ in (41) as

$$\mathcal{M} = \begin{bmatrix} G^TT^T & 0 \\ F^TT^T & 0 \\ 0 & I \\ 0 & 0 \end{bmatrix} \text{ with } G = \begin{bmatrix} G^{11} & 0 \\ G^{21} & G^{22} \end{bmatrix}$$

C. H_∞ static output feedback control for uncertain systems

Theorem 2. *If there exist symmetric positive matrices P_{ij}, V_{ij}, and matrices G_{ij}, F_{ij}, Y with the following structure*

$$G_{ij} = \begin{bmatrix} G^{11} & 0 \\ G_{ij}^{21} & G_{ij}^{22} \end{bmatrix} Y = \begin{bmatrix} Y_1 & 0 \end{bmatrix} F_{ij} = \begin{bmatrix} \lambda G^{11} & 0 \\ F_{ij}^{21} & F_{ij}^{22} \end{bmatrix}$$

$$V_{ij} = \begin{bmatrix} V_{ij}^{11} & * & * & * \\ V_{ij}^{21} & V_{ij}^{22} & * & * \\ V_{ij}^{31} & V_{ij}^{32} & V_{ij}^{33} & * \\ V_{ij}^{41} & V_{ij}^{42} & V_{ij}^{43} & V_{ij}^{44} \end{bmatrix}$$

satisfying one of the following LMI conditions:

$$\textit{T2.1}: \qquad \Phi_i < 0, \quad 1 \le i \le N \tag{28}$$

$$\textit{T2.2}: \qquad \Psi_i < 0, \quad 1 \le i \le N \tag{29}$$

$$\textit{T2.3}: \ \Upsilon_{ij} = \begin{bmatrix} \Upsilon_{ij}^{11} & * & * & * & * \\ \Upsilon_{ij}^{21} & \Upsilon_{ij}^{22} & * & * & * \\ \Upsilon_{ij}^{31} & \Upsilon_{ij}^{32} & \Upsilon_{ij}^{33} & * & * \\ \Upsilon_{ij}^{41} & \Upsilon_{ij}^{42} & \Upsilon_{ij}^{43} & \Upsilon_{ij}^{44} & * \\ \Upsilon_{ij}^{51} & \Upsilon_{ij}^{52} & \Upsilon_{ij}^{53} & \Upsilon_{ij}^{54} & \Upsilon_{ij}^{55} \end{bmatrix} < 0, \\ 1 \le i,j \le N \\ V_{ij} + V_{ji} \ge 0, 1 \le j < i \le N \\ \sum_{i=1}^{N} (V_{ij} + V_{ji}) \le 0, j = 1, ..., N \tag{30}$$

where

$$\begin{aligned} &\Upsilon_{ij}^{11} = -T_iG_i - G_iT_i^T; && \Upsilon_{ij}^{22} = -I + V_{ij}^{11} \\ &\Upsilon_{ij}^{21} = 0; && \Upsilon_{ij}^{32} = B_{1i} + V_{ij}^{21} \\ &\Upsilon_{ij}^{31} = A_iT_iG_i + B_{2i}Y; && \Upsilon_{ij}^{42} = D_{11i} + V_{ij}^{31} \\ &\Upsilon_{ij}^{41} = C_{1i}T_iG_i + D_{12i}Y; && \Upsilon_{ij}^{52} = V_{ij}^{41} \\ &\Upsilon_{ij}^{51} = P_j && \end{aligned}$$

$$\begin{aligned} &\Upsilon_{ij}^{33} = -P_i + V_{ij}^{22}; && \Upsilon_{ij}^{44} = -\gamma^2 I + V_{ij}^{33} \\ &\Upsilon_{ij}^{43} = V_{ij}^{32}; && \Upsilon_{ij}^{54} = V_{ij}^{43} \\ &\Upsilon_{ij}^{53} = V_{ij}^{42}; && \Upsilon_{ij}^{55} = -P_j + V_{ij}^{44} \end{aligned}$$

where Φ_i(28) Ψ_i(29) can be obtained by Φ (22) Ψ (23) by replacing

$$(A, B_1, B_2, C_1, D_{11}, D_{12}, C_2, P, G_{21}, G_{22}, F_{21}, F_{22})$$

as

$$(A, B_{1i}, B_{2i}, C_{1i}, D_{11i}, D_{12i}, C_{2i}P_i, G_{21i}, G_{22i}, F_{21i}, F_{22i})$$

where the transformation matrices $T_i, i = 1, ..., N$ are selected similar with (19)i.e.

$$T_i = \begin{bmatrix} C_{2i}^T(C_{2i}C_{2i}^T)^{-1} & C_{2i}^{\perp} \end{bmatrix}$$

then the static output feedback controller (16) with $K = Y_1G_{11}^{-1}$ renders the the closed-loop system (18) robust stable with $\|H(z)\|_\infty < \gamma$.

Proof. Conditions *T2.1* and *T2.2* are trivial extensions of Theorem 1, and the proof of them are omitted here.

Denote $W_i = (T_i G_i)^{-1}$, firstly, pre- and post-multiplying Υ_{ij} of (30) by

$$\mathbb{W}_{ij} = \begin{bmatrix} W_i^T & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix}$$

and its transpose, it follows that:

$$\mathbb{W}_{ij}\Upsilon_{ij}\mathbb{W}_{ij}^T < 0 \tag{31}$$

Multiplying (31) by $\alpha_i\alpha_j$ and summing them, we have

$$\Omega_1 + \Omega_2 < 0 \tag{32}$$

where

$$\Omega_1 = \begin{bmatrix} -\mathscr{W}^T - \mathscr{W} & * & * & * & * \\ 0 & -I & * & * & * \\ \mathscr{A} + \mathscr{B}_2 K \mathscr{C}_2 & \mathscr{B}_1 & -\mathscr{P} & * & * \\ \mathscr{C}_1 + \mathscr{D}_{12} K \mathscr{C}_2 & \mathscr{D}_{11} & 0 & -\gamma^2 I & 0 \\ \mathscr{P}\mathscr{W} & 0 & 0 & 0 & -\mathscr{P} \end{bmatrix} \tag{33}$$

$$\Omega_2 = \begin{bmatrix} 0 & * & * & * & * \\ 0 & & & & \\ 0 & & \mathscr{V} & & \\ 0 & & & & \\ 0 & & & & \end{bmatrix} \tag{34}$$

where $\mathscr{A}, \mathscr{B}_2, \mathscr{C}_1, \mathscr{C}_2, \mathscr{D}_{12}$ are same as in (3), and

$$\mathscr{V} = \sum_{i=1}^{N}\sum_{j=1}^{N} \alpha_i\alpha_j \begin{bmatrix} V_{ij}^{11} & * & * & * \\ V_{ij}^{21} & V_{ij}^{22} & * & * \\ V_{ij}^{31} & V_{ij}^{32} & V_{ij}^{33} & * \\ V_{ij}^{41} & V_{ij}^{42} & V_{ij}^{43} & V_{ij}^{44} \end{bmatrix}$$

$$\mathscr{P} = \sum_{j=1}^{N} \alpha_i P_i$$

$$\mathscr{W} = \sum_{j=1}^{N} \alpha_i T_i G_i$$

Now, let pre- and post-multiply (35) by

$$\begin{bmatrix} \mathscr{W}^{-T} & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix}$$

and its transpose,

$$\Pi_1 + \Pi_2 < 0 \tag{35}$$

where

$$\Pi_1 = \begin{bmatrix} -\mathscr{W}^{-T} - \mathscr{W}^{-1} & * & * & * & * \\ 0 & -I & * & * & * \\ (\mathscr{A} + \mathscr{B}_2 K \mathscr{C}_2)\mathscr{W}^{-1} & \mathscr{B}_1 & -\mathscr{P} & * & * \\ (\mathscr{C}_1 + \mathscr{D}_{12} K \mathscr{C}_2)\mathscr{W}^{-1} & \mathscr{D}_{11} & 0 & -\gamma^2 I & * \\ \mathscr{P} & 0 & 0 & 0 & -\mathscr{P} \end{bmatrix} \tag{36}$$

$$\Pi_2 = \begin{bmatrix} 0 & * & * & * & * \\ 0 & & & & \\ 0 & & \mathscr{V} & & \\ 0 & & & & \\ 0 & & & & \end{bmatrix} < 0 \tag{37}$$

On the other hand, applying lemma 2, we have:

$$\Pi_2 < 0 \tag{38}$$

From (35) and (38), we have $\Pi_1 < 0$, applying Schur complement for $\Pi_1 < 0$, it directly leads to:

$$\begin{bmatrix} \mathscr{P} - \mathscr{W}^{-T} - \mathscr{W}^{-1} & * & * & * \\ 0 & -I & * & * \\ (\mathscr{A} + \mathscr{B}_2 K \mathscr{C}_2)\mathscr{W}^{-1} & \mathscr{B}_1 & -\mathscr{P} & * \\ (\mathscr{C}_1 + \mathscr{D}_{12} K \mathscr{C}_2)\mathscr{W}^{-1} & \mathscr{D}_{11} & 0 & -\gamma^2 I \end{bmatrix} < 0 \tag{39}$$

Which is equivalent to the following inequality

$$\mathbb{P} + He(\mathbb{M}\mathbb{H}) < 0 \tag{40}$$

where

$$\mathbb{P} = \begin{bmatrix} \mathscr{P} & 0 & 0 & 0 \\ 0 & -\mathscr{P} & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & -\gamma^2 I \end{bmatrix}$$

$$\mathbb{M} = \begin{bmatrix} \mathscr{W}^{-T} & 0 \\ 0 & 0 \\ 0 & I \\ 0 & 0 \end{bmatrix} \tag{41}$$

$$\mathbb{H} = \begin{bmatrix} -I & 0 \\ \mathscr{A} + \mathscr{B}_2 K \mathscr{C}_2 & \mathscr{B}_1 \\ 0 & -I \\ \mathscr{C}_1 + \mathscr{D}_{12} K \mathscr{C}_2 & 0 \end{bmatrix}^T$$

From Finsler's lemma, we can obtain that the closed-loop system (18) is robust stable with $\|H(z)\|_\infty < \gamma$. Moreover, form (30), we can deduce that the matrix G_{ij} is positive-definite (not necessarily symmetric) which implies that the matrix G_{ij}, and implicitly G_{11} is invertible. Then $G_{11}K = Y_1$ admits the solution of the controller gain $K = G_{11}^{-1}Y_1$. Thus, the proof is complete. □

Remark 4. *The matrices V_{ij} are additional slack variables introduced here for further reduction of conservatism by considering the relations of parameter α_i. However, in order to obtain convex conditions, the technique is only used for one part of the global matrix inequality, (i.e., this type of slack variables only emergence on the blocks (2,2), (2,3) (2,4), (2,5), (3,2) (3,3), (3,4), (3,5), (4,2), (4,3)(4,4), (4,5), (5,2), (5,3), (5,4), (5,5) of (30)). If this type of slack variables in (30) is applied to the global matrix inequality, less conservative result may be given. But, the corresponding conditions will become nonlinear, and, hence, not convex.*

IV. Numerical examples

Example 1 Consider the following unstable plant model which is borrowed from [1]

$$\left[\begin{array}{c|c|c} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ \hline C_2 & & \end{array}\right] = \left[\begin{array}{ccc|cc|cc} \alpha & 0.3 & 2 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0.3 & 0.6 & -0.6 & 0 & 0 & 1 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ \hline 1 & 1 & 0 & & & & \end{array}\right] \quad (42)$$

where the positive number α is a unknown constant. Table I shows the numerical results. In Table I, K is the feedback gains, $\|H_{z\omega}\|_\infty$ is the actual H_∞ performance achieved by K. It is shown that our method leads to less conservative results for any given α. The merit of reducing the conservatism lies on the slack variables G, F.

		Theorem 1	LMI Method [1]
$\alpha = 3.0$	K	$\begin{bmatrix} -0.9866 \\ -0.1907 \end{bmatrix}$	$\begin{bmatrix} -1.0192 \\ -0.3760 \end{bmatrix}$
	γ	10.1240	33.5548
	$\|H_{z\omega}\|_\infty$	9.1140	23.9512
	λ	-0.28	—
$\alpha = 3.1$	K	$\begin{bmatrix} -1.0097 \\ -0.1564 \end{bmatrix}$	$\begin{bmatrix} -1.0442 \\ -0.3441 \end{bmatrix}$
	γ	13.0519	97.3011
	$\|H_{z\omega}\|_\infty$	11.1999	48.2416
	λ	-0.29	—
$\alpha = 3.2$	K	$\begin{bmatrix} -1.0318 \\ -0.1224 \end{bmatrix}$	×
	γ	17.5058	×
	$\|H_{z\omega}\|_\infty$	13.9340	×
	λ	-0.30	—
$\alpha = 3.3$	K	$\begin{bmatrix} -1.0530 \\ -0.0891 \end{bmatrix}$	×
	γ	25.2006	×
	$\|H_{z\omega}\|_\infty$	17.6014	×
	λ	-0.31	—

TABLE I
Numerical results of Example 1 with different α

V. Conclusions

New LMI conditions for designing H_∞ static output feedback controller for discrete-time systems are proposed. The relationships between the proposed methods and the existing results have been clarified, which show that our new results are of less conservatism. Extensions of our results to polytopic uncertain systems are also included.

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