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Robust $\mathcal{H}_2$ Filtering for Discrete LTI Systems with Linear Fractional Representation

Rubens H. Korogui and José C. Geromel

Abstract— **This paper introduces a new approach to $\mathcal{H}_2$ robust filtering design for discrete LTI systems subjected to linear fractional parameter uncertainty representation. We calculate a performance certificate in terms of the gap between the lower and the upper bounds of a minimax programming problem, which defines the optimal robust filter and the associated equilibrium cost. The calculations are performed through convex programming methods, applying slack variables, known as multipliers, to handle the fractional dependence of the plant transfer function with respect to the parameter uncertainty. The theory is illustrated by means of an example borrowed from the literature and a practical application involving the design of a robust filter for the load voltage estimation on a transmission line with a stub feeding an unknown resistive load.**

I. INTRODUCTION

Over the last years the problem of robust estimation for uncertain dynamic systems in the context of convex programming has been widely studied. One approach to this problem is to consider it in the Kalman filtering context [1], in which the uncertain system is assumed to be subjected to white noise, leading to the formulation of the $\mathcal{H}_2$ robust filter design problem. In other words, the goal is to design a unique linear filter in order to minimize the worst case mean square estimation error, able to cope with different models generated by a set of uncertain parameters.

Many works in the literature, as for example [3], [4], [5], [12], [13], address the considered problem when its state space matrices representation depend linearly on the uncertain parameters. On the other hand, some recent works faced the more general problem of robust filter design for uncertain linear fractional transformation (LFT) systems (see [6], [8] and [11]). For continuous-time systems [11] presents necessary and sufficient conditions to design a robust $\mathcal{H}_2$ filter, but under the difficulty of an appropriate choice of the multiplier order and dynamics. In this paper we consider discrete LTI systems whose parameter uncertainty, supposed to belong to a polytopic set, is described in the LFT form. This assumption enables us to take into account the nonlinear dependence of the state space matrices with respect to the parameter uncertainty, a situation that often occurs in practice, as in the transmission line model presented afterwards.

The procedure adopted here is based on the recent results of [4] and [8], where we propose to determine lower and upper bounds to the optimal $\mathcal{H}_2$ cost for the robust filtering problem, as a way to certify the optimality gap and, by consequence, the distance from a particular filter to the optimal robust one. In a first step we calculate the lower bound of the cost and provide a filter, prior of eventual poles and zeros cancellations, of order equal to the order of the plant times the the number of vertices of the uncertainty convex polytopic domain. In a second step we determine a robust filter of order equal to the order of the one associated to the lower bound using the previous calculations and well known results on multiplier theory developed in [7]. From [4] we conclude that the greater order of the filter compared to the order of the plant appears to be essential to reduce conservatism, yielding more accurate results against the previous design procedures.

In the next section we state the minimax $\mathcal{H}_2$ robust filtering problem and the model for the uncertain system to be dealt with along the text. In sections III and IV we proceed by the calculation of lower and upper bounds, respectively, to the equilibrium solution of the minimax problem, by means of LMIs [2], and the corresponding optimistic and robust filter parametrization. In Section V we analyze the application of the results obtained so far to an example borrowed from the literature. Section VI is devoted to transmission lines modelling under load uncertainty and to the problem of its one-end voltage estimation when it is subject to reflections due to impedance mismatch and stub connection. Finally, Section VII contains the conclusion and final remarks.

The notation used throughout is standard. Capital letters denote matrices and small letters denote vectors. For scalars, small Greek letters are used and $\mathbb{N} = \{1, \cdots, N\}$. For real matrices or vectors $(')$ indicates its transpose. For square matrices $\mathrm{Tr}(X)$ denotes the trace function of X being equal to the sum of its eigenvalues and $\mathrm{diag}(X, Y)$ generates a block diagonal matrix in whose main diagonal are the matrices X and Y. For the sake of easing the notation of partitioned symmetric matrices, the symbol $(\bullet)$ denotes generically each of its symmetric blocks. For matrices or transfer functions X_λ denotes the convex combination $X_\lambda := \sum_{i=1}^{N} \lambda_i X_i$, where λ belongs to the unitary simplex

$$\Lambda = \left\{ \lambda \in \mathbb{R}^N \ : \ \sum_{i=1}^{N} \lambda_i = 1 \ , \quad \lambda_i \geq 0 \right\} \tag{1}$$

Finally, the following notation

$$G(\zeta) = C(\zeta I - A)^{-1} B + D = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right] \tag{2}$$

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The authors are with DSCE/School of Electrical and Computing Engineering, UNICAMP, CP 6101, 13083 - 970, Campinas, SP, Brazil korogui,geromel@dsce.fee.unicamp.br

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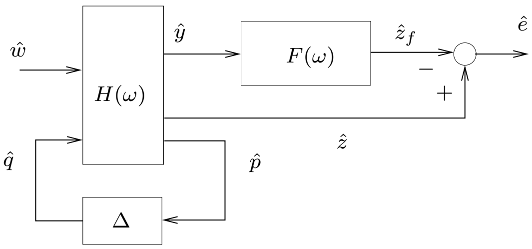

Fig. 1. Filtering Structure

is used for transfer functions, where the real matrices A, B, C and D of compatible dimensions define a possible state space realization. Besides, $G(\omega)$ denotes $G(\zeta)$ calculate at $\zeta = e^{j\omega}$, where $\omega \in \mathbb{R}$. For any real signal ξ, defined in the discrete-time domain, $\hat{\xi}$ denotes its $\mathcal{Z}$ transform.

II. PROBLEM FORMULATION

Figure 1 shows the basic filtering structure design in terms of transfer functions, where $F(\omega)$ denotes the filter transfer function to be designed and $H(\omega)$ denotes the transfer function of an LTI system subject to structured uncertainties characterized by the following state space model

$$\begin{array}{rcl} x(k+1) & = & Ax + Eq + Bw \\ p & = & C\left[\begin{array}{c} x \\ w \end{array}\right] + Dq \\ q & = & \Delta p \quad , \quad \Delta \in \Xi \\ y & = & C_y x + D_y w \\ z & = & C_z x + D_z w \end{array} \tag{3}$$

where $x \in \mathbb{R}^n$ is the state, $q \in \mathbb{R}^m$ and $p \in \mathbb{R}^r$ are internal variables, $w \in \mathbb{R}^{m_w}$ is the external disturbance, $y \in \mathbb{R}^{r_y}$ is the measured output, $z \in \mathbb{R}^{r_z}$ is the output to be estimated and Ξ is the set of all feasible parameters uncertainty, defined by the convex hull

$$\Xi = \mathrm{co}\{\Delta_i \ : \ i \in \mathbb{N}\} \tag{4}$$

generated by N known matrices Δ_i for all $i \in \mathbb{N}$. Hence, any element of the set Ξ can be written in the form Δ_λ for some $\lambda \in \Lambda$. Furthermore, all matrices are supposed to be of compatible dimensions, yielding the following definition of the transfer function $H(\omega)$

$$H(\lambda,\omega) = \left[\begin{array}{c} T(\lambda,\omega) \\ S(\lambda,\omega) \end{array}\right] = \left[\begin{array}{c|c} A_\Delta(\lambda) & B_\Delta(\lambda) \\ \hline C_y & D_y \\ C_z & D_z \end{array}\right] \tag{5}$$

where

$$[A_\Delta(\lambda) \ B_\Delta(\lambda)] = [A \ B] + E(I - \Delta_\lambda D)^{-1}\Delta_\lambda C \tag{6}$$

makes clear the nonlinear dependence of the state space representation of the plant, with respect to $\lambda \in \Lambda$, whenever $D \neq 0$. It is assumed that $\det(I - \Delta_\lambda D) \neq 0$ for all $\lambda \in \Lambda$. Notice that this model is quite general and reduces to the structured LFT description considered in [6] from a particular choice of matrices C, D, E and the structure of $\Delta \in \Xi$. For this system, the filter transfer function $F(\omega)$ has to be designed in such a way that its output is the best estimate of $\hat{z}$ that can be obtained from the data contained in $\hat{y}$. Formally, the problem is expressed as

$$\min_{F \in \mathcal{F}} \max_{\lambda \in \Lambda} J(F(\omega), H(\lambda,\omega)) \tag{7}$$

where $J(F(\omega), H(\lambda,\omega)) = \|E_F(\lambda,\omega)\|_2^2$ is the $\mathcal{H}_2$ squared norm of the transfer function from the exogenous input $\hat{w}$ to the estimation error $\hat{e}$, that is $E_F(\lambda,\omega) = S(\lambda,\omega) - F(\omega)T(\lambda,\omega)$, and the set $\mathcal{F}$ is used to impose some desired characteristics to the optimal filter as, for instance, asymptotical stability.

However, the equilibrium solution of (7) is very difficult to calculate (see [10]). The main reason is the highly nonlinear dependence of the transfer function $H(\lambda,\omega)$ with respect to $\lambda \in \Lambda$, which makes the $\max$ problem in (7) hard to solve. To circumvent this difficulty, in this paper we adopt the same reasoning presented in [4] and we generalize those results, as done in [8], to cope with the linear fractional representation of the uncertainty. First we determine a lower bound to (7), by solving a problem that can be written in terms of LMIs. The optimal optimistic filter obtained by this way has order equal to the order of the plant times the number of vertices of the unitary simplex Λ (see [4], [5]), putting aside eventual poles and zeros cancellations. Afterwards, the filter associated to the lower bound defines a parametrization which enables us to determine a robust filter with a certification of the distance to the optimal robust filter provided by the equilibrium solution of problem (7).

III. OPTIMISTIC PERFORMANCE

In this section our purpose is to calculate a lower bound to the equilibrium cost (7), since in the general case of uncertain polytopic systems its global solution is virtually impossible to be exactly calculated. A lower bound of (7) is determined from

$$\begin{array}{l} \displaystyle\min_{F \in \mathcal{F}} \max_{\lambda \in \Lambda} J(F(\omega), H(\lambda,\omega)) \geq \\ \quad \geq \ \displaystyle\min_{F \in \mathcal{F}} \max_{i \in \mathbb{N}} J(F(\omega), H(e_i,\omega)) \\ \quad \geq \ \displaystyle\min_{F \in \mathcal{F}} \max_{\lambda \in \Lambda} \left\| \sum_{i=1}^{N} \lambda_i \left(S(e_i,\omega) - F(\omega)T(e_i,\omega)\right) \right\|_2^2 \end{array} \tag{8}$$

where e_i is the i-th row of the $N \times N$ identity matrix and it defines one of the N vertices of the parameter polytope Λ. The first inequality follows from the fact that the set of all vertices of Λ is a subset of Λ and the last one comes from the convexity of the functional $\|\cdot\|_2^2$, implying that the indicated maximum is attained at one vertex of the convex polytope Λ.

Using the results of [4], the minimax problem on the right hand side of (8) can be exactly solved. Thus, a lower bound of the equilibrium solution of (7) can be stated as

$$J_L = \min_{F \in \mathcal{F}} \max_{\lambda \in \Lambda} \|E_{F\lambda}(\omega)\|_2^2 \tag{9}$$

where the error transfer function $E_{F\lambda}(\omega) = S_\lambda(\omega) - F(\omega)T_\lambda(\omega)$ depends linearly on $\lambda \in \Lambda$. Considering the filter state space realization

$$F_L(\omega) = \left[\begin{array}{c|c} A_L & B_L \\ \hline C_L & D_L \end{array}\right] \tag{10}$$

and the matrices of compatible dimensions $A_E = \mathrm{diag}(A_\Delta(e_1), \cdots, A_\Delta(e_N))$, $C_Y = [C_y, \cdots, C_y]$, $C_Z = [C_z, \cdots, C_z]$ and $B(\lambda) = [\lambda_1 B_\Delta(e_1)', \cdots, \lambda_N B_\Delta(e_N)']'$, the error transfer function $E_{F\lambda}(\omega)$ produced by the filter (10) is given by

$$E_{F\lambda}(\omega) = \left[\begin{array}{cc|c} A_E & 0 & B(\lambda) \\ B_L C_Y & A_L & B_L D_y \\ \hline C_Z - D_L C_Y & -C_L & D_z - D_L D_y \end{array}\right] \tag{11}$$

where it is noticed that only the input matrix $B(\lambda)$ is affected by the parameter uncertainty $\lambda \in \Lambda$ and matrix A_E is of dimension $nN \times nN$, in accordance to the fact that the transfer functions $S_\lambda(\omega)$ and $T_\lambda(\omega)$ are of order nN (prior to possible poles and zeros cancellations). Under the error state space realization (11), the solution of problem (9) is addressed by the next theorem.

Theorem 1: The filtering design problem (9) is equivalent to the convex programming problem

$$J_L = \inf_{\sigma, W_i, X, L, K} \{\sigma \ : \ \mathrm{Tr}(W_i) < \sigma \ , \ i \in \mathbb{N}\} \tag{12}$$

where W_i and X are symmetric matrices and K, L are matrices of compatible dimensions satisfying

$$\begin{bmatrix} W_i & \bullet & \bullet \\ XB(e_i) + LD_y & X & \bullet \\ D_z - KD_y & 0 & I \end{bmatrix} > 0 \tag{13}$$

for all $i \in \mathbb{N}$ and

$$\begin{bmatrix} X & \bullet & \bullet \\ XA_E + LC_Y & X & \bullet \\ C_Z - KC_Y & 0 & I \end{bmatrix} > 0 \tag{14}$$

Proof: The proof of this theorem is based on the previous result of [4] and [5]. ■

Concerning this theorem, some facts must be pointed out. First, the linearity of the error transfer function $E_{F\lambda}(\omega)$ with respect to $\lambda \in \Lambda$ is crucial to obtain the result of Theorem 1 since, as proven in [5], for the special class of polytopic systems, such that $H(\lambda, \omega)$ depends linearly on $\lambda \in \Lambda$, it provides the global optimal solution of problem (7), and (8) holds with equality. Second, from the optimal solution of the convex problem (12), the optimal optimistic filter is given by (see [5], [4])

$$F_L(\omega) = \left[\begin{array}{c|c} A_E + X^{-1} L C_Y & -X^{-1} L \\ \hline C_Z - KC_Y & K \end{array}\right] \tag{15}$$

Although this obtained filter has order nN it has been verified in [4], by means of several examples, that due to eventual poles and zeros cancellations its order is, in general, sensibly smaller than nN. However, generally, the order of $F_L(\omega)$ remains greater than n, a fact that decisively contributes to improve performance. Finally, for perfectly known systems ($N = 1$) Theorem 1 provides the celebrated Kalman filter [1].

Based on the previous calculations, the next section is devoted to determine a robust filter and the associated upper bound to the $\mathcal{H}_2$ optimal equilibrium cost which defines, together with the lower bound, a certificate for robust performance.

IV. ROBUST PERFORMANCE

Once we have determined a filter $F_L(\omega)$ associated to the minimum lower bound of the filter design problem (7), our goal in this section is to design a robust filter $F_H(\omega) \in \mathcal{F}_H \subset \mathcal{F}$ associated to a robust performance level J_H guaranteed for all $\lambda \in \Lambda$. Note that $F_L(\omega)$ is not a robust filter, since its performance level J_L can not be guaranteed for all $\lambda \in \Lambda$ or, in other words, for all $\Delta \in \Xi$.

Adopting the same reasoning presented in [4], we propose to choose the set $\mathcal{F}_H$ as the set of all LTI causal filters of the form

$$F_H(\omega) = \left[\begin{array}{c|c} A_L & B_L \\ \hline C_H & D_H \end{array}\right] \tag{16}$$

where A_L and B_L are the matrices from the already determined state space realization of the optimistic filter $F_L(\omega)$, in (15), and C_H and D_H, of compatible dimensions, are to be determined. The rationale behind this approach is that $F_L(\omega) \in \mathcal{F}_H$ for an appropriate choice of matrices C_H and D_H. Thus, we can define an upper bound to problem (7) as being

$$\min_{F \in \mathcal{F}} \max_{\lambda \in \Lambda} \|E_F(\lambda, \omega)\|_2^2 \leq \min_{F \in \mathcal{F}_H} \max_{\lambda \in \Lambda} \|E_F(\lambda, \omega)\|_2^2 \tag{17}$$

where $E_F(\lambda, \omega) = S(\lambda, \omega) - F(\omega)T(\lambda, \omega)$ is the estimation error transfer function produced by a filter $F(\omega) \in \mathcal{F}_H$. The main difficulty we have to face in order to solve the problem stated on the right hand side of (17) stems from the nonlinear dependence of transfer functions $S(\lambda, \omega)$ and $T(\lambda, \omega)$ with respect to the uncertain parameter $\lambda \in \Lambda$. Considering the state space realization of any feasible filter $F_H(\omega) \in \mathcal{F}_H$, given by (16), and taking into account the state space equations (3), the dynamic of the estimation error is governed by

$$\begin{aligned} x(k+1) &= \mathcal{A}\tilde{x} + \mathcal{E}q + \mathcal{B}w \\ p &= \mathcal{C}\begin{bmatrix} \tilde{x} \\ w \end{bmatrix} + \mathcal{D}q \\ q &= \Delta p \ , \ \Delta \in \Xi \\ e &= \mathcal{C}_e \tilde{x} + \mathcal{D}_e w \end{aligned} \tag{18}$$

where $\tilde{x}' = [x_F' \ x']$ is the state vector composed by the state vectors of the filter and the plant, respectively, and the indicated matrices are given by

$$\mathcal{A} = \begin{bmatrix} A_L & B_L C_y \\ 0 & A \end{bmatrix}, \ \mathcal{E} = \begin{bmatrix} 0 \\ E \end{bmatrix}, \ \mathcal{B} = \begin{bmatrix} B_L D_y \\ B \end{bmatrix} \tag{19}$$

$$\mathcal{C} = \begin{bmatrix} 0 & C \end{bmatrix}, \ \mathcal{D} = D \tag{20}$$

and

$$\mathcal{C}_e = \begin{bmatrix} -C_H & C_z - D_H C_y \end{bmatrix}, \ \mathcal{D}_e = D_z - D_H D_y \tag{21}$$

Thus, the transfer function from the external disturbance $\hat{w}$ to the estimation error $\hat{e}$ can be readily calculated as being

$$E_F(\lambda,\omega) = \left[\begin{array}{c|c} \mathcal{A}_\Delta(\lambda) & \mathcal{B}_\Delta(\lambda) \\ \hline \mathcal{C}_e & \mathcal{D}_e \end{array}\right] \quad (22)$$

where

$$[\mathcal{A}_\Delta(\lambda)\ \mathcal{B}_\Delta(\lambda)] = [\mathcal{A}\ \mathcal{B}] + \mathcal{E}(I - \Delta_\lambda \mathcal{D})^{-1}\Delta_\lambda \mathcal{C} \quad (23)$$

which makes clear the mentioned nonlinear dependence of $S(\lambda,\omega)$ and $T(\lambda,\omega)$ with respect to $\Delta \in \Xi$ and, consequently, to $\lambda \in \Lambda$. Taking into account this state space realization for the estimation error $E_F(\lambda,\omega)$, the following theorem gives an upper bound to problem (7) and the corresponding robust filter.

Theorem 2: Consider the filter $F_H(\omega)$ given in (16), a symmetric multiplier Π satisfying the infinity dimensional linear constraint

$$\begin{bmatrix} I \\ \Delta_\lambda \end{bmatrix}' \Pi \begin{bmatrix} I \\ \Delta_\lambda \end{bmatrix} > 0\ ,\ \forall \lambda \in \Lambda \quad (24)$$

and positive definite matrices $\mathcal{P}_i$ and W_i, of appropriate dimensions, satisfying the LMIs

$$\begin{bmatrix} \mathcal{P}_i & \bullet & \bullet & \bullet \\ \mathcal{A}'\mathcal{P}_i & \mathcal{P}_i & \bullet & \bullet \\ \mathcal{B}'\mathcal{P}_i & 0 & I & \bullet \\ \mathcal{E}'\mathcal{P}_i & 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ \mathcal{C}' & 0 \\ \mathcal{D}' & I \end{bmatrix} \Pi \begin{bmatrix} 0 & 0 \\ \mathcal{C}' & 0 \\ \mathcal{D}' & I \end{bmatrix}' > 0 \quad (25)$$

$$\begin{bmatrix} W_i & \bullet & \bullet \\ \mathcal{C}_e' & \mathcal{P}_i & \bullet \\ \mathcal{D}_e' & 0 & I \end{bmatrix} > 0 \quad (26)$$

for all $i \in \mathbb{N}$. Then, the $\mathcal{H}_2$ squared norm of the estimation error satisfies $\|E_F(\lambda,\omega)\|_2^2 < \sum_{i=1}^N \lambda_i \mathrm{Tr}\,(W_i)$ for all $\lambda \in \Lambda$.

Proof: This proof depends on several algebraic manipulations being omitted here due to space limitations. ■

At this point some remarks are appropriate. First, the result of this theorem provides a slight generalization of the previous results on multiplier theory from [7], since non-independent parameter uncertainties acting on both matrices A and B can be handled with no additional difficulty. Second, the constraints (24) represent a set of infinity linear matrix inequalities, one for each $\lambda \in \Lambda$. However it can be converted into a set of N LMIs, each one corresponding to a vertex of the unitary simplex Λ, considering the set of multipliers of the form

$$\Pi_i = \begin{bmatrix} R_i & -G \\ -G' & -Q \end{bmatrix},\ \forall i \in \mathbb{N} \quad (27)$$

where the indicated matrices are of compatible dimensions and $Q > 0$. Then it is verified that the LMIs

$$\begin{bmatrix} I \\ \Delta_i \end{bmatrix}' \Pi_i \begin{bmatrix} I \\ \Delta_i \end{bmatrix} > 0,\ \forall i \in \mathbb{N} \quad (28)$$

assure that (24) holds. As a consequence of this fact the robust filter design problem can be stated as

$$J_H = \inf_{\sigma, W_i, \mathcal{P}_i, \Pi_i, C_H, D_H} \{\sigma\ :\ \mathrm{Tr}(W_i) < \sigma\ ,\ i \in \mathbb{N}\} \quad (29)$$

where the indicated matrices satisfy (25) , (26) and (28), for all $i \in \mathbb{N}$.

V. EXAMPLE

For comparison purpose we consider the second order discrete-time system presented in [6], where its LFT representation (3) is given. In the second column of Table I we present the guaranteed cost J_u given by [6] for LFT and NFT (Nonlinear Fractional Transformation) descriptions of the plant, each of them connected to proper and strictly proper filters. The following columns show the upper bound J_H, the exact $\mathcal{H}_2$ squared guaranteed cost calculated by gridding and the lower bond J_L. While in [6] the obtained filters are of order 2, for this example we have no cancellations of poles and zeros on $F_L(\omega)$, implying that the robust filters are of order 4. From Table I, we can see that the best result provided in [6] for proper filters presents the $\mathcal{H}_2$ guaranteed cost $J_u = 0.1524$ while our method provides a proper filter with $\mathcal{H}_2$ guaranteed cost $J_H = 0.0867$. Hence, in this case, the cost reduction is about 43% and the gap between lower and upper bounds is approximatively 23%.

For strictly proper filters the results are more accurate. Indeed, our method is able to determine the (almost) optimal $\mathcal{H}_2$ robust filter (see problem (7)) since the difference between J_H and J_L is less than 0.5%.

VI. PRACTICAL APPLICATION

At this point we propose to apply the previous robust filtering design method to the estimation of the load-end voltage on a transmission line. For this we consider a transmission line with negligible losses, of length $2l = 0.4$ km, whose parameters are $C = 95$ nF/km and $L = 0.26$ mH/km, which define Z_0, the characteristic impedance [9]. The line is connected to a voltage source of internal impedance Z_g, which feeds a resistive load Z_c. At a distance l from the voltage source there is a stub of length l, with characteristic impedance Z_0, feeding another resistive load Z_d.

It is assumed that the voltage on the load Z_c is sampled with period $T_s = 1\ \mu$s, corresponding to the interval of time for a voltage wave to travel from the voltage source to the middle of the line. We denote, at each instant of time $k \geq 0$, $v_{ci}(k)$ and $v_{cr}(k)$ as the incident and reflected voltages on the resistive load Z_c, $v_{di}(k)$ and $v_{dr}(k)$ as the incident and reflected voltages on the stub load Z_d and $v_{gi}(k)$ and $v_{gr}(k)$ as the incident and reflected voltage waves at the voltage source terminal of the line. The voltage provided by the source is indicated as $V(k)$. For the discrete-time modelling of the transmission line we also need to define the reflection coefficients at the source, at the load end, at the stub end and at the middle of the line as being, respectively, Γ_g, Γ_c, Γ_d and Γ_m (see [9]). Since Γ_m depends only of the characteristics impedance of the line and the stub, it is given by $\Gamma_m = -1/3$.

Considering the total incident wave at the load end of the line, at the load end of the stub and at the voltage source, for each instant of time $k \geq 0$ we have

$$v_{ci}(k+2) = (1+\Gamma_m)\left[v_{dr}(k) + v_{gr}(k)\right] + \Gamma_m v_{cr}(k) + \\ +(1+\Gamma_m)\frac{Z_0}{Z_0+Z_g}V(k) \quad (30)$$

TABLE I

DISCRETE-TIME EXAMPLE

Method	J_u [6]	J_H	$\max_{\lambda\in\Lambda} J(F_H(\omega), H(\lambda,\omega))$	J_L
NFT/proper	0.1524	—	—	—
LFT/proper	1.2089	0.0867	0.0782	0.0703
NFT/strictly proper	4.1534	—	—	—
LFT/strictly proper	—	4.0556	4.0415	4.0374

$$v_{di}(k+2) = (1+\Gamma_m)\left[v_{cr}(k)+v_{gr}(k)\right] + \Gamma_m v_{dr}(k) + +(1+\Gamma_m)\frac{Z_0}{Z_0+Z_g}V(k) \tag{31}$$

$$v_{gi}(k+2) = (1+\Gamma_m)\left[v_{cr}(k)+v_{dr}(k)\right] + \Gamma_m v_{gr}(k) + +\Gamma_m\frac{Z_0}{Z_0+Z_g}V(k) \tag{32}$$

Defining the new variables $v_c(k) = v_{ci}(k) + v_{cr}(k) = (1+\Gamma_c)v_{ci}(k)$, $v_d(k) = v_{di}(k)+v_{dr}(k) = (1+\Gamma_d)v_{di}(k)$ and $v_g(k) = v_{gi}(k) + v_{gr}(k) = (1+\Gamma_g)v_{gi}(k)$, some algebraic manipulations on equations (30)-(32) yield a discrete-time state space model of the form

$$\eta(k+1) = A_\Delta \eta(k) + B_\Delta V(k) \tag{33}$$

where the state vector is defined by

$$\eta(k) = [v_c(k)\ v_d(k)\ v_g(k)\ v_c(k+1)\ v_d(k+1)\ v_g(k+1)]' \tag{34}$$

The state variables have the following interpretation: $v_c(k)$ is the total voltage on the resistive load Z_c, $v_d(k)$ is the total voltage on the stub load Z_d and $v_g(k)$ is the sum of the incident and the reflected voltage waves at the voltage source terminal of the line. Furthermore, the state space matrices in (33) have the following structure

$$A_\Delta = \begin{bmatrix} 0 & I \\ \Phi_\Delta & 0 \end{bmatrix},\ B_\Delta = \begin{bmatrix} 0 \\ \Psi_\Delta \end{bmatrix} \tag{35}$$

where the inner block matrices are given by

$$\Phi_\Delta = (1+\Gamma_m)\begin{bmatrix} \frac{\Gamma_m\Gamma_c}{1+\Gamma_m} & \frac{(1+\Gamma_c)\Gamma_d}{1+\Gamma_d} & \frac{(1+\Gamma_c)\Gamma_g}{1+\Gamma_g} \\ \frac{(1+\Gamma_d)\Gamma_c}{1+\Gamma_c} & \frac{\Gamma_m\Gamma_d}{1+\Gamma_m} & \frac{(1+\Gamma_d)\Gamma_g}{1+\Gamma_g} \\ \frac{(1+\Gamma_g)\Gamma_c}{1+\Gamma_c} & \frac{(1+\Gamma_g)\Gamma_d}{1+\Gamma_d} & \frac{\Gamma_m\Gamma_g}{1+\Gamma_m} \end{bmatrix} \tag{36}$$

and

$$\Psi_\Delta = \frac{Z_0}{Z_0+Z_g}\begin{bmatrix} (1+\Gamma_m)(1+\Gamma_c) \\ (1+\Gamma_m)(1+\Gamma_d) \\ \Gamma_m(1+\Gamma_g) \end{bmatrix} \tag{37}$$

As previously indicated, the subindex Δ is used to cope with parameter uncertainty to be precisely defined in the sequel.

Clearly, the next important step is to validate the proposed model (33) by comparing its behavior with the transmission line model available in Matlab/Simulink environment. To this end we consider $V(k)$ being a pulse sequence with amplitude 10 V, period 1 ms and duty cycle 50%. For simulation purposes we assume that $Z_g = 1\ \mathrm{k}\Omega$, $Z_c = 2\ \mathrm{k}\Omega$ and the stub end is open implying that $\Gamma_d = 1$. In Figure 2 the dashed line denotes the Matlab model output for the voltage $v_c(k)$ and the continuous line corresponds to the same output of the proposed model (33). The quality of the proposed model is apparent since only small errors are observed during the line transient.

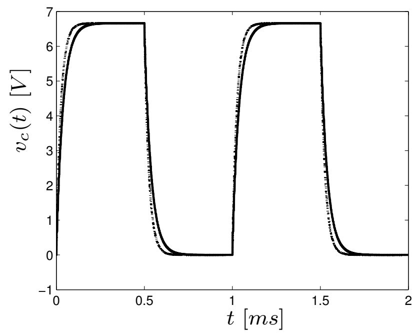

Fig. 2. Transmission line model simulation

Let us now assume that the transmission line operates at $\eta_0(k)$, due to a known input $V_0(k)$, then from the linearity of the model (33) we can define $x(k) = \eta(k) - \eta_0(k)$ and $w(k) = V(k) - V_0(k)$ as new state and input signals. Furthermore, assuming also that all impedances but Z_d are known, the reflection coefficient belongs to a pre-defined interval $[\Gamma_{d_{min}}, \Gamma_{d_{max}}]$. The $\mathcal{H}_2$ filtering design problem is solved in order to determine a linear time invariant filter to estimate the voltage $v_d(k)$ on the stub end from the voltage measurements at the load end of the line $v_c(k)$. This problem fits exactly as the one considered before (3) by taking as the nominal matrices

$$A = \begin{bmatrix} 0 & I \\ \Phi & 0 \end{bmatrix},\ B = \begin{bmatrix} 0 & 0 \\ \Psi & 0 \end{bmatrix} \tag{38}$$

where the indicated inner matrices are now given by

$$\Phi = (1+\Gamma_m)\begin{bmatrix} \frac{\Gamma_m\Gamma_c}{1+\Gamma_m} & 0 & \frac{(1+\Gamma_c)\Gamma_g}{1+\Gamma_g} \\ \frac{\Gamma_c}{1+\Gamma_c} & 0 & \frac{\Gamma_g}{1+\Gamma_g} \\ \frac{(1+\Gamma_g)\Gamma_c}{1+\Gamma_c} & 0 & \frac{\Gamma_m\Gamma_g}{1+\Gamma_m} \end{bmatrix} \tag{39}$$

and

$$\Psi = \frac{Z_0}{Z_0+Z_g}\begin{bmatrix} (1+\Gamma_m)(1+\Gamma_c) \\ 1+\Gamma_m \\ \Gamma_m(1+\Gamma_g) \end{bmatrix} \tag{40}$$

Here the second column added to matrix B refers to the measurement noise, also included in the vector of exogenous disturbance $w(k)$. To highlight the dependence of (33) with respect to the uncertain parameter Γ_d we define

$$D = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix},\ \Delta = \begin{bmatrix} \Gamma_d & 0 \\ 0 & \Gamma_d \end{bmatrix} \tag{41}$$

TABLE II

TRANSMISSION LINE FILTERING SIMULATIONS FOR $Z_g = 1$ kΩ AND $Z_c = 2$ kΩ

$\Gamma_{d_{min}}$	$\Gamma_{d_{max}}$	Order	J_H	$\max_{\lambda\in\Lambda} J(F_H(\omega), H(\lambda,\omega))$	J_L
0.8	0.9	12	0.4024	0.3977	0.3804
0.5	0.9	12	0.5117	0.4860	0.3963
0.3	0.9	12	0.5760	0.5356	0.4000
0.0	0.9	10	0.6701	0.6104	0.4012
−0.3	0.9	12	0.7658	0.6691	0.4010
0.5	0.8	6	0.3597	0.3493	0.3215
0.0	0.8	6	0.4333	0.4116	0.3216
−0.3	0.8	6	0.4981	0.4423	0.3216

and determine the other matrices in order to cope with the parameter uncertainty in both matrices A and B, that is

$$E = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ \frac{1+\Gamma_g}{1+\Gamma_c} & 0 \end{bmatrix}, \ C' = (1+\Gamma_m)\begin{bmatrix} 0 & \frac{\Gamma_c}{1+\Gamma_c} \\ \frac{1}{1+\Gamma_c} & \frac{\Gamma_m}{1+\Gamma_m} \\ 0 & \frac{\Gamma_g}{1+\Gamma_g} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & \frac{Z_0}{Z_0+Z_g} \\ 0 & 0 \end{bmatrix} \tag{42}$$

Under the assumption that the load end voltage $v_c(k) - v_{c0}(k)$ is measured by a sensor with static gain $\kappa = 10$ and that it is corrupted by an additive noise, we define

$$C_y = \begin{bmatrix} \kappa & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \ D_y = \begin{bmatrix} 0 & 1 \end{bmatrix} \tag{43}$$

Finally, our goal is to estimate the stub end voltage $v_d(k) - v_{d0}(k)$, amplified by the same static gain, which yields the output matrices

$$C_z = \begin{bmatrix} 0 & \kappa & 0 & 0 & 0 & 0 \end{bmatrix}, \ D_z = \begin{bmatrix} 0 & 0 \end{bmatrix} \tag{44}$$

For several admissible values of $\Gamma_{d_{min}}$ and $\Gamma_{d_{max}}$ satisfying $-1 < \Gamma_{d_{min}}$ and $\Gamma_{d_{max}} < 1$, we have collected in Table II the order of the robust filter, the upper bound J_H to the $\mathcal{H}_2$ cost, the exact value of the guaranteed cost calculated by gridding and the optimistic $\mathcal{H}_2$ cost J_L. We want to stress that for $\Gamma_{d_{max}}$ close to the maximum value 1 (meaning that $Z_d \to \infty$) an apparent singularity in the upper bound problem appears since it becomes ill conditioned. Moreover, in this situation, the order of the robust filter equals twice the order of plant as it can be verified for almost all cases with $\Gamma_{d_{max}} = 0.9$, meaning that no cancellation of poles and zeros was possible. When $\Gamma_{d_{max}}$ is reduced to 0.8 we can notice that the cancellations of 6 poles and zeros were performed on $F_L(\omega)$, resulting a robust filter $F_H(\omega)$ of order equal to the order of the transmission line model (33). The comparison of J_H, J_L and the exact value of the guaranteed cost given in Table II puts in evidence that the proposed robust filtering design method performs well for this particular problem derived from an actual practical application.

VII. CONCLUSIONS

In this paper a new approach to $\mathcal{H}_2$ robust filter design for discrete LTI systems subject to linear fractional parameter uncertainty representation has been proposed. It is based on the determination of lower and upper bounds of the equilibrium solution of a minimax problem. A robust filter is constructed from the optimal solution of the problems defined by both bounds. The most interesting characteristic of the design method proposed is that these problems are expressed in terms of linear matrix inequalities without imposing the order of the filter to be equal to the order of the plant. Moreover, it was possible to certify the performance of the robust filter from the determination of the optimality gap. An example borrowed from the literature and a practical application involving a stub connected to a transmission line with uncertain reflection coefficient have been considered for illustration.

Some points deserve more attention in the future. First, the determination of the robustness properties of the filter $F_L(\omega)$ associated to the lower bound since this could avoid the computational effort needed to determine the filter associated to the minimum upper bound. Second, the generalization of the present results to cope with $\mathcal{H}_\infty$ norm.

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