A Probabilistic Approach to Process Identification and Control: A Case Study in Pulp Bleaching Focused on the Stationary Probability Density Function

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Abstract— **This paper uses the operation of a pulp bleaching reactor as a platform for the illustration and discussion of several ideas involving use of the stationary probability density function in process control. A computer simulation of the partial differential equations, derived from first-principles, describing the bleaching reactor is used to represent the true process. Empirical, discrete-time models of the reactor are then constructed using sampled input and output time series. In the model building phase, the influence of data distribution on final model quality is emphasized. Once the discrete-time model has been derived, two control laws are synthesized. One law is based on established design techniques while the other takes advantage of less recognized probabilistic concepts. The controllers are then applied to the true process model the relative performances discussed.**

I. INTRODUCTION

Viewing process behaviour in a probabilistic framework allows for the inclusion of random disturbances and calculation of expected long term process trends. Stochastic process models allow for uncertainty to be recognized in control designs and compensated for – or at least accommodated – in process operations. The standard approach to these situations, as reported in the process control literature, uses linear process models, Gaussian disturbances, and assumes quadratic objectives. These strategies focus on reduction of variance about a setpoint or reduction of variance so that mean process operation can be moved closer to constraints. If any of the assumptions underlying this approach do not hold in practice then there is the potential for inadequate system performance and economic losses.

Some recent works have investigated a probabilistic approach that includes nonlinear process models and non-quadratic objective functions. In [1], an approximation of the integral equation governing the relationship between process nonlinearity and the stationary probability density function (PDF) is described. In [3] this approximation technique is applied to design control laws for first-order process so that stationary PDF of the closed-loop process takes a pre-specified Gram-Charlier form. Another application of this technique is presented in [2], where the optimal control design with respect to a nonsymmetric/nonquadratic expectation-type objective function is approximated. The role of data distribution in determining the quality of empirical plant models is discussed in [4].

The aim of the present work is to apply these ideas to the modelling and control of a pulp bleaching reactor. A nonlinear continuous time model is assumed as the underlying true process from which measurements are made and manipulated variables are adjusted at discrete time intervals. The study begins with the goal of improving the control of product quality (pulp brightness – as inferred from lignin content) with respect to minimum brightness specifications. For analysis and design purposes a discrete-time model is developed to accurately capture the behaviour of the nonlinear process across the entire region of operation. The identification relies on generating data that is uniformly distributed across the region of operation so that model accuracy remains constant for the entire region. An objective function reflecting the economic value of the pulp brightness is the main performance indicator and used as the goal for controller synthesis. Two control designs are performed with respect to this objective. Simulations for the two control strategies demonstrate the relative strengths of each.

This case study illustrates a number of points. The control of industrial processes can be much more challenging than is often reflected in the control literature. In this case, the process is distributed in nature, inducing long delays in the input-output data. It is also nonlinear, and subject to random variability, which makes it difficult to model using standard linear model identification approaches. Likewise, industrially relevant economic cost functions are often nonquadratic and nonsymmetric, so standard control designs based on quadratic costs are inappropriate. As a result of these challenges, different approaches must be sought. This case study demonstrates potential success to be realized through using probabilistic techniques for process modelling and control. In both the identification and the design stage, improvements over the standard approaches are made, so that the end result is better process performance due to a better control design based on a better model.

In §II the equations used to describe the 'true' bleaching reactor are given along with information on the numerical simulation technique used in their solution. In §III time series data from the reactor is used to construct a discrete-time model of the plant behaviour. In §IV two different

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control laws are designed based on the identified discrete time model. These are then tested through simulation of the 'true' bleaching reactor and the relative merits of each are discussed.

II. The Bleaching Reactor

The bleaching reactor model used as the true plant for this investigation is taken from [8]. It consists of two partial differential equations describing the evolution through distance, z, along the reactor and time, t, of the lignin, $L(z,t)$, and bleach, $C(z,t)$, concentrations, as well as appropriate boundary conditions. The complete model, as used in this paper, is:

$$\frac{\partial C}{\partial t} = -v\frac{\partial C}{\partial z} + D\frac{\partial^2 C}{\partial z^2} - k_C C^3 L^3, \tag{1}$$

$$\frac{\partial L}{\partial t} = -v\frac{\partial L}{\partial z} + D\frac{\partial^2 L}{\partial z^2} - k_L C^3 L^3, \tag{2}$$

$$\frac{\partial L(0,t)}{\partial z} = \frac{v}{D}\left(L(0,t) - L_{in}(t)\right), \tag{3}$$

$$\frac{\partial C(0,t)}{\partial z} = \frac{v}{D}\left(C(0,t) - C_{in}(t)\right), \tag{4}$$

$$\frac{\partial L(1,t)}{\partial z} = 0, \tag{5}$$

$$\frac{\partial C(1,t)}{\partial z} = 0, \tag{6}$$

$$L(z,0) = 1.7, \tag{7}$$

$$C(z,0) = 0, \tag{8}$$

where C_{in} is the concentration of bleach (in %) and L_{in} is the concentration of lignin (in g/l) in the inlet streams. The equations have been normalized to a reactor length of 1 meter and the parameters are given in Table I. With the given convection and diffusion terms the Peclet number is 10^4.

Numerical simulations of this reactor model were performed using the sequencing method proposed in [8], with spatial discretization of 0.005 meters and temporal discretization of 0.1 minutes . It is assumed, for the purposes of this case study, that inlet lignin concentration is a measured disturbance variable varying over a range of $[1.6, 2.4]$. To give this input variable a drifting-type nature, the following sum of sinusoids is used:

$$L_{in} = 2 + 0.3\sin\left(\frac{7}{5}2\pi\frac{t}{T}\right) + 0.2\sin\left(\frac{7}{5}16\pi\frac{t}{T}\right) + 0.02v_t, \tag{9}$$

where v_t is the normal random variable sampled and held at each time step. This same inlet sequence is used for all of the simulations in this paper. While this is not possible in a real plant situation, it provides a method for ensuring that identification and control benchmarks are uniformly assessed.

TABLE I
Parameters for the Bleaching Reactor

Parameter	Value
v	0.05
D	5×10^{-6}
k_C	5.5×10^{-5}
k_L	5.7×10^{-7}

In the subsequent section, values of the inlet and outlet concentrations of the lignin and the bleach are periodically sampled. These are denoted as follows:

$$C_k^{in} = C_{in}(k\Delta t), \tag{10}$$

$$L_k^{in} = L_{in}(k\Delta t), \tag{11}$$

$$C_k^{out} = C(1, k\Delta t), \tag{12}$$

$$L_k^{out} = L(1, k\Delta t), \tag{13}$$

where Δt is the sampling interval.

III. Model Identification

For the purposes of this case study, it is assumed that measurements of the inlet and outlet concentrations of the lignin and bleach may be made at periodic intervals, but that no other measurements are available. The goal of this section of the paper is to build a discrete-time model based on input-output data from the plant. Initial step tests reveal that there is a delay of 20 minutes between input and output changes in the lignin and bleach concentrations and that the time constant for the process is approximately 1 minute. Based on these observations, process input changes and outlet measurements are made every 0.1 minutes ($\Delta t = 0.1$). To generate data for identification a PRBS input sequence for the inlet bleach concentration is used, with the chosen value (either 40 or 120) held over the time interval. Since lignin inlet concentration cannot be manipulated, it is allowed to drift, as described in the previous section.

Based on the data obtained (see Figure 1), the following time series model is constructed:

$$\tilde{C}_{k+b}^{out} = c_1\tilde{C}_k^{in} + c_2\tilde{L}_k^{in} + c_3\tilde{C}_{k-1}^{in} + c_4\tilde{L}_{k-1}^{in}, \tag{14}$$

$$\tilde{L}_{k+b}^{out} = l_1\tilde{C}_k^{in} + l_2\tilde{L}_k^{in} + l_3\tilde{C}_{k-1}^{in} + l_4\tilde{L}_{k-1}^{in}, \tag{15}$$

where $b = 201$ is the delay before the output response. Here, the tilde notation (*e.g.* $\tilde{C}$) refers to a centered and scaled variable. The centering is performed by removing the mean, while the scaling is done by dividing by half of the spread of the variable. Prediction error methods, such as are given in [7], are used to calculate the parameter values.

In [4], it is demonstrated that to ensure that model errors are equally weighted for all areas of the operating region, it is necessary to have uniformly distributed identification data. In this case the operating region is $[1.6, 2.4]$ for the lignin inlet concentration and $[40, 120]$ for the bleach concentration. Since the model structure determined from

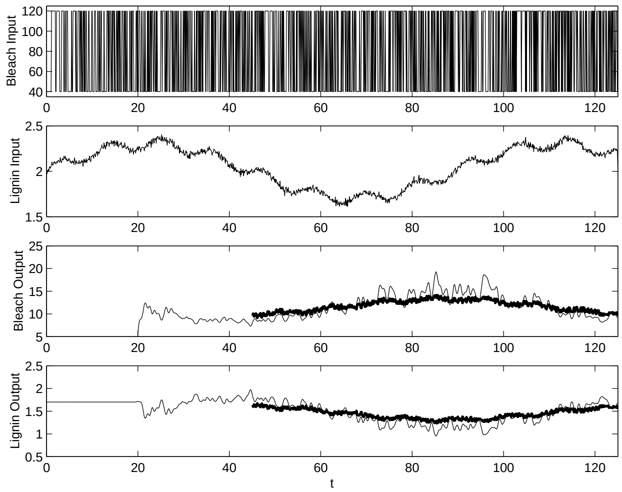

Fig. 1. Time series data and predictions used for the initial model building attempt. The predictions are given by the heavier lines.

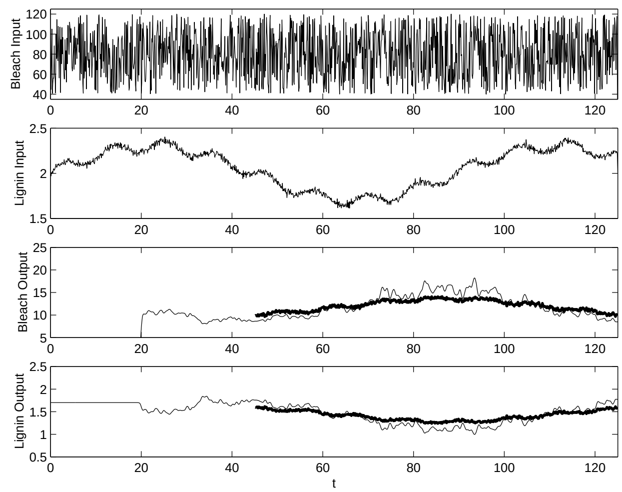

Fig. 2. Time series data and predictions used for the second model building attempt. The predictions are given by the heavier lines.

TABLE II
NORMALIZING VALUES FOR THE SECOND DATA SET

Variable	Mean	Spread/2
C^{in}	80	40
L^{in}	2.0	0.39
C^{out}	12	5.1
L^{out}	1.4	0.42

the first set of data is not autoregressive, the distribution of the outlet data is unimportant.

Nothing can be done to affect the distribution of the inlet lignin concentration, since it is a disturbance variable, although it is hoped that it will drift through a full range of values during the plant experiment. The distribution of the chlorine inlet concentration can easily be made uniform simply by adjusting the input sequence.

To generate a second data set, the inlet bleach concentration is again randomly adjusted at each time step; however, in this case it is allowed to vary uniformly over the region $[40, 120]$ This data is given in Figure 2.

The parameters obtained in each case are given in Table III.

TABLE III
PARAMETERS FOR THE LINEAR MODELS

Parameter	Model 1		Model 2	
	Value	3σ	Value	3σ
c_1	0.078	± 0.016	0.100	± 0.023
c_2	-0.37	± 0.22	-0.48	± 0.19
c_3	0.073	± 0.016	0.094	± 0.023
c_4	-0.39	± 0.22	-0.42	± 0.19
l_1	-0.072	± 0.013	-0.089	± 0.017
l_2	0.39	± 0.18	0.51	± 0.14
l_3	-0.068	± 0.013	-0.084	± 0.017
l_4	0.41	± 0.18	0.42	± 0.14

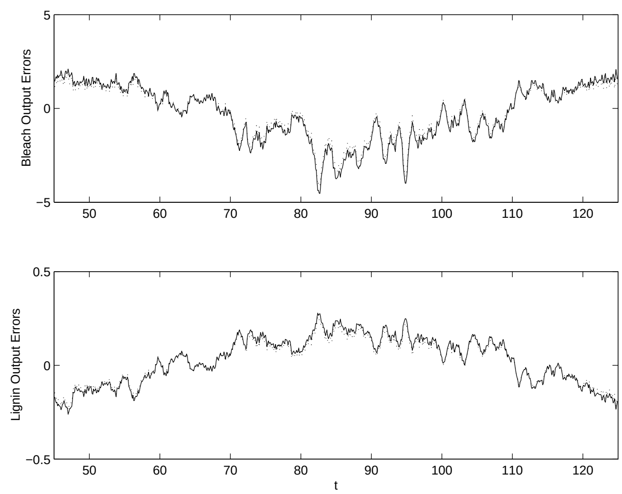

Fig. 3. Time series data used for model validation. The solid line gives the (unscaled) errors for the first model, and the dotted line gives the (unscaled) errors for the second model.

In Figure 3 the prediction errors for the outlet concentrations of bleach and lignin are compared for the two models. While the two sets of errors are similar, the errors for the second model are almost always smaller than those of the first. Comparing the sums of the squares of the errors (SSE) shows that the SSE for the first model is 29% higher for outlet bleach predictions and 28% higher for outlet lignin predictions. This shows that although the difference is not visually striking, consistent better predictions add up to better performance over time.

For the second data set, the centering and scaling values for the variables are given in Table II. The second model

TABLE IV

PARAMETERS FOR THE NONLINEAR MODEL

Parameter	Value
c_1	0.097
c_2	-0.52
c_3	0.094
c_4	-0.36
c_5	-0.076
c_6	-0.062
c_7	0.16
l_1	-0.088
l_2	0.57
l_3	-0.084
l_4	0.41
l_5	-0.085

can be refined by adding some nonlinear terms:

$$\tilde{C}^{out}_{k+b} = c_1\tilde{C}^{in}_k + c_2\tilde{L}^{in}_k + c_3\tilde{C}^{in}_{k-1} + c_4\tilde{L}^{in}_{k-1} + c_5\tilde{C}^{in^4}_k + c_6\tilde{C}^{in^4}_{k-1} + c_7\tilde{L}^{in^4}_{k-1} + w^C_k, \quad (16)$$

$$\tilde{L}^{out}_{k+b} = l_1\tilde{C}^{in}_k + l_2\tilde{L}^{in}_k + l_3\tilde{C}^{in}_{k-1} + l_4\tilde{L}^{in}_{k-1} + l_5\tilde{L}^{in^3}_k + w^L_k. \quad (17)$$

These parameter values are given in Table IV. It is interesting to note that in attempting to add nonlinear terms to the models using the different sets of data, the type of nonlinearity that is significant varies between the data sets. This suggests that depending on the input signal used, different aspects of nonlinear plant behaviour are revealed.

A. Disturbance Distributions

To complete the modelling process, and provide all the information necessary to the controller design portion of this study, it is necessary to estimate the distributions of the model data. By inspection of their histograms, the prediction errors (w^C and w^L) for the outlet bleach and lignin concentration models appear to be normally distributed. The mean of each variable is calculated to be zero and the variances are 0.014 and 0.0087, respectively.

For the measured input disturbance, L^{in}, it is not as clear how it can be statistically described. The sum-of-sinusoids signal used in the simulations is meant to emulate a drifting type variation over the operating region $[1.6, 2.4]$. For this reason, inlet lignin concentration is modelled as being uniformly distributed. A histogram of the inlet lignin concentration, as it varies over one cycle of the dominant sinusoid, confirms that this is a reasonable model. The histogram is given be Figure 4.

Throughout this paper, the PDF of a specific variable will be identified with the variable written as a subscript. For example, the PDF of w^C is $p_{w^C}\left(w^C\right)$.

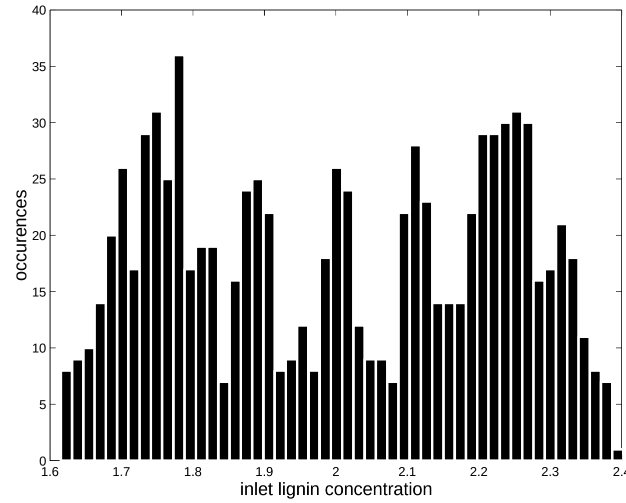

Fig. 4. A histogram of the inlet lignin concentration.

IV. CONTROL DESIGN

The objective for controller synthesis is the nonsymmetric, nonquadratic objective function:

$$J = E\left[\ell\left(L^{out}_{k+b}\right) + \frac{\left(C^{in}_k - 40\right)^2}{4000}\right], \quad (18)$$

where:

$$\ell\left(L^{out}_{k+b}\right) = \begin{cases} 1.4 - L^{out}_{k+b} & L^{out}_{k+b} \le 1.4 \\ 5 + \left(L^{out}_{k+b} - 1.4\right)^2 & L^{out}_{k+b} > 1.4 \end{cases} \quad (19)$$

This reflects the goal of keeping outlet lignin concentration below its maximum acceptable value, but without using excessive quantities of the bleach. Since this objective is written in terms of the unscaled variables, it must be rewritten in the scaled variables as:

$$\tilde{J} = E\left[\tilde{\ell}\left(\tilde{L}^{out}_{k+b}\right) + 0.4\left(\tilde{C}^{in}_k + 1\right)^2\right], \quad (20)$$

with:

$$\tilde{\ell}\left(\tilde{L}^{out}_{k+b}\right) = \begin{cases} -0.42\tilde{L}^{out}_{k+b} & \tilde{L}^{out}_{k+b} \le 0 \\ 5 + \left(0.42\tilde{L}^{out}_{k+b}\right)^2 & \tilde{L}^{out}_{k+b} > 0 \end{cases} \quad (21)$$

Only the lignin outlet concentration is of interest for this case study. Controller design is then based on equation (17). The actual control law is then implemented for the true process, by reversing the scaling and centering, *i.e.*,

$$C^{in}_k = 40\tilde{C}^{in}_k + 80. \quad (22)$$

Estimations of process performance for a given control strategy can be made from the sampled input-output data using:

$$\hat{J} = \sum_{k=1}^{N}\left(\ell\left(L^{out}_{k+b}\right) + \frac{\left(C^{in}_k - 40\right)^2}{4000}\right). \quad (23)$$

A. Modified-Objective Design

Even with a purely linear process model, the objective (20) cannot be directly minimized with respect to the input to yield any closed-form solution for the control law. One paper where this type of problem has been addressed is [5]. The idea is to replace expectation with conditional expectation to yield the modified problem:

$$\tilde{J}_{MO} = E^{*}\left[\tilde{\ell}\left(\tilde{L}^{out}_{k+b}\right)\right] + 0.4\left(\tilde{C}^{in}_{k}+1\right)^{2}, \tag{24}$$

where $E^{*}[\cdot]$ represents expectation conditioned on the available data.

Rewriting the process equation (17) as:

$$\tilde{L}^{out}_{k+b} = \hat{\tilde{L}}^{out}_{k+b} + w^{L}_{k}, \tag{25}$$

allows for the following control law derivation. The modified objective is expanded to:

$$\begin{aligned}\tilde{J}_{MO} = &\int_{-\infty}^{-\hat{\tilde{L}}^{out}_{k+b}} -0.42\left(\hat{\tilde{L}}^{out}_{k+b}+w^{L}_{k}\right)p_{w_L}\left(w^{L}_{k}\right)dw^{L}_{k} \\ &+\int_{-\hat{\tilde{L}}^{out}_{k+b}}^{\infty}\left(5+\left(0.42\left(\hat{\tilde{L}}^{out}_{k+b}+w^{L}_{k}\right)\right)^{2}\right)p_{w_L}\left(w^{L}_{k}\right)dw^{L}_{k} \\ &+0.4\left(\frac{\hat{\tilde{L}}^{out}_{k+b}-l_2\tilde{L}^{in}_{k}-l_3\tilde{C}^{in}_{k-1}-l_4\tilde{L}^{in}_{k-1}-l_5\tilde{L}^{in^3}_{k}}{l_1}+1\right)^{2},\end{aligned} \tag{26}$$

and then differentiated with respect to the b-step ahead prediction, $\hat{\tilde{L}}^{out}_{k+b}$, as follows

$$\begin{aligned}\frac{\partial\tilde{J}_{MO}}{\partial\hat{\tilde{L}}^{out}_{k+b}} = &\int_{-\infty}^{-\hat{\tilde{L}}^{out}_{k+b}} -0.42\,p_{w_L}\left(w^{L}_{k}\right)dw^{L}_{k} \\ &+\int_{-\hat{\tilde{L}}^{out}_{k+b}}^{\infty} 0.84\left(\hat{\tilde{L}}^{out}_{k+b}+w^{L}_{k}\right)p_{w_L}\left(w^{L}_{k}\right)dw^{L}_{k} \\ &+5p_{w_L}\left(-\hat{\tilde{L}}^{out}_{k+b}\right) \\ &+0.8\frac{\hat{\tilde{L}}^{out}_{k+b}-l_2\tilde{L}^{in}_{k}-l_3\tilde{C}^{in}_{k-1}-l_4\tilde{L}^{in}_{k-1}-l_5\tilde{L}^{in^3}_{k}+l_1}{l_1^{2}}.\end{aligned} \tag{27}$$

This is set equal to zero, and the integration performed to give:

$$\begin{aligned}&-0.21\,\mathrm{erf}\left(0.667\tilde{C}^{in}_{k}-4.32\tilde{L}^{in}_{k}+0.637\tilde{C}^{in}_{k-1}\right. \\ &\left.-3.11\tilde{L}^{in}_{k-1}+0.644\tilde{L}^{in^3}_{k}\right)-9.30-9.13\tilde{C}^{in}_{k} \\ &+0.239\tilde{L}^{in}_{k}-0.0353\tilde{C}^{in}_{k-1}+0.172\tilde{L}^{in}_{k-1} \\ &-0.0357\tilde{L}^{in^3}_{k}-0.42\left(-0.088\tilde{C}^{in}_{k}+0.57\tilde{L}^{in}_{k}\right. \\ &\left.-0.084\tilde{C}^{in}_{k-1}+0.41\tilde{L}^{in}_{k-1}-0.085\tilde{L}^{in^3}_{k}\right)\mathrm{erf}\left(0.667\tilde{C}^{in}_{k}\right. \\ &\left.-4.32\tilde{L}^{in}_{k}+0.637\tilde{C}^{in}_{k-1}-3.11\tilde{L}^{in}_{k-1}+0.644\tilde{L}^{in^3}_{k}\right) \\ &+21.4\exp\left(-57.5\left(-0.088\tilde{C}^{in}_{k}+0.57\tilde{L}^{in}_{k}\right.\right. \\ &\left.\left.-0.084\tilde{C}^{in}_{k-1}+0.41\tilde{L}^{in}_{k-1}-0.085\tilde{L}^{in^3}_{k}\right)^{2}\right)=0.\end{aligned} \tag{28}$$

This control law cannot be further simplified. It is implemented by numerically solution at each time step. A simulation of this control strategy is illustrated in Figure 5.

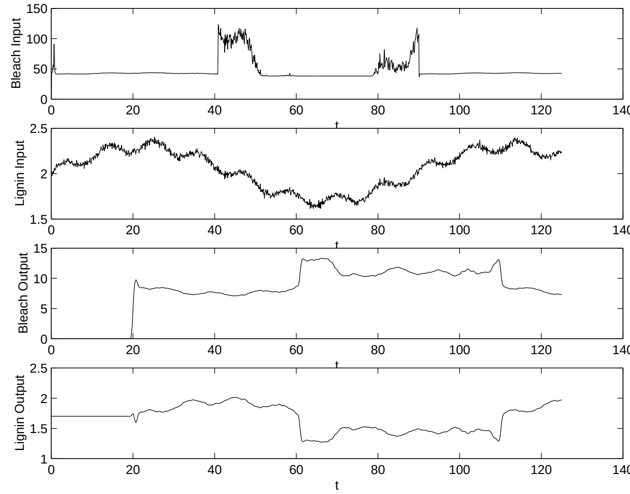

Fig. 5. Time series for the process governed by the first control law.

The value of the objective function for this control law and simulation is $J = 3221$.

B. An Approximate Probabilistic Design

As discussed in [1], very few closed-form analytical results for optimal control design with respect to the objective function (20) are available. An important part of the approximation of the optimal solution is to parameterize the input. In this case, the parameterization is designed to linearize the dynamics:

$$\tilde{C}^{in}_{k} = \alpha_0 + \alpha_1\tilde{C}^{in}_{k-1} + \alpha_2\tilde{L}^{in}_{k-1} + \alpha_3\tilde{L}^{in}_{k} - \frac{l_5}{l_1}\tilde{L}^{in^3}_{k}, \tag{29}$$

where the α_i are constants to be determined. In order to evaluate the objective function, the PDF's of $\tilde{L}^{out}_{k+b}$ and $\tilde{C}^{in}_{k}$ are required. For both, it is necessary to consider the control law. To do so the control law (29) is rewritten as the augmented dynamic system:

$$\begin{aligned}\tilde{C}_k^{in} &= \alpha_0 + \alpha_1 \tilde{C}_{k-1}^{in} + \alpha_2 \tilde{X}_{k-1}^{in} + \alpha_3 \tilde{L}_k^{in} - \frac{l_5}{l_1}\tilde{L}_k^{in^3} \\ \tilde{X}_k^{in} &= \tilde{L}_k^{in}.\end{aligned} \tag{30}$$

This system has one random input and two states; therefore some manipulations are required to derive the joint PDF. The change of variables:

$$\begin{aligned}&\tilde{Z}_k^{in} = \tilde{C}_k^{in} - \alpha_3 \tilde{L}_k^{in} + \frac{l_5}{l_1}\tilde{L}_k^{in^3} = \alpha_0 + \alpha_1 \tilde{C}_{k-1}^{in} + \alpha_2 \tilde{X}_{k-1}^{in} \\ &\Rightarrow \tilde{C}_{k-1}^{in} = \tilde{Z}_{k-1}^{in} + \alpha_3 \tilde{X}_{k-1}^{in} - \frac{l_5}{l_1}\tilde{X}_{k-1}^{in^3}\end{aligned} \tag{31}$$

is introduced and the system becomes:

$$\begin{aligned}\tilde{Z}_k^{in} &= \alpha_0 + \alpha_1 \left(\tilde{Z}_{k-1}^{in} + \alpha_3 \tilde{X}_{k-1}^{in} - \frac{l_5}{l_1}\tilde{X}_{k-1}^{in^3} \right) + \alpha_2 \tilde{X}_{k-1}^{in} \\ \tilde{X}_k^{in} &= \tilde{L}_k^{in}.\end{aligned} \tag{32}$$

It is clear that these two states are independent, with PDFs:

$$p_{\tilde{X}^{in}}\left(\tilde{X}_k^{in}\right) = p_{\tilde{L}^{in}}\left(\tilde{X}_k^{in}\right), \tag{33}$$

and

$$\begin{aligned}&p_{\tilde{Z}^{in}}\left(\tilde{Z}_k^{in}\right) \\ &= \int \delta\left(\tilde{Z}_k^{in} - \alpha_0 - \alpha_1 \tilde{Z}_{k-1}^{in} - \alpha_1\alpha_3 \tilde{L}_{k-1}^{in} + \alpha_1 \frac{l_5}{l_1}\tilde{L}_{k-1}^{in^3}\right. \\ &\left. - \alpha_2 \tilde{L}_{k-1}^{in}\right) p_{\tilde{Z}^{in}}\left(\tilde{Z}_{k-1}^{in}\right) p_{\tilde{L}^{in}}\left(\tilde{L}_{k-1}^{in}\right) d\tilde{L}_{k-1}^{in} d\tilde{Z}_{k-1}^{in} \\ &= \int_{-1}^{1} \frac{1}{2|\alpha_1|} p_{\tilde{Z}^{in}}\left(\frac{1}{\alpha_1}\left(\tilde{Z}_k^{in} - \alpha_0 - \alpha_1\alpha_3 \tilde{L}_{k-1}^{in}\right.\right. \\ &\left.\left. + \alpha_1 \frac{l_5}{l_1}\tilde{L}_{k-1}^{in^3} - \alpha_2 \tilde{L}_{k-1}^{in}\right)\right) d\tilde{L}_{k-1}^{in}.\end{aligned} \tag{34}$$

The controller parameterization (29) and the change of variables (31) are substituted into the process model (17) to give:

$$\begin{aligned}\tilde{L}_{k+b}^{out} = &\, l_1\alpha_0 + l_1\alpha_1 \tilde{Z}_{k-1}^{in} + l_1\alpha_1\alpha_3 \tilde{L}_{k-1}^{in} - l_1\alpha_1 \frac{l_5}{l_1}\tilde{L}_{k-1}^{in^3} \\ &+ l_1\alpha_2 \tilde{L}_{k-1}^{in} + l_1\alpha_3 \tilde{L}_k^{in} + l_2 \tilde{L}_k^{in} + l_3 \tilde{Z}_{k-1}^{in} \\ &+ l_3\alpha_3 \tilde{L}_{k-1}^{in} - l_3 \frac{l_5}{l_1}\tilde{L}_{k-1}^{in^3} + l_4 \tilde{L}_{k-1}^{in} + w_k^L.\end{aligned} \tag{35}$$

Therefore the PDF $p_{\tilde{L}^{out}}\left(\tilde{L}^{out}\right)$ can be written as a function of independent variables.

In both this case, and for equation (34), there is no way to manipulate this equation to get the PDF in an explicit form; however, approximation techniques, such as those described in [1] can be used. Here the PDF's are approximated as Gaussian by calculating the mean and variance of each. These are:

$$\mu_Z = \frac{\alpha_0}{1-\alpha_1} \tag{36}$$

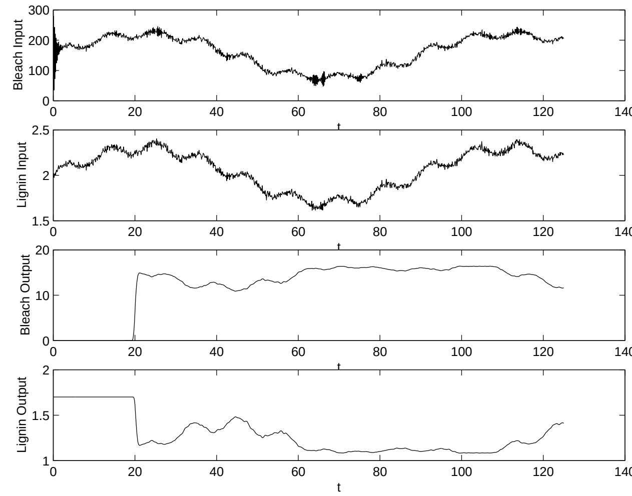

Fig. 6. Time series for the process governed by the second control law.

$$\begin{aligned}\sigma_Z^2 = &\, \left(15\alpha_1^2 l_5^2 + 35\alpha_1^2\alpha_3^2 l_1^2 - 42\alpha_1^2\alpha_3 l_5 l_1\right. \\ &\frac{\left. + 70\alpha_1\alpha_3\alpha_2 l_1^2 - 42\alpha_1 l_5 \alpha_2 l_1 + 35\alpha_2^2 l_1^2\right)}{105 l_1^2 (1-\alpha_1^2)}\end{aligned} \tag{37}$$

$$\mu_{L^{out}} = l_1\alpha_0 + (l_1\alpha_1 + l_3)\frac{\alpha_0}{1-\alpha_1} \tag{38}$$

The expression for σ_L^2 is quite large. The objective function is expanded and the change of variables is used to obtain:

$$\begin{aligned}\tilde{J} = &\, E\left[\tilde{\ell}\left(\tilde{L}_{k+b}^{out}\right)\right] \\ &+ E\left[0.4\left(\tilde{Z}_k^{in} + \alpha_3 \tilde{L}_k^{in} - \frac{l_5}{l_1}\tilde{L}_k^{in^3} + 1\right)^2\right],\end{aligned} \tag{39}$$

and the definition of expectation is then applied, yielding:

$$\begin{aligned}\tilde{J} = &\int_{-\infty}^{0} -0.42\left(\tilde{L}_{k+b}^{out}\right) p_{L^{out}}\left(\tilde{L}_{k+b}^{out}\right) d\tilde{L}_{k+b}^{out} \\ &+ \int_{0}^{\infty}\left(5 + \left(0.42\left(\tilde{L}_{k+b}^{out}\right)\right)^2\right) p_{L^{out}}\left(\tilde{L}_{k+b}^{out}\right) d\tilde{L}_{k+b}^{out} \\ &+ 0.16\sigma_Z^2 + 0.16 + 0.32\mu_Z + 0.053\alpha_3^2 \\ &- 0.064\frac{\alpha_3 l_5}{l_1} + 0.0229\frac{l_5^2}{l_1^2}.\end{aligned} \tag{40}$$

Evaluating this objective function assuming Gaussian PDFs with the calculated parameters leads to an algebraic function of the controller parameters. This is then minimized to give the parameter values: $\alpha_0 = 3.1601, \alpha_1 = -0.8436, \alpha_2 = 2.0942, \alpha_3 = 2.7699$.

A simulation of the process under this control strategy is performed and the resulting time series are given in Figure 6.

The value of the cost function for this set of data is $J = 2497$.

C. Discussion

The behaviours of the two different control strategies are quite different. In the first case, the modification of the objective function leads to a bounded control law. Additionally this control law tends to take no action if the predicted result is costly for the perceived benefit (at each time step). By using the conditional expectation, the designed control strategy tends to be myopic. While it is good that the controller is conservative in the control action, this is somehow inadvertently imposed by the modification of the control law. No such constraints on the control were imposed as part of the problem specification.

The second strategy gives a much more aggressive control law that keeps the value of the lignin outlet concentration lower at all times.

A major conclusion to be drawn from these examples is that using a probabilistic technique that considers the unconditional expectation leads to control designs that outperform designs based on conditional expectation. There is a 22% decrease in the cost for the second control law in this example.

V. CONCLUSIONS AND FUTURE WORKS

A. Conclusions

It has been demonstrated, through the study of a pulp bleaching reactor, how probabilistic concepts can be used to benefit identification and control in a process control setting. For identification of process models, ideas based on probabilistic concepts lead to improved experimental designs for the generation of identification data. Data generated in this way leads to improved process models. For regulatory controller design for processes, such as the bleaching reactor, where random variation is significant, the use of unconditional expectation leads to better long term performance.

B. Future Works

While there has been increased activity in recent years investigating the use of probabilistic concepts for system identification and control, there are many open and potentially fruitful research areas. The idea of shaping the distribution of identification data – to improve the parameter value estimates – is a continuing interest of the authors. The unusual properties of the control design based on the modified objective function deserve further consideration. Somehow, this type of design, based on conditional expectation, places unnecessary hard constraints on the input. As well, the impact of random feedforward variables on the PDF of a process output can be analyzed in general.

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