# Design of Uncertain Discrete Time Systems with Constructive Nonlinear Dynamics Methods

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September 8, 2006

**Abstract**

This contribution presents a new methodology for the optimization based design of uncertain discrete time systems. Constructive nonlinear dynamics methods have been developed for the optimization of dynamical systems under uncertainty over the past few years by the authors [2, 4–8]. The present contribution deals with the extension of these methods to discrete time systems.

## 1 Introduction

In a typical application, constructive nonlinear dynamics (CNLD) allow to take parametric uncertainty with respect to stability properties into account in process optimization [2, 4–8]. Originally, the development of these methods was motivated by the need to impose constraints for robust stability on dynamical systems modeled by systems of continuous time differential-algebraic equations (DAE systems). Several types of dynamical systems that arise in science and engineering cannot be modeled as DAE systems, however. Here we address the extension to discrete time systems, that is, models that consist of algebraic and difference equations. Models of this type arise under several circumstances. For one, processes exist that are intrinsically discrete in time. Systems of difference equations also arise naturally when oscillating processes are described with the aid of Poincaré mappings [3]. This type of description is particularly useful when studying the stability and robustness properties of periodically operated systems. A third important class of examples comprises discrete time systems that result from sampling continuous time processes. Since digital control systems use sampled data, this class of models is particularly important in the modeling for control.

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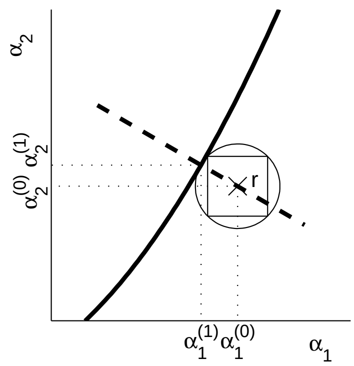


Figure 1: Nominal point of operation $\alpha^{(0)}$ with stability boundary (thick line). The critical point $\alpha^{(1)}$ nearest to the nominal point is in the direction of the normal vector $r$ (dashed thick line). The distance between the nominal point and the nearest critical point must be larger than the radius of the circle around the nominal point.

## 2 Constructive Nonlinear Dynamics

The method presented in this contribution is based on bifurcation theory and nonlinear programming. Bifurcation theory is employed to state formal descriptions of critical points for nonlinear dynamical systems, for example, critical points for stability [6]. Under mild mathematical conditions, critical points for stability form critical manifolds. These critical stability manifolds can be thought of as boundaries that separate those parts of the parameter space in which the dynamical system is stable from those parts in which it is unstable. Beyond stability boundaries, the concept of a critical manifold can be used to describe a broad variety of points at which process behavior changes qualitatively, such as feasibility boundaries, and various boundaries that characterize the dynamic behavior of nonlinear systems [6].

Based on a formal description of critical manifolds from applied bifurcation theory, systems of equations for normal vectors to the critical manifold can be derived [6]. These normal vectors allow measuring the distance between a candidate point of operation in the process parameter space and the critical manifold. By imposing a lower bound on the distance to all critical manifolds, robustness with respect to parametric uncertainty of the dynamical system is guaranteed in the sense sketched in Fig. 1.

In Fig. 1, the critical boundary and the normal direction to it are sketched as a bold full line and a bold dashed line, respectively. The shortest distance between the nominal point at the center of the circle and the critical manifold occurs along the dashed normal direction. By requiring this distance to be at least as large as, or larger than, the radius of the circle, robustness can be guaranteed, since the critical boundary will not be crossed, regardless of the actual values the parameters $\alpha_i$ attain within the region of uncertainty represented by the box of side lengths $\Delta\alpha_i$ around the nominal point. Loosely speaking, the optimization software pushes the robustness ball along any critical manifold. This way, robustness with respect to critical manifolds can be taken into account in the optimization of dynamical systems in spite of parametric uncertainty. While the sketch in Fig. 1 is only two-dimensional, the idea of measuring distance along normal vectors generalizes to arbitrary finite dimensional spaces of uncertain parameters.

The underlying mathematical foundations for discrete time systems are similar to those used for continuous time systems with some fundamental differences, however. Most importantly, the underlying bifurcation theory is different for the two system classes. Continuous time systems can, for example, experience a loss of stability due to two generic one parameter bifurcations (saddle-node and Hopf), while in discrete time systems, three types of bifurcation points can cause a loss of stability (Neimark-Sacker, flip or period doubling, and fold bifurcation, for details see textbooks on bifurcation theory, e.g., [3]). In order to characterize stability boundaries, three types of critical points therefore have to be taken into account in discrete time systems as opposed to only two in continuous time systems. To this end, normal vector systems can be derived from systems of equations for critical manifolds of discrete time systems known from numerical bifurcation theory [6]. Based on these normal vector systems, constraints for parametric robustness with respect to stability boundaries can be included in process optimization.

# 3 Applications

The use of the CNLD method is illustrated with a model for periodic harvesting of a fish population with periodic reproduction events [1, 9]. This model serves as an example for processes that operate on renewable resources. Renewable resources naturally involve periodic processes. Crop production, for example, proceeds as a cycle of seeding, growth, and harvesting much like the behavior of the model treated here.

Figure 2 shows typical time series of a simulation with the model for the periodically harvested fish population. These time series demonstrate that the point in time in which harvesting occurs is critical for the stability of the system. In this particular example, the stability boundary consists of period doubling bifurcations [3]. When optimizing the process we are interested in finding those values for the harvesting time and harvesting effort that are optimal with respect to the profit from harvested fish while guaranteeing that the process does not undergo period doubling in order to ensure stability and sustainability.

Figure 3a shows the profit $Y$ as a function of the harvesting effort $E$ and the harvesting time $T$. For details on the cost function the reader is referred to the literature [1, 9]. The thick red line marks the period doubling bifurcation of the system. In order to ensure stability, the point of operation must be located

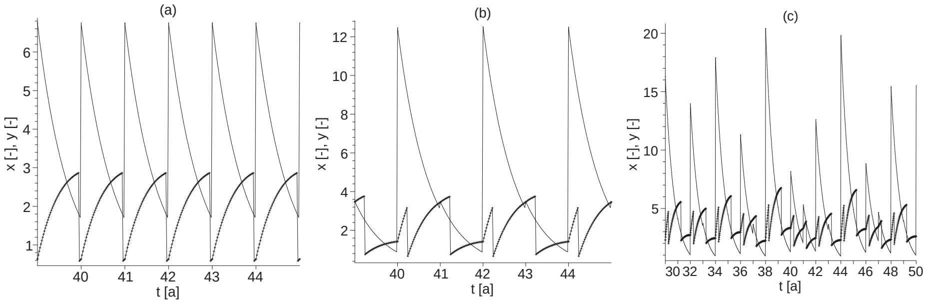


Figure 2: Dynamic behavior of a model of a periodically harvested fish population. The dimensionless quantities $x$ (dotted line) and $y$ (continuous line) are measures for the number of mature and young individuals in the population [1, 9]. In (a) the period is one year. After changing the time at which harvesting takes place, period doubling occurs (b). Cascades of period doublings can lead to chaotic behavior (c).

to the left and below the stability boundary in Fig. 3a and b. The result of the optimization with the CNLD constraints is shown in Fig. 3b. The back off from the critical boundary ensures robust stability of the optimal point of operation.

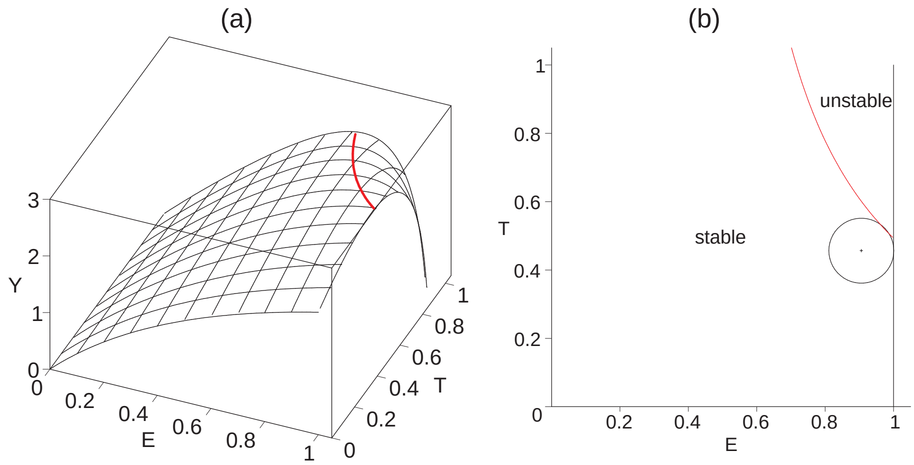


Figure 3: (a) Profit $Y$ as a function of harvesting time $T$ and harvesting effort $E$. The red line marks the stability boundary. (b) Stability boundary in the $(E, T)$ plane and result of the optimization with CNLD constraints.

# 4 Conclusion

We demonstrated that the concept of constructive nonlinear dynamics that had previously been developed by the authors for the robust optimization of continuous time processes with dynamic constraints, can be extended to the important class of discrete time systems. Towards this end we developed extended systems of equations for normal vectors on period doubling bifurcations. These normal vectors were successfully employed to optimize a model of the periodic harvesting of a fish population.

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