COMPUTING SENSOR LOCATION FOR NONLINEAR SYSTEMS UNDER THE INFLUENCE OF DISTURBANCES

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1.0 Introduction

Sensor network design is a topic which has received a large amount of attention in recent years. Most of the performed work focuses either on maximizing some norm of observability of a system by choosing a sensor network or on using the Kalman filter error covariance matrix for computing optimal sensor locations. While initial approaches focused on linear systems or linearized nonlinear systems (Muske & Georgakis, 2003; Van den Berg et al., 2000) more recent work has also dealt with nonlinear processes without the requirement of linearization (Wouwer et al., 2000; Alonso et al., 2004; Singh & Hahn, 2005a; 2006). However, the topic of determining a sensor network for nonlinear systems under the influence of disturbances has received comparatively little attention.

This work extends the technique presented by Singh & Hahn (2005a) for designing sensor networks for nonlinear systems under the influence of disturbances. The focus of the method is not simply on determining the disturbance itself, but on designing sensor networks that allow for reliable state reconstruction in addition to computing the magnitude of the disturbance. This is achieved by balancing the empirical observability and controllability gramians of the system (Hahn & Edgar, 2002), where the disturbances act as inputs to the system. The empirical controllability gramian contains the information describing the effect that the disturbances have on the states of the system, while the empirical observability gramians contains the information about the state-to-output behavior of the process. Both empirical gramians will be balanced and the sum of the resulting Hankel singular values serves as a measure for the "quality" of a sensor location for state and disturbance estimation: (1) directions in state space which are not uniquely affected by the disturbances will automatically be ignored for the sensor network design as this is not reflected in the empirical controllability gramian; (2) directions in the state space which cannot be uniquely observed for a chosen sensor network design will also not be considered; (3) balancing the empirical controllability and observability gramians ensures that the measure describing the sensor network will simultaneously reflect state reconstruction and disturbance estimation. The presented technique has been applied to two examples: a distillation column described by 32 nonlinear differential equations and a fixed bed reactor model described by nonlinear partial differential equations.

2.0 Balancing of gramians

2.1 Controllability gramian

For a linear time-invariant system of the form:

$$\dot{x} = Ax + Bu \tag{1a}$$

$$y = Cx + Du \tag{1b}$$

the controllability gramian is given by Fairman (1998):

$$W_C = \int_0^\infty e^{At} B B e^{A^T t} dt \tag{2}$$

If the controllability gramian has full rank then the system (1) is controllable. However, if the matrices are rank deficient then some of the states (or directions in state space) cannot be controlled or excited by the input.

2.2 Empirical controllability gramian

For a nonlinear system defined as:

$$\dot{x} = f(x(t), u(t)) \tag{3a}$$

$$y = h(x(t)) \tag{3b}$$

the empirical controllability gramian is defined as (Lall et al., 2002):

$$W_C = \sum_{l=1}^{r}\sum_{m=1}^{s}\sum_{i=1}^{p}\frac{1}{(rsc_m^2)}\int_0^\infty \Phi^{ilm}(t)dt \tag{4}$$

where $\Phi^{ilm}(t) \in R^{n \times n}$ corresponds to $\Phi_{ij}^{lm}(t) = (x^{ilm}(t) - x_{ss}^{ilm})^T (x^{jlm}(t) - x_{ss}^{jlm}), x^{ilm}(t)$ is the state of the nonlinear system corresponding to the impulse input $u(t) = c_m T_l e_i \delta(t) + u_0$, and x_0^{ilm} is the steady state of the system. The other variables in equation (4) are defined as follows:

$$T^n = \{T_1, \ldots\ldots\ldots, T_r; T_i \in R^{n \times n}, T_i^T T_i = I, i = 1, \ldots\ldots\ldots, r\}$$

$$M = \{c_1, \ldots\ldots\ldots\ldots, c_s; c_i \in R, c_i > 0, i = 1, \ldots\ldots\ldots\ldots, s\}$$

$$E^n = \{e_1, \ldots\ldots\ldots\ldots, e_n; \text{ standard unit vectors in } R^n \}$$

with r being the number of matrices for the perturbation directions, s the number of different perturbation sizes for each direction, and *n* the number of states of the system. These matrices can be used for controllability analysis of nonlinear systems (Singh & Hahn, 2005b).

2.3 Observability gramian

Observability refers to the property of a system that allows the reconstruction of the state variables given the outputs. For a linear time-invariant systems of the form (1) the observability gramian:

$$W_{O,linear} = \int_0^\infty e^{A^T t} C^T C e^{A\,t} dt \tag{5}$$

can be computed in order to determine the observability of the system. If the observability gramian has full rank then the system (1) is observable. However, if the matrices are rank deficient then the system will not be observable and some of the states (or directions in state space) cannot be reconstructed from the output data.

2.4 Empirical observability gramian

While an observability test for linear systems is straightforward, determining observability for nonlinear systems is usually too complex to be interpreted for all but very simple systems. One alternative is to use the relatively new concepts of empirical observability gramians. The empirical observability gramian:

$$W_O = \sum_{l=1}^{r}\sum_{m=1}^{s}\frac{1}{(rsc_m^2)}\int_0^{\infty} T_l \Psi^{lm}(t) T_l^T dt \tag{6}$$

can be computed for stable nonlinear systems of the form of equation (3), where $\Psi^{lm}(t) \in R^{n\times n}$ corresponds to $\Psi_{ij}^{lm}(t) = (y^{ilm}(t) - y_{ss})^T (y^{jlm}(t) - y_{ss}), y^{ilm}(t)$ is the output of the system corresponding to the initial condition $x(0) = c_m T_l e_i + x_{ss}$, and y_{ss} is the steady state output of the system. The other variables in equation (6) are as defined in section 2.2.

2.5 Balanced system and Hankel singular value

Let (A,B,C,D) be a minimal realization of a stable transfer function G(s), then the realization (A,B,C,D) is balanced, if the controllability gramian (W_C) and observability gramian (W_O) of the system are equal such that (Skogestad & Postlethwaite, 1997):

$$W_C = W_O = \Sigma = \mathrm{diag}(\sigma_1, \sigma_2, \ldots\ldots\ldots, \sigma_n) \tag{7}$$

where, $\sigma_1 \geq \sigma_2 \geq \ldots\ldots \sigma_n$.

The σ_i's are the Hankel singular values of the system. The σ_i can be further defined as:

$$\sigma_i = \lambda_i^{1/2}(W_O W_C), i = 1,2,\ldots,n \tag{8}$$

3.0 Computing sensor location for nonlinear systems under the influence of disturbances

In this section, a new technique for sensor placement that takes into account process disturbances is presented for a nonlinear system. The motivation for this work is that measuring some of the process disturbances may not be possible due to practical constraints and these disturbances may have to be inferred from secondary measurements. However, these unknown process disturbances may have adverse effects on process monitoring. For example if an estimator is used for state estimation, these disturbances can result in poor estimates. As a result, computing measurement based on nominal parameters alone may not be sufficient for process monitoring. Therefore, it is important for sensor network design that state reconstruction and disturbance estimation is possible.

In order to incorporate the influence of disturbances in the sensor network design the effect of disturbances on the system states is analyzed. This is done by the computing the empirical controllability gramian where the disturbances are the inputs to the system.

The amount of energy transferred to the states by excitation with the inputs can be defined as (Fairman, 1998):

$$E_C = \int_0^{\infty} x^T(t)x(t)dt \tag{9}$$

In the special case where impulse inputs are used and the system initially starts out at steady state, the energy E_C is given by (Fairman, 1998; Wicks & Decarlo, 1988):

$$E_C = \mathrm{trace}(W_C) \tag{10}$$

Similarly, for nonlinear systems, the energy transferred from input to states can be approximated by replacing the controllability gramian in equation (10) with the empirical controllability gramian. This approximation will hold locally as the empirical controllability gramian reduce to the linear gramians for linearized system (Lall et al., 2002). Also, the energy transferred by the input to the states can be interpreted as:

$$E_C = \sum_{i=1}^{n} W_{C,ii} \tag{11}$$

This alternate definition of the energy expression has the advantage that, in addition to the influence of inputs on the overall system, the effect on individual physical states can be inferred from the controllability gramians. The diagonal elements of the gramian represent the variance of individual states due to excitation with by the input. Therefore, states with relatively large diagonal entries are strongly excited by the perturbations. In comparison, the states with a small diagonal entry are almost uncontrollable or almost unaffected by process disturbances. If measurements are placed at states that are not influenced by a disturbance or are weakly excited by the inputs, then the disturbances or the influence of the disturbances may not be reconstructable from the measurements. The resulting system will have a poor degree of observability in the presence of disturbances and consequently, poor estimator convergence.

After computing the empirical controllability gramian, observability analysis of the nonlinear system is carried out in the next step. The empirical observability gramian is computed for each sensor network under consideration. The computed controllability gramian is then balanced with the computed observability gramians of the nonlinear system, where each different sensor network design results in a different empirical observability gramian. The reason for balancing the matrices is that in the balanced coordinates only the observable and controllable directions will be reflected. The balanced gramians, thus capture the information about the states most excited by the disturbances and the directions in states space that can be more readily reconstructed.

In case of balanced coordinates, the observability gramian and controllability gramian is given by (Fairman, 1998):

$$\widetilde{W}_O = T^T W_O T \tag{12a}$$

$$\widetilde{W}_C = T^T W_O T \tag{12b}$$

where 'T' is the transformation matrix that balances the gramians.

However, in balanced coordinates observability and controllability gramians are equal and the total output energy in balanced coordinates is given by:

$$E_O = \mathrm{trace}(\Sigma_b)\,; \quad \Sigma_b = \widetilde{W}_O = \widetilde{W}_C \tag{13}$$

Therefore, in order to place sensors in the system the output energy defined by (equation 14) is maximized over the entire set of possible sensor combinations:

$$J = \max_{i=1,..,n}\left(x_i\left(\mathrm{trace}(\Sigma_b)\right)_i\right) \tag{14}$$

$$x_i = 0,1 \quad \forall i \in \{1, \quad \ldots, \quad n\}$$

where $x_i = 1$, if measurement is placed at a location and $x_i = 0$, if state 'i' is not measured.

In this technique, the sensors are placed such that the output energy or information about the observable directions in the states space is maximized. However, these observable directions pertain to part of the observable subspace influenced by the input excitations or controllable subspace. Therefore, the directions in the states space that are most influenced by the disturbance are more heavily weighted and the directions that are not influenced get a weighting of zero. However, for the special case, where all the directions in the states space are equally influenced, i.e., the controllability gramian is given by an identity matrix ($W_C = I$), the sensor location problem defined in (14) reduces to the sensor location problem with nominal operating conditions presented in Singh & Hahn (2005a):

$$J = \max_{i=1,..,n}\left(x_i\left(\mathrm{trace}(W_O)\right)_i\right) \tag{15}$$

$$x_i = 0,1 \quad \forall i \in \{1, \quad \ldots, \quad n\}$$

4.0 Case study

This section illustrates the presented technique by performing optimal sensor location for a distillation column model as well as a fixed-bed reactor. For evaluating the predicted results for different sensor networks, an extended Kalman filter has been implemented for the distillation column. The performance of the extended Kalman filter has been compared for optimal and non-optimal sensor locations.

4.1 Process models

4.1.1 Catalytic fixed bed reactor

The reactor model is an important industrial process for vapor phase oxidation of o-xylene to phthalic anhydride (Singh & Hahn, 2005a). The reaction is highly exothermic and is carried out in a fixed-bed reactor. The reactor model is assumed to be one-dimensional pseudo-homogenous. The model consists of two partial differential equations, one each for the mass and the energy balance along the length of the reactor. The boundary conditions used for this model are $p(t,0) = 0.015$ atm and $T(t,0) = 625\,K$, corresponding to the inlet reactant partial pressure and inlet reactant fluid temperature. The reactor wall temperature is assumed to be equal to the inlet reactant fluid temperature. The infinite dimensional reactor model is discretized in space using finite differences converting the original model into a set of $2 \times n$ nonlinear ordinary differential equations, where n is the number of discretization points in space. For this work 30 discretization points were chosen, resulting in a set of 60 nonlinear ordinary differential equations.

4.1.2 Distillation column

A distillation column model with 30 trays for the separation of a binary mixture of cyclo-hexane and n-heptane has been considered (Singh & Hahn, 2005a). The column has 32 states and is assumed to have a constant relative volatility of 1.6. The feed is introduced in the middle compartment (17th tray) as a saturated liquid. The feed stream has a composition of $x_f = 0.5$, distillate and bottom product purities are $x_D = 0.935$ and $x_B = 0.065$, respectively. The boiling points of cyclo-hexane and n-heptane are 353 k and 371 k, respectively, at constant column pressure. The reflux ratio is held at a constant value of 3.0. The column model is described by a set of 32 nonlinear ordinary differential equations with temperatures as state variables and 33 explicit algebraic equations.

4.2 Optimal sensor location

4.2.1 Catalytic fixed bed reactor

The optimal location for a temperature sensor for a 10% disturbance in the feed composition to the reactor is investigated. The empirical controllability gramian, as defined in Section 2.2, is computed where the feed composition is chosen as the input to the system. In a next step, the empirical observability gramian, as defined in Section 2.4, is computed for all the thirty possible sensor locations in the reactor. Then, a balanced gramian is computed for every location in the reactor by computing a suitable transformation matrix 'T', defined in equation (12). The best location for a temperature sensor for the case where there are disturbances in the feed composition is computed by maximizing equation (14) over the set of thirty possible temperature locations. The optimal sensor location for 1a 0% disturbance in the feed composition is determined to be 0.6m from the reactor inlet (Figure 1). In order to verify the computed sensor location, the diagonal elements of the controllability gramian have been plotted along the reactor. The diagonal elements of the controllability gramian can provide an indication of the influence of process disturbances on the individual states. It can be concluded that the states which are strongly influenced by process disturbances will have larger diagonal entries. In comparison, the states which are hardly influenced by the input will have diagonal entries close to zero. From Figure 2, it can be concluded that the states around the hot spot are most affected by the disturbance in the feed composition. The computed optimal sensor location of 0.6 m (Figure 1), points to the state most affected by a disturbance in the feed composition. As result of the process disturbance, the steady state hotspot location moves from its nominal value of 0.4m to 0.5m (Figure 3). Therefore, the computed sensor location corresponds to the most sensitive location in the reactor under the influence of disturbances.

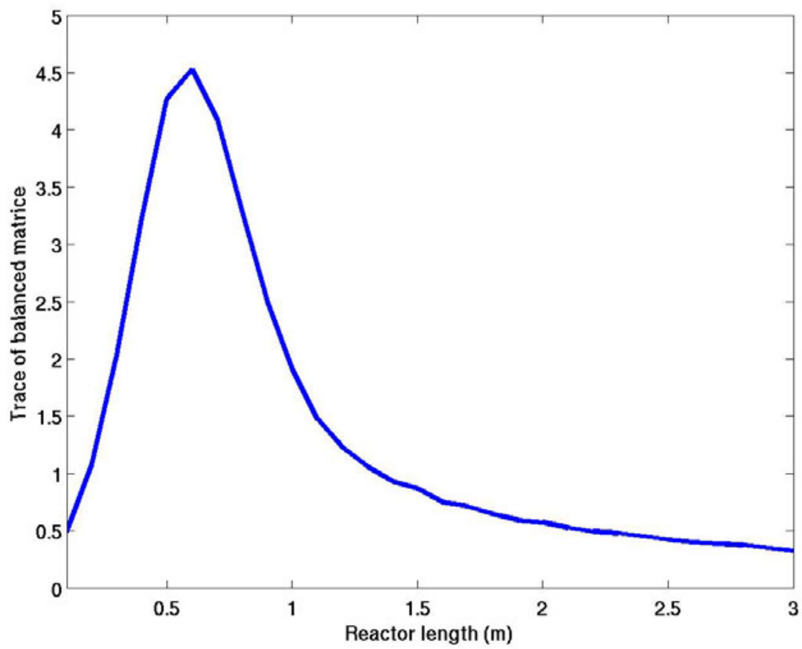

Fig. 1. Measure for placing one sensor in the reactor with 10% disturbance in feed composition

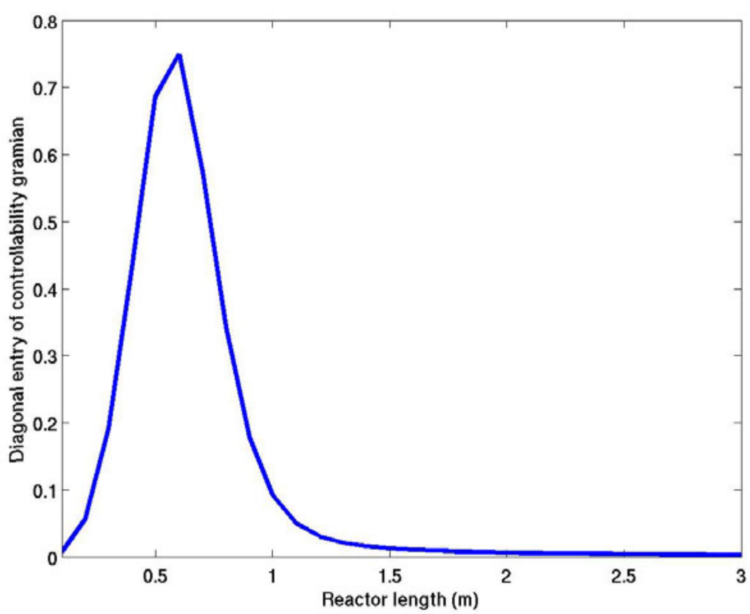

Fig. 2. Diagonal entry of states, i.e. variances of the states, in the empirical controllability gramian for 10% disturbance in feed composition for the fixed bed reactor

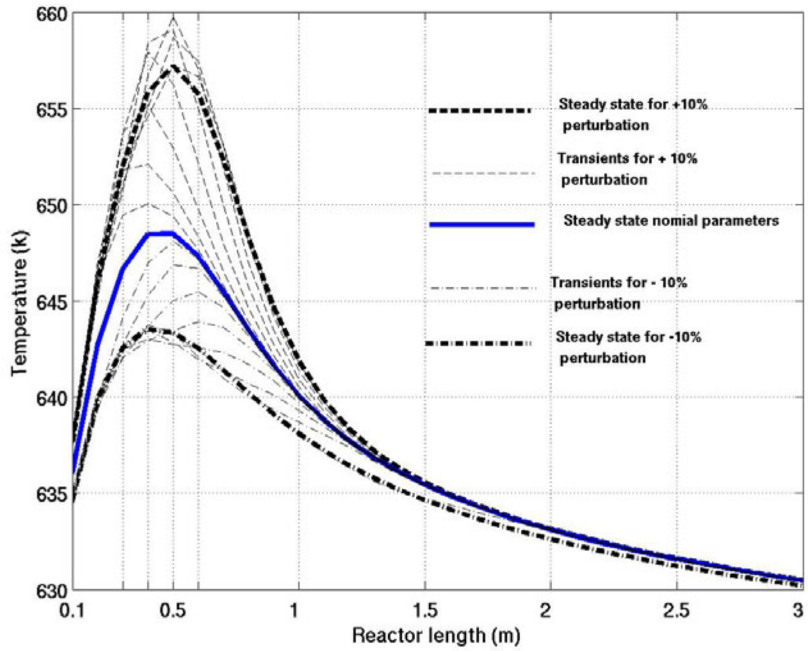

Fig.3. Steady state and transient profile for nominal feed composition and ±10% disturbance in the feed composition

In comparison, if only the observability gramian, i.e. the disturbance is not taken into account for sensor location, is used for sensor location in the reactor (by solving the sensor location problem given by equation (15)), then the optimal location is determined to be 0.4 m from the reactor inlet (Figure 4). The reason for this is that by using only the observability, the movement in the hotspot location in the reactor would not be taken into account (see Figure 3).

4.2.2 Distillation column

In the second example, the location of a temperature sensor has been computed for a nonlinear distillation column. In order to investigate the influence of the feed disturbance, the diagonal elements of the controllability gramian have been plotted for a 10% disturbance in feed composition (Figure 5).

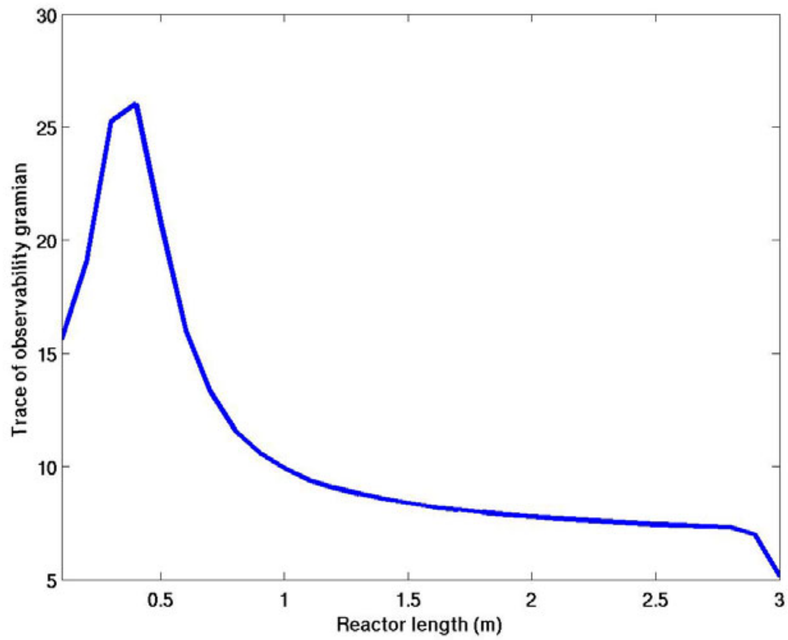

Fig. 4. Measure for placing one sensor in the reactor for nominal operating parameters

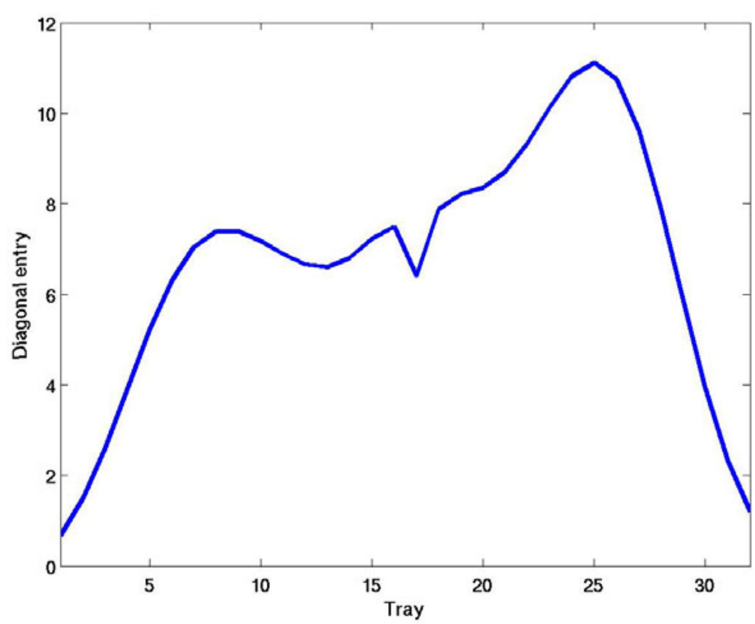

Fig. 5. Diagonal entry of states (variance of states) in the empirical controllability gramian for 10% disturbance in feed composition for the distillation column

It can be concluded from Figure 5 that for 10% disturbances in the feed composition, that many of the states of the column are affected by this disturbance. The states around the 26th tray in the stripping section are most affected by the perturbation in the feed composition, while the states near the top and bottom of the column are least sensitive to perturbations in the feed composition. The states near the 6th tray in the distillation column also seem to be good locations for placing a temperature sensor. However, the states near the column top and the column bottom are not the good choices for placing measurement in the system. In addition, the states near the feed tray are also not good locations for placing measurements in the column. The reason for this is, although the feed tray is highly sensitive to disturbances in the feed composition, the feed tray is insensitive to perturbations in any column state. As a result, a measurement at the feed tray will result in a poor degree of observability of the system. Since the Hankel singular values are the product eigenvalues of controllability and observability gramian, feed tray has a small Hankel singular value due to the poor system observability for a measurement at the feed tray. In order to further corroborate the computed sensor locations in the distillation column, an extended Kalman filter has been implemented for optimal (26th tray) and non-optimal measurements (1st and 17th tray). The extended Kalman filter has been tuned for similar process and measurement noise. The filter performance has been compared for 10% sinusoidal disturbance in the feed composition and the process disturbance is an unknown input for the estimator.

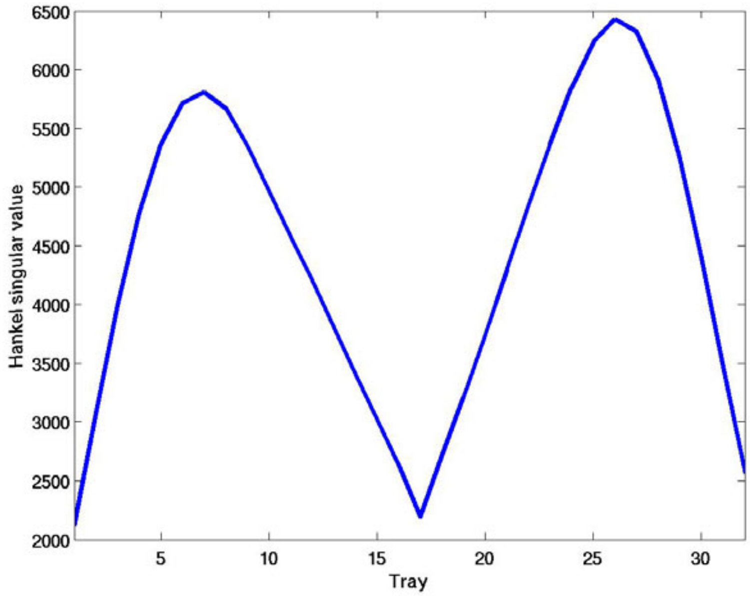

Fig. 6 Measure for placing one sensor in the distillation column with 10% disturbance in feed composition

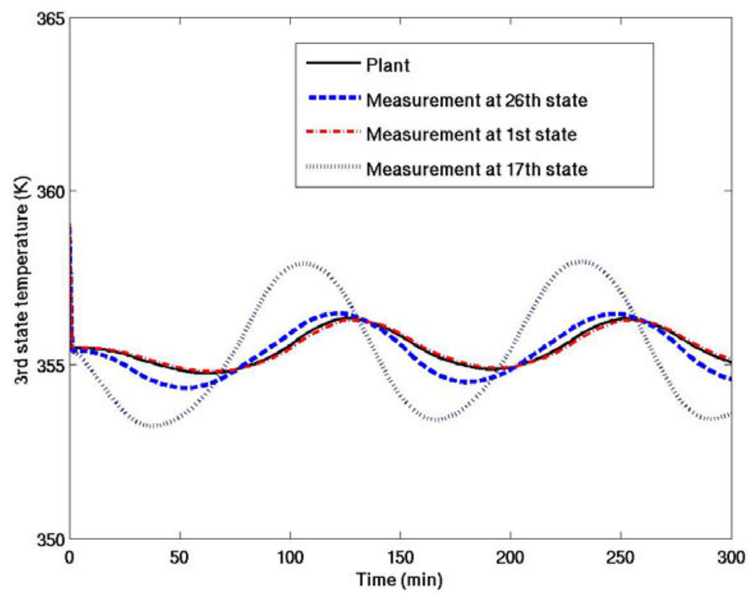

Fig. 7. Reconstruction of 3rd state in the distillation column, for optimal (26th tray) and non-optimal (1st and 17th), for sinusoidal process disturbance of 10% in the feed composition

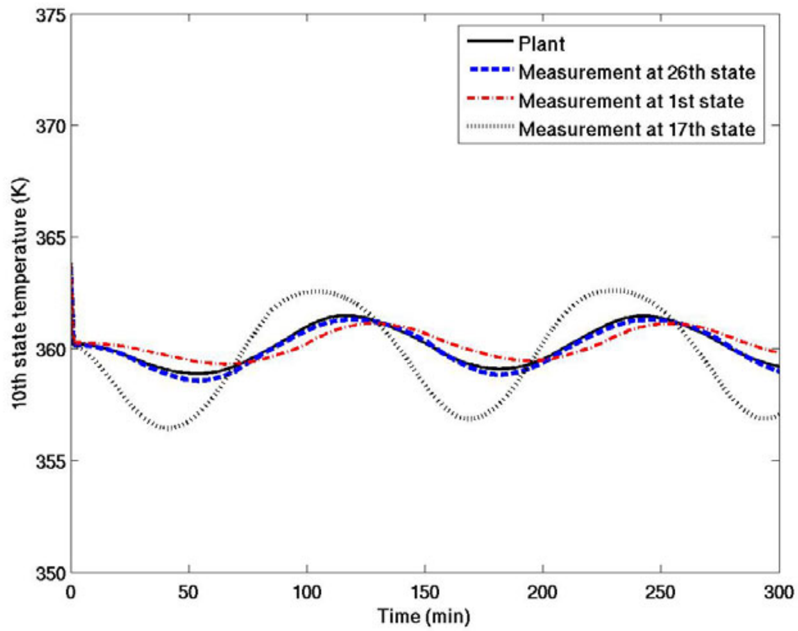

Fig. 8. Reconstruction of 10^{th} state in the distillation column, for optimal (26^{th} tray) and non-optimal (1^{st} and 17^{th}), for sinusoidal process disturbance of 10% in the feed composition

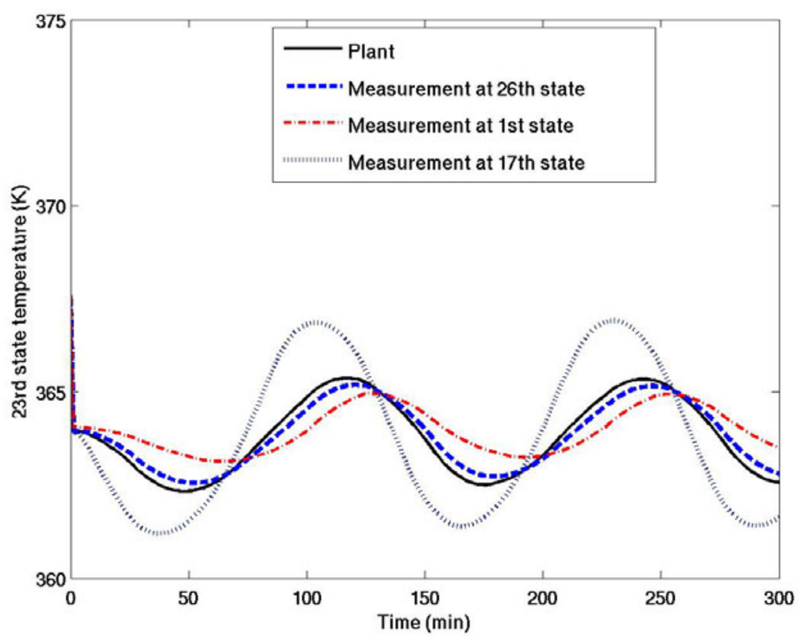

Fig. 9. Reconstruction of 23rd state in the distillation column, for optimal (26^{th} tray) and non-optimal (1^{st} and 17^{th}), for sinusoidal process disturbance of 10% in the feed composition

It can be concluded from the results that for states very close to the measurement result in good reconstruction of the states. As shown in Figure 7, the estimation of state 3 return better results for a measurement at the 1st state compared to the measurements placed at the 26th state or the17th state. However, for the optimal location (26th tray) good overall reconstruction is obtained throughout the distillation column as can be seen from the reconstruction of the 10th state (Figure 8), the 23rd state (Figure 9) and the 30th state (Figure 10) in the distillation column. A good reconstruction of the 10th state (Figure 8) in the distillation column is obtained for a measurement at the 26th state. In comparison, the performance of the extended Kalman filter worsens, if the measurement is placed at the 1st state. For the case of a measurement at the 17th state, the reconstruction of the 10th state with an extended Kalman filter is extremely poor (Figure 8). Similarly, good reconstruction of the 23rd state (Figure 9) and the 30th state (Figure 10) in the distillation column is obtained for a measurement at the 26th state. In comparison, reconstruction by extended Kalman filter results in large estimation error if the measurement is placed at non-optimal locations, i.e., 1st or 17th state.

In addition, the performance of the extended Kalman filter has been compared for step disturbance of 10% in the feed composition. Figure 11 shows the steady state reconstruction for the entire distillation column profile for optimal and non-optimal measurement locations. In case of measurement at the 1st state, good estimation is obtained at the upper end of the column. However, the estimation error increases as one moves away from the measurement. In case of measurement at the 17th state, there is a large estimation error in all the states of the column, with the exception being the 17th

state which is directly measured. In comparison, for the optimal measurement (26th state), reliable state estimation is possible for the entire distillation column.

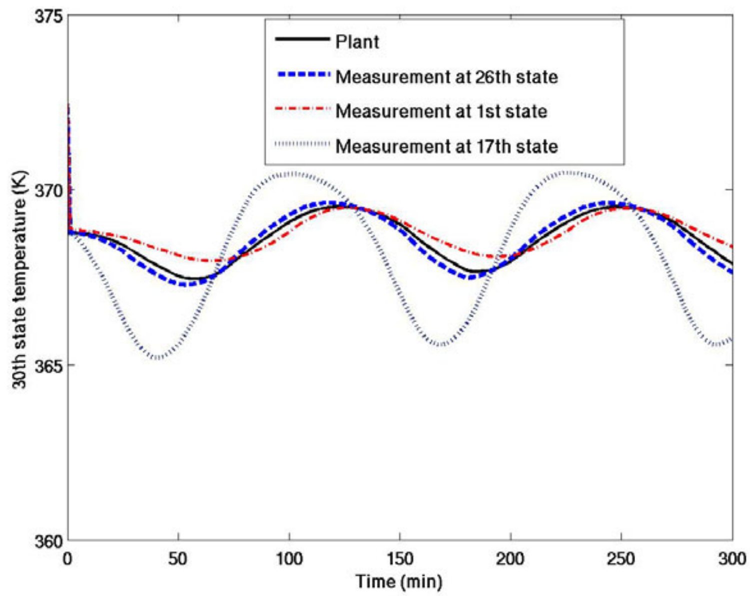

Fig. 10. Reconstruction of 30th state in the distillation column, for optimal (26th tray) and non-optimal (1st and 17th), for sinusoidal process disturbance of 10% in the feed composition

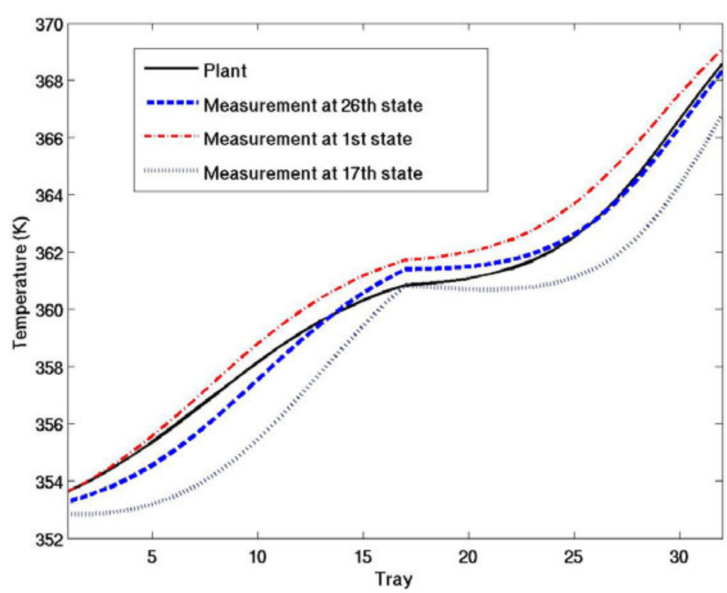

Fig. 11. Reconstruction of distillation column profile, for optimal (26th tray) and non-optimal (1st and 17th), for 10% step disturbance in the feed composition

5.0 Conclusions

In this work a new technique for sensor location for nonlinear dynamic systems has been presented that takes into account the effect of disturbances on the system. This is achieved by considering the disturbances as inputs to the system and computing an empirical controllability gramian for this augmented set of inputs, followed by balancing the empirical controllability gramian with the observability gramians resulting from different sensor network configurations. The optimal sensor network for this case is determined to be the one which results in the largest sum of the Hankel singular values.

The presented technique has been applied to two nonlinear systems, a fixed bed reactor and a binary distillation column models. The computed results have been corroborated by comparing the performance of an extended Kalman filter for optimal and non-optimal measurement locations.

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