**89d Computation of Equilibrium States and Bifurcations in Ecosystem Models Using Interval Analysis**

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A problem of frequent interest in the fields of science and engineering is the analysis of nonlinear ordinary differential equation (ODE) systems. These systems often display rich mathematical behavior, including varying numbers of steady states (equilibria). The number and stability of equilibria in a model may change in a bifurcation as one or more model parameters change. Bifurcation diagrams concisely represent a large amount of information concerning the number and stability of equilibria in an ODE model in terms of two model parameters. Bifurcations of equilibria are generally found by solving a nonlinear algebraic equation system consisting of the equilibrium (steady-state) conditions, plus one (in the case of codimension-1 bifurcations) or two (codimension-2) additional conditions. Typically this equation system is solved using some continuation-based strategy (Kuznetsov, 1991). However, in general, these methods do not provide any guarantee that all bifurcations will be found, and may be initialization sensitive. Thus, without some a priori knowledge of system behavior, one may not know with complete certainty if all bifurcation curves have been identified and explored.

Interval mathematics (e.g., Kearfott, 1996) provides tools with which one can resolve this issue with computational certainty. Specifically, in solving a nonlinear algebraic equation system, the interval-Newton approach combined with generalized bisection (IN/GB) provides a mathematical and computational guarantee that all solutions will be found (or more precisely enclosed within a very narrow interval). Thus, using IN/GB, all bifurcations within parameter intervals of interest can be located without need for initialization or a priori knowledge of system behavior. In previous work, this method was applied to the problem of locating equilibrium states and bifurcations in a simple ecosystem model, specifically a tritrophic Rosenzweig-MacArthur food-chain model (Gwaltney, *et. al*., 2004). Our interest in ecosystem modeling is motivated by its use as one tool in studying the impact on the environment of the industrial use of newly discovered materials, such as ionic liquids (Brennecke and Maginn, 2001).

In this presentation, we consider more complex ecosystem models than addressed previously, and apply the interval methodology to rigorously locate all equilibrium states and codimension-1 and codimension-2 bifurcations of equilibria. The first ecosystem model considered is an experimentally verified, age-structured food chain model of a planktonic rotifer population feeding on unicellular green algae (Fussmann et. al. 2000). The second ecosystem model considered is a food web model with up to 7 species. In a model with $n$ state variables, there are $2^n$ different sets of equations to consider when determining feasible steady-states, thus there is potentially a very large number of feasible equilibria. Using the interval methodology all the feasible equilibrium states can be found simultaneously without the need to address each different set of equations individually. These examples demonstrate the effectiveness of the interval-Newton approach in the analysis of nonlinear dynamical systems.

References

Brennecke, J. F., & Maginn, E. J. Ionic liquids: Innovative fluids for chemical processing. *AIChE Journal* 47, 2384 (2001).

Fussmann, G.F., S.P. Ellner, K.W. Shertzer, and N.G. Hairston Jr, "Crossing the Hopf Bifurcation in a Live Predator-Prey System", *Science* 290, 1358 (2000).

Gwaltney, C.R., M.P. Styczynski, and M.A. Stadtherr, "Reliable computation of equilibrium states and bifurcations in food chain models", *Computers in Chemical Engineering*, 28, 1981 (2004).

Kearfott, R.B., *Rigorous Global Search: Continuous Problems*, Kluwer Academic Publishers, Dordrecht, 1996.

Kuznetsov, Y.A., *Elements of Applied Bifurcation Theory*, Springer-Verlag, New York, 1991.