# BOUNDARY PREDICTIVE CONTROL OF PARABOLIC PDEs

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*Abstract*— **This work focuses on boundary model predictive control of linear parabolic partial differential equations (PDEs) with input and state constraints. Various predictive control formulations are presented and their ability to enforce stability and constraint satisfaction in the infinite-dimensional closed-loop system is analyzed. A numerical example of a linear parabolic PDE with unstable steady state and flux boundary control subject to state and control constraints is used to demonstrate the implementation and effectiveness of the predictive controllers.**
**Key words: Diffusion-reaction processes, Dissipative systems, Boundary control, Parabolic PDEs, Model predictive control, Input/state constraints.**

## I. INTRODUCTION

The model predictive control framework is widely used in the control of process systems due to its ability to explicitly handle manipulated input and state variable constraints. Manipulated input constraints express limits on the capacity of the control actuators and state constraints usually express desired specifications on the operating range of the process state variables. Despite the significant efforts on the development of model predictive control methods for lumped parameter processes described by linear/nonlinear ordinary differential equation (ODE) systems ([15], [18]), at this stage, few results are available on the model predictive control of distributed parameter systems.

On the other hand, control of various classes of nonlinear highly dissipative distributed parameter systems, arising in the modeling of transport-reaction processes, particulate processes and fluid dynamic systems, has attracted a lot of attention in the last ten years. Specifically, motivated by the property of highly-dissipative distributed parameter systems that their dominant dynamic behavior is low-dimensional in nature, research has focused on the development of a general framework for the synthesis of nonlinear low-order controllers for nonlinear parabolic partial differential equation (PDE) systems with distributed control – and other highly dissipative PDE systems that arise in the modeling of spatially-distributed processes – on the basis of nonlinear low-order ODE models derived through combination of Galerkin's method (using analytical or empirical basis functions) with the concept of inertial manifolds [7]. Using these order reduction techniques, a number of control-relevant problems - such as nonlinear and robust controller design, dynamic optimization, and the control under actuator saturation - have been addressed for various classes of dissipative PDE systems (e.g., see [2], [1] and the book [7] for results and references in this area). In addition to the above results which deal with dissipative PDEs subject to distributed control (i.e., the manipulated inputs enter directly into the PDE), significant research has been carried out on boundary-controlled linear distributed parameter systems (see, for example, [14], [21], [8], [13]) and necessary conditions for stabilization under state and output feedback control have been derived. More recently, results on boundary control of distributed parameter systems include the use of singular functions for identification and control [5], boundary control of nonlinear distributed parameter systems by means of static and dynamic output feedback regulation [4] and the development of feedback control laws based on the backstepping methodology [17], [3]. Referring to these results, it is important to point out that they do not address the issue of state stabilization subject to state and control constraints.

Recently, we have initiated a research effort trying to develop computationally-efficient predictive control algorithms for parabolic PDEs subject to state and control constraints. Specifically, in [10], we considered linear parabolic PDEs with distributed control and derived predictive controller formulations that systematically handle the objectives of state and input constraints satisfaction and stabilization of the infinite dimensional system; subsequently, we extended these results to linear parabolic PDEs under output feedback control [9] and quasi-linear parabolic PDEs [11]. Other results in this area include model predictive control of first-order hyperbolic PDE systems [20], [6] and state feedback model predictive control of a diffusion reaction process on the basis of finite-dimensional approximations derived by the finite difference method [12]. However, in all these works, the attention is focused on PDEs in which the manipulated inputs enter directly into the PDE (distributed control).

Motivated by these considerations, this work focuses on boundary model predictive control of linear parabolic PDEs with input and state constraints. A standard transformation is initially used to rewrite the original boundary control problem as a distributed control problem that involves the presence of both the input and its time-derivative in the PDE and has homogeneous boundary conditions. Then, modal decomposition techniques are applied to the transformed system to decompose it into an interconnection of a finite-dimensional subsystem, capturing the dominant dynamics

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of the parabolic PDE (slow subsystem), with an infinite-dimensional (fast) subsystem. Subsequently, various predictive control formulations are proposed and their ability to enforce stability and constraint satisfaction in the infinite-dimensional closed-loop system is analyzed. Finally, an example of boundary control of a linear parabolic PDE, with spatially-uniform unstable steady state and flux boundary control, subject to state and control constraints, is considered. Simulations are carried out to demonstrate the ability of the predictive controllers in enforcing closed-loop system stability and state constraint satisfaction.

## II. PRELIMINARIES

### A. Parabolic PDEs with boundary control

In this work, we consider linear parabolic PDEs of the following form:

$$\frac{\partial \bar{x}(z,t)}{\partial t} = \bar{b}\frac{\partial^2 \bar{x}(z,t)}{\partial z^2} + \bar{c}\bar{x}(z,t) \tag{1}$$

with the following boundary and initial conditions:

$$\begin{aligned} \bar{b}_1\frac{\partial \bar{x}}{\partial z}(0,t) + \bar{c}_1\bar{x}(0,t) &= 0, \\ \bar{b}_2\frac{\partial \bar{x}}{\partial z}(l,t) + \bar{c}_2\bar{x}(l,t) &= u(t), \\ \bar{x}(z,0) &= \bar{x}_0(z) \end{aligned} \tag{2}$$

subject to the following input and state constraints:

$$u^{min} \leq u(t) \leq u^{max} \tag{3}$$

$$\chi^{min} \leq \int_0^l r(z)\bar{x}(z,t)dz \leq \chi^{max} \tag{4}$$

where $\bar{x}(z,t)$ denotes the state variable, $z \in [0,l]$ is the spatial coordinate, $t \in [0,\infty)$ is the time, $u(t) \in \mathbb{R}$ denotes the constrained manipulated input; $u^{min}$ and $u^{max}$ are real numbers representing the lower and upper bounds on the manipulated input, respectively, and $\chi^{min}$ and $\chi^{max}$ are real numbers representing the lower and upper state constraints, respectively. The term $\frac{\partial^2 \bar{x}}{\partial z^2}$ denotes the second-order spatial derivative of $\bar{x}(z,t)$; $\bar{b}$, $\bar{c}$, $\bar{b}_1, \bar{c}_1, \bar{b}_2, \bar{c}_2$ are constant coefficients with $\bar{b} > 0$, $\bar{b}_1^2 + \bar{c}_1^2 \neq 0$, $\bar{b}_2^2 + \bar{c}_2^2 \neq 0$ and $\bar{x}_0(z)$ is a sufficiently smooth function of $z$. In Eq.4, the function $r(z) \in L_2(0,l)$ is a "state constraint distribution" function which is square-integrable and describes how the state constraint is enforced in the spatial domain $[0,l]$. Whenever the state constraint is applied at a single point of the spatial domain $z_c$, with $z_c \in [0,l]$, the function $r(z)$ is taken to be nonzero in a finite spatial interval of the form $[z_c-\mu, z_c+\mu]$, where $\mu$ is a small positive real number, and zero elsewhere in $[0,l]$. Throughout the paper, the notation $|\cdot|$ will be used to denote the standard Euclidian norm in $\mathbb{R}^n$, while the notation $|\cdot|_Q$ will be used to denote the weighted norm defined by $|x|_Q^2 = x'Qx$, where $Q$ is a positive–definite matrix and $x'$ denotes the transpose of $x$.

In order to simplify the notation and the presentation of the theoretical results, the PDE of Eqs.1-2-3-4 is formulated as an infinite dimensional system in the state space $\mathcal{H} = L_2(0,l)$, with the inner product and norm:

$$(\omega_1,\omega_2) = \int_0^l \omega_1(z)\omega_2(z)\, dz\ , \quad \|\omega_1\|_2 = (\omega_1,\omega_1)^{\frac{1}{2}} \tag{5}$$

where $\omega_1, \omega_2$ are any two elements of $L_2(0,l)$.

To this end, we define the state function $x(t)$ on the state-space $\mathcal{H} = L_2(0,l)$ as:

$$x(t) = \bar{x}(z,t), \quad t > 0, \quad 0 < z < l \tag{6}$$

the operator $\mathcal{F}$ as:

$$\mathcal{F}\phi = \bar{b}\frac{d^2\phi}{dz^2} + \bar{c}\phi, \quad 0 < z < l \tag{7}$$

where $\phi(z)$ is a smooth function on $[0,l]$, with the following dense domain

$$\mathcal{D}(\mathcal{F}) = \{\phi(z) \in L_2(0,l) : \phi(z), \frac{d\phi(z)}{dz}\ \mathit{are\ absolutely\ continuous}, \mathcal{F}\phi(z) \in L_2(0,l), \bar{b}_1\frac{d\phi}{dz}(0) + \bar{c}_1\phi(0) = 0\} \tag{8}$$

the boundary operator $\mathcal{B} : L_2(0,l) \mapsto \mathbb{R}$ as:

$$\begin{aligned} &\mathcal{B}\phi(z) = \bar{b}_2\frac{d\phi(l)}{dz} + \bar{c}_2\phi(l), \quad \mathit{with} \\ &\mathcal{D}(\mathcal{B}) = \{\phi(z) \in L_2(0,l) : \phi(z)\ \mathit{is\ absolutely\ continuous}, \frac{d\phi(z)}{dz} \in L_2(0,l)\} \end{aligned} \tag{9}$$

and the state constraint as:

$$\chi^{min} \leq (r, x(t)) \leq \chi^{max} \tag{10}$$

Using the above definitions, the system of Eqs.1-2-3-4 can be written as follows

$$\begin{aligned} \dot{x}(t) &= \mathcal{F}x(t), \quad x(0) = x_0 \\ \mathcal{B}x(t) &= u(t) \\ u^{min} &\leq u(t) \leq u^{max} \\ \chi^{min} &\leq (r, x(t)) \leq \chi^{max} \end{aligned} \tag{11}$$

on $\mathcal{H} = L_2(0,\pi)$. However, the above equation has inhomogeneous boundary conditions owing to the presence of $u(t)$ in the boundary conditions. To be able to transform this boundary control problem into an equivalent distributed control problem (i.e., the manipulated input $u(t)$, and possibly its time derivative $\dot{u}(t)$, enter directly into the differential equation and do not appear in the boundary condition), we follow [14], [8] and assume that a function $B(z)$ exists such that for all the $u(t)$, $Bu(t) \in \mathcal{D}(\mathcal{F})$ and the following holds:

$$\mathcal{B}Bu(t) = u(t), \tag{12}$$

The requirement of existence of $B$, together with the assumption that the input $u(t)$ is sufficiently smooth, allow us to define the following transformation [14], [8] $p(t) = x(t) - Bu(t)$ which leads to the following equation:

$$\dot{p}(t) = \mathcal{A}p(t) + \mathcal{F}Bu(t) - B\dot{u}(t), \quad p(0) = p_0 \in \mathcal{D}(\mathcal{A}) \tag{13}$$

where the operator $\mathcal{A}$ on $\mathcal{H}$ is defined as:

$$\begin{aligned}&\mathcal{A}\phi(z) = \mathcal{F}\phi(z) \quad and \quad \mathcal{D}(\mathcal{A}) = \mathcal{D}(\mathcal{F}) \cap ker(\mathcal{B}) = \\ &\{\phi(z) \in L_2(0,l) : \phi(z), \frac{d\phi(z)}{dz} \ are \ abs. \ cont., \\ &\mathcal{A}\phi(z) \in L_2(0,l), \ and \ \bar{b}_1 \frac{d\phi}{dz}(0) + \bar{c}_1\phi(0) = 0\} \\ &\bar{b}_2 \frac{d\phi}{dz}(l) + \bar{c}_2\phi(l) = 0\}\end{aligned} \tag{14}$$

Eq.13 has a well defined mild solution since $\mathcal{A}$ is the infinitesimal generator of a $C_0$-semigroup and the operators $B$ and $\mathcal{F}B$ are bounded. Specifically, the operator $\mathcal{A}$ generates a $C_0$–strongly continuous semigroup $\mathcal{T}(t)$ given by:

$$\mathcal{T}(t) = \sum_{n=0}^{\infty} e^{\lambda_n t} \left(\cdot, \phi_n(z)\right) \psi_n(z) \tag{15}$$

such that $\sup_{n\geq 1} \ Re(\lambda_n) \leq \infty$, where $\lambda_n \ \{n \geq 1\}$, are simple eigenvalues of $\mathcal{A}$, and $\phi_n$ and $\psi_n$ are the corresponding eigenfunctions of $\mathcal{A}$ and $\mathcal{A}^*$, respectively, such that $(\phi_n, \psi_m) = \delta_{nm}$. When $\bar{b}_1 = 1, \bar{c}_1 = 0, \bar{b}_2 = 1, \bar{c}_2 = 0$, the eigenvalues and eigenfunctions of $\mathcal{A}$ are obtained by solving the eigenvalue problem analytically and are of the form,

$$\begin{aligned}&\lambda_n = \bar{b} - \bar{c}(n\ \pi/l)^2, \ n \geq 0, \ \phi_0 = \frac{1}{\sqrt{l}}, \\ &\phi_n(z) = \sqrt{\frac{2}{l}} cos(n\pi z/l), \quad n = 1, \cdots, \infty\end{aligned} \tag{16}$$

In this case, $\psi_n(z) = \phi_n(z)$ because the operator $\mathcal{A}$ is symmetric.

### *B. Modal decomposition*

Referring to the system of Eq.13, let $\mathcal{H}_s$ and $\mathcal{H}_f$ be modal subspaces of $\mathcal{A}$, defined as $\mathcal{H}_s = span\{\phi_1, \phi_2, \ldots, \phi_m\}$ and $\mathcal{H}_f = span\{\phi_{m+1}, \phi_{m+2}, \ldots\}$ (the existence of $\mathcal{H}_s$, $\mathcal{H}_f$ follows from the properties of $\mathcal{A}$ and $m$ is chosen such that $\lambda_{m+1} < 0$). Defining the orthogonal projection operators, $\mathcal{P}_s$ and $\mathcal{P}_f$, such that $p_s(t) = \mathcal{P}_s p(t)$, $p_f(t) = \mathcal{P}_f p(t)$, the state $p(t)$ of the system of Eq.13 can be decomposed as

$$p(t) \ = \ \mathcal{P}_s p(t) + \mathcal{P}_f p(t) \ = \ p_s(t) + p_f(t) \tag{17}$$

Applying $\mathcal{P}_s$ and $\mathcal{P}_f$ to the system of Eq.13 and using the above decomposition for $x(t)$, the system of Eq.13 can be written in the following equivalent form:

$$\begin{aligned}\frac{dp_s}{dt} &= \mathcal{A}_s p_s + (\mathcal{F}B)_s u - B_s \dot{u} \\ \frac{dp_f}{dt} &= \mathcal{A}_f p_f + (\mathcal{F}B)_f u - B_f \dot{u}\end{aligned} \tag{18}$$

where $\mathcal{A}_s = \mathcal{P}_s\mathcal{A}$, $(\mathcal{F}B)_s = \mathcal{P}_s\mathcal{F}B$, $B_s = \mathcal{P}_s B$, $\mathcal{A}_f = \mathcal{P}_f\mathcal{A}$, $(\mathcal{F}B)_f = \mathcal{P}_f\mathcal{F}B$, $B_f = \mathcal{P}_f B$. In the above system, $\mathcal{A}_s$ is a diagonal matrix of dimension $m \times m$ of the form $\mathcal{A}_s = diag\{\lambda_k\}$ ($\lambda_k$ are, possibly unstable, eigenvalues of $\mathcal{A}_s$, $k = 1, ..., m$) and $\mathcal{A}_f$ is an infinite dimensional operator which is exponentially stable (following from the fact that $\lambda_{m+1} < 0$ and the selection of $\mathcal{H}_s, \mathcal{H}_f$). In the remainder of the paper, we will refer to the $p_s(t)$- and $p_f(t)$-subsystems in Eq.18 as the slow and fast subsystems, respectively. Both the slow and fast subsystems in Eq.18 include a derivative of the control term which is undesirable and it cannot be handled by the standard model predictive control formulation. Therefore, we introduce a new variable $\tilde{u} = \dot{u}$, and rewrite the system of Eq.18 in the following form:

$$\begin{aligned}\begin{bmatrix} \dot{u}(t) \\ \dot{p}_s(t) \\ \dot{p}_f(t) \end{bmatrix} &= \begin{bmatrix} 0 & 0 \\ (\mathcal{F}B)_s & \mathcal{A}_s \\ (\mathcal{F}B)_f & \mathcal{A}_f \end{bmatrix} \begin{bmatrix} u(t) \\ p_s(t) \\ p_f(t) \end{bmatrix} + \\ &+ \begin{bmatrix} 1 \\ -B_s \\ -B_f \end{bmatrix} \tilde{u}(t)\end{aligned} \tag{19}$$

The state constraint of Eq.11 in terms of the state variables of Eq.19 takes the form:

$$\chi^{min} \leq (Bu(t), r(z)) + (p_s(t) + p_f(t), r(z)) \leq \chi^{max} \tag{20}$$

Finally, a finite dimensional approximation of the system of Eq.19 can be obtained by neglecting the $p_f(t)$ subsystem in Eq.19 and has the form:

$$\begin{aligned}\begin{bmatrix} \dot{u}(t) \\ \dot{p}_s(t) \end{bmatrix} &= \begin{bmatrix} 0 & 0 \\ (\mathcal{F}B)_s & \mathcal{A}_s \end{bmatrix} \begin{bmatrix} u(t) \\ p_s(t) \end{bmatrix} + \\ &+ \begin{bmatrix} 1 \\ -B_s \end{bmatrix} \tilde{u}(t)\end{aligned} \tag{21}$$

## III. Model predictive control

In this section, we present various predictive control formulations. We begin with a predictive control formulation constructed on the basis of the finite-dimensional approximation of Eq.21, subject to the input constraints of Eq.3 and the state constraints given by Eq.20 of the form:

$$\begin{aligned}&\min_{\tilde{u}} \int_t^{t+T} [u(\tau)\ p_s(\tau)]' Q \begin{bmatrix} u(\tau) \\ p_s(\tau) \end{bmatrix} d\tau \\ &\quad + F([u(t+T)\ p_s(t+T)])\end{aligned} \tag{22}$$

$$\begin{aligned}&s.t. \begin{bmatrix} \dot{u}(\tau) \\ \dot{p}_s(\tau) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ (\mathcal{F}B)_s & \mathcal{A}_s \end{bmatrix} \begin{bmatrix} u(\tau) \\ p_s(\tau) \end{bmatrix} + \\ &\quad + \begin{bmatrix} 1 \\ -B_s \end{bmatrix} \tilde{u}(\tau) \\ &u^{min} \leq u(\tau) \leq u^{max}, \ \tau \in [t, t+T] \\ &\chi^{min} \leq (Bu(\tau), r(z)) + (p_s(\tau), r(z)) \leq \chi^{max}\end{aligned} \tag{23}$$

where $Q$ is a positive definite real block diagonal matrix of the form $Q = \begin{pmatrix} Q_u & 0 \\ 0 & Q_{ps} \end{pmatrix}$ where the $Q_u > 0$ entry in the matrix $Q$ denotes the weight associated with the control input and $Q_{ps}$ is an $(m \times m)$ positive definite matrix expressing the weights imposed on the slow modes. Referring to the model predictive control formulation of Eqs.22-23, we note that it may not be necessarily stabilizing for the closed-loop finite-dimensional system (Eq.21 under the controller of Eq.22-23). To address this potential problem, a number of terminal constraints have been proposed in the literature [19], which, if the initial condition $(u(0), p_s(0))$ is one for which the predictive control problem of Eqs.22-23 has a solution for all future times, guarantee stability of the closed-loop finite-dimensional system. Terminal inequality constraints include, for example, $[u(t+T)\ p_s(t+T)] \in \mathcal{W}_s$ where $\mathcal{W}_s$ is an invariant set centered around the origin or simply setting $[u(t+T)\ p_s(t+T)] = [0\ 0]$. Furthermore, in the predictive controller of Eqs.22-23 the control action applied to the process is penalized in the cost with $Q_u$ being the weight and is subjected to the constraints $u^{min} \leq u(\tau) \leq u^{max}$, which appears as a state constraint since the optimization is done with respect to the auxiliary input $\tilde{u}(t)$ (resulting from the dynamic extension $\dot{u} = \tilde{u}$). This type of input penalty and input constraint is consistent with the practical implementation of the computed control action since what is applied to the PDE is $u(t)$ and not $\tilde{u}(t)$. Finally, since the predictive controller of Eqs.22-23 is independent of the fast states $p_f(t)$, it generates an implicit control law of the form $\tilde{u}(t) = M(p_s(t), u(t))$, and thus, it can be shown that if a terminal equality constraint is added to this controller and the initial conditions are chosen such that the resulting predictive controller enforces stability in the closed-loop finite dimensional system, then the closed-loop infinite-dimensional system under the same predictive controller is also asymptotically stable. However, since the $p_f(t)$ states are not included either in the cost functional or in the state constraints, there is no guarantee that the state constraints imposed on the infinite-dimensional system will be satisfied for all times (i.e., satisfaction of $\chi^{min} \leq (r, Bu(t) + p_s(t)) \leq \chi^{max}$ does not guarantee that $\chi^{min} \leq (r, x(t)) \leq \chi^{max}$). So, unlike the stabilization objective, which is achieved independently of the fast subsystem, the additional objective of state constraint satisfaction requires that the MPC design accounts in some way for the contribution of the fast states.

To deal with this problem, we present in the following two MPC formulations which explicitly account for the evolution of the $p_f(t)$-subsystem. In particular, one way to account for the effect of the fast states on the state constraints of the infinite–dimensional system is to incorporate the fast states explicitly into the state constraint inequality. The control action, under the resulting predictive control law in this case, is computed by solving the following optimization problem:

$$\min_{\tilde{u}} \int_t^{t+T} [u(\tau)\ p_s(\tau)]' Q \begin{bmatrix} u(\tau) \\ p_s(\tau) \end{bmatrix} d\tau + \\ + F([u(t+T)\ p_s(t+T)]) \tag{24}$$

$$\begin{aligned} s.t.\ & \begin{bmatrix} \dot{u}(\tau) \\ \dot{p_s}(\tau) \\ \dot{p_f}(\tau) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ (\mathcal{F}B)_s & \mathcal{A}_s \\ (\mathcal{F}B)_f & \mathcal{A}_f \end{bmatrix} \begin{bmatrix} u(\tau) \\ p_s(\tau) \\ p_f(\tau) \end{bmatrix} + \\ & + \begin{bmatrix} 1 \\ -B_s \\ -B_f \end{bmatrix} \tilde{u}(\tau), \\ & u^{min} \leq u(\tau) \leq u^{max},\ \tau \in [t, t+T] \\ & \chi^{min} \leq (Bu(\tau), r(z)) + (p_s(\tau), r(z)) + \\ & \quad +(p_f(\tau), r(z)) \leq \chi^{max} \end{aligned} \tag{25}$$

Note that given any initial condition, for which the above formulation is initially and successively feasible, stabilization and state constraint satisfaction for the infinite–dimensional system are achieved. Stabilization of the infinite–dimensional closed–loop system under the formulation of Eqs.24-25 can be proved using an argument similar to the one used above for the formulation of Eqs.22-23. The implementation of the above controller, however, requires computation of the infinite-dimensional state $p_f(t)$, which can only be done approximately in practice. The key feature of this formulation is that it underscores the fact that even when using a sufficiently high number of modes to simulate the dynamics of the fast modes, the fast modes need not be part of the cost function, thereby keeping the computational requirement low.

The drawback of incorporating the fast states directly in the state constraints equation is that the set of initial conditions for which the optimization problem is feasible becomes infinite–dimensional, and therefore impossible to compute or even estimate. The realization that stability of the slow subsystem is sufficient to ensure stability of the infinite–dimensional system justifies the use of only the slow modes in the cost functional and the stability constraint, thereby substantially reducing the computational requirement.

To further reduce some of the computational load associated with solving the $p_f(t)$-subsystem in the formulation of Eqs.24-25, we now present another MPC formulation that approximates the effect of the fast dynamics by exploiting the two time-scale separation between the slow and fast subsystems and deriving an approximate model that describes the evolution of the fast subsystem. We define $\epsilon := \dfrac{|Re\{\lambda_1\}|}{|Re\{\lambda_{m+1}\}|}$ and multiply the $p_f(t)$-subsystem of Eq.19 by $\epsilon$ to obtain the following system:

$$\begin{aligned} \frac{du}{dt} &= \tilde{u}(t) \\ \frac{dp_s(t)}{dt} &= \mathcal{A}_s p_s(t) + (\mathcal{F}B)_s u(t) - B_s \tilde{u}(t) \\ \epsilon \frac{dp_f(t)}{dt} &= \mathcal{A}_{f\epsilon} p_f(t) + \epsilon(\mathcal{F}B)_f u(t) - B_{\epsilon f} \tilde{u}(t) \end{aligned} \tag{26}$$

where $\mathcal{A}_{f\epsilon}$ is an infinite dimensional bounded differential operator defined as $\mathcal{A}_{f\epsilon} = \epsilon \mathcal{A}_f$ and bounded operator

$B_{f\epsilon} = \epsilon B_f$. Introducing the fast time scale $\hat{\tau} = \frac{t}{\epsilon}$ and setting $\epsilon = 0$, the fast subsystem takes the form:

$$\frac{dp_f(\hat{\tau})}{d\hat{\tau}} = \mathcal{A}_{f\epsilon} p_f(\hat{\tau}) + B_{f\epsilon}\tilde{u}(\hat{\tau}) \quad (27)$$

In the above equation Eq.27, the term $\mathcal{B}_{f\epsilon}\tilde{u}(\tau)$ is kept because we do not impose any constraint on the evolution of $\tilde{u}(\tau)$, so we improve the accuracy of the fast subsystem by keeping this term; we do remove the term $\epsilon(\mathcal{F}B)_f u(t)$ because $u(\tau)$ is bounded because of the constraints of Eq.3. This approximation leads to the following predictive control formulation:

$$\min_{\tilde{u}} \int_t^{t+T} [u(\tau)\ p_s(\tau)]' Q \begin{bmatrix} u(\tau) \\ p_s(\tau) \end{bmatrix} d\tau + \quad (28)$$
$$+F([u(t+T)\ p_s(t+T)])$$

$$s.t. \begin{bmatrix} \dot{u}(\tau) \\ \dot{p_s}(\tau) \\ \dot{p_f}(\tau) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ (\mathcal{F}B)_s & \mathcal{A}_s \\ 0 & \mathcal{A}_f \end{bmatrix} \begin{bmatrix} u(\tau) \\ p_s(\tau) \\ p_f(\tau) \end{bmatrix} +$$
$$+ \begin{bmatrix} 1 \\ -B_s \\ -B_f \end{bmatrix} \tilde{u}(\tau),$$
$$u^{min} \le u(\tau) \le u^{max},\ \tau \in [t, t+T]$$
$$S^{min} \le (Bu(\tau), r(z)) + (p_s(\tau), r(z)) +$$
$$+(p_f(\tau), r(z)) \le S^{max} \quad (29)$$

where $\tau \in [t, t+T]$, and $S^{min} = \chi^{min} + \alpha$ and $S^{max} = \chi^{max} - \alpha$, where $\alpha$ is a small parameter. We denote the set of initial conditions for which the predictive controller of Eqs.28-29 achieves stabilization of the $[u(t)\ p_s(t)\ p_f(t)] = [0\ 0\ 0]$ solution of the corresponding closed–loop infinite–dimensional system (Eq.26 with $\epsilon(\mathcal{F}\mathcal{B}_f u(t) = 0$ under the predictive controller of Eqs.28-29) by $\Omega'$ and assume that $\Omega'$ is non-empty and contains the origin. State constraints satisfaction for the infinite–dimensional system is achieved by revising the state constraints (through the $\alpha$ parameter) in the controller formulation of Eqs.28-29 because of the error (due to neglecting the effect of the term $\epsilon(\mathcal{F}B)_f u(t)$ on the evolution of the fast modes) in the prediction of the fast state dynamics. Theorem 1 below states precise conditions under which the predictive controller of Eqs.28-29 enforces stability and constraint satisfaction in the infinite-dimensional closed-loop system of Eq.26.

**Theorem 1:** *Consider the system of Eq.26 under the predictive control law of Eqs.28-29 with $\alpha > 0$ and pick a compact set $\Omega''$ within $\Omega'$. Then, if the initial condition $[u(0)\ p_s(0)\ p_f(0)] \in \Omega''$, there exists an $\epsilon^*$ such that if $\epsilon \in (0, \epsilon^*]$, the $x(t) = 0$ solution of the infinite-dimensional closed-loop system (Eqs.26-28-29) is asymptotically stable and $\chi^{min} \le (r, x(t)) \le \chi^{max}$ for all $t \ge 0$.*

**Proof of Theorem 1**: Let $\tilde{u}(t) = M(u(t), p_s(t), p_f(t))$ be the implicit control law generated by the predictive controller of Eqs.28-29 for $[u(0)\ p_s(0)\ p_f(0)] \in \Omega''$. Under this control law the system of Eq.26 take the form:

$$\begin{aligned} \frac{du}{dt} &= M(u(t), p_s(t), p_f(t)) \\ \frac{dp_s(t)}{dt} &= \mathcal{A}_s p_s(t) + (\mathcal{F}B)_s u(t) - \\ &\quad -B_s M(u(t), p_s(t), p_f(t)) \\ \epsilon\frac{dp_f(t)}{dt} &= \mathcal{A}_{f\epsilon} p_f(t) + \epsilon(\mathcal{F}B)_f u(t) - \\ &\quad -B_{\epsilon f} M(u(t), p_s(t), p_f(t)) \end{aligned} \quad (30)$$

In the above system, when the term $\epsilon(\mathcal{F}B)_f u(t)$ is not present, the solution $[u(t)\ p_s(t)\ p_f(t)] = [0\ 0\ 0]$ is asymptotically stable and the term $\|\epsilon(\mathcal{F}B)_f u(t)\| \le \epsilon M$ where $M$ is a positive constant (this follows from the boundedness of $(\mathcal{F}B)_f$ and the constraints of Eq.3) and furthermore, $\|\epsilon(\mathcal{F}B)_f u(t)\|$ converges to zero when $[u(t)\ p_s(t)\ p_f(t)]$ converges to zero. Therefore, given an initial condition $[u(0)\ p_s(0)\ p_f(0)] \in \Omega'' \subset \Omega'$ there exists an $\epsilon^{**}$ such that if $\epsilon \in (0, \epsilon^{**}]$ the closed-loop system of Eq.30 is asymptotically stable; this can be shown using a Lyapunov argument similar to the one used for stability of nonlinear finite-dimensional systems subject to sufficiently small and vanishing additive perturbations (see, for example, [16],[7]). Furthermore, since the term $\epsilon(\mathcal{F}B)_f u(t)$ is of order $\epsilon$, given $\alpha > 0$, we can always find an $\epsilon^* \le \epsilon^{**}$ such that if $\epsilon \in (0, \epsilon^*]$, then $\chi^{min} + \alpha \le (Bu(t), r(z)) + (p_s(t), r(z)) + (\tilde{p}_f(t), r(z)) \le \chi^{max} - \alpha$ implies $\chi^{min} \le (Bu(t), r(z)) + (p_s(t), r(z)) + (p_f(t), r(z)) \le \chi^{max}$.

## IV. Numerical example

We consider the boundary-controlled parabolic PDE of the form:

$$\begin{aligned} \frac{\partial \bar{x}(z,t)}{\partial t} &= \bar{b}\frac{\partial^2 \bar{x}(z,t)}{\partial z^2} + \bar{c}\bar{x}(z,t), \quad \bar{x}(z,0) = \bar{x}_0 \\ \frac{d\bar{x}(0,t)}{dz} &= 0 \\ \frac{d\bar{x}(1,t)}{dz} &= u(t) \\ u^{min} &\le u(t) \le u^{max} \\ \chi^{min} &\le \int_0^1 r(z)\bar{x}(z,t)dz \le \chi^{max} \end{aligned} \quad (31)$$

where $\bar{c} = 0.66$ and $\bar{b} = 1.0$, $[u^{min}, u^{max}] = [-18, 18]$ and $\chi^{min} = -0.1$ and $\chi^{max} = 2.5$. The state constraint distribution function is given by the function $r(z) = \frac{1}{2\mu}$ for $z \in [z_c - \mu, z_c + \mu]$ where $z_c = 0.11$ and $\mu = 0.006$ and is zero elsewhere in $z \in [0, 1]$.

We first formulate the PDE of Eq.31 into the infinite-dimensional equation of the form of Eq.11 by formulating the operator $\mathcal{F}$=$\bar{b}\frac{d^2}{dz^2} + \bar{c}$ with

$$\mathcal{D}(\mathcal{F}) = \{\phi(z) \in L_2(0,1) : \phi(z), \frac{d\phi(z)}{dz},$$
$$are\ abs.\ cont., \mathcal{F}\phi(z) \in L_2(0,1)\ and\ \phi'(0) = 0\} \quad (32)$$

and the boundary operator defined by $\mathcal{B}\phi = \frac{d\phi}{dz}(1)$. Furthermore, we select $B(z) = \frac{1}{2}z^2$ which satisfies $\mathcal{B}Bu(t) = u(t)$ and use the transformation $p(t) = x(t) - Bu(t)$ to end up with Eq.18 where the operator $\mathcal{A} = \bar{b}\frac{d^2}{dz^2} + \bar{c}$ with domain

$$\begin{aligned} \mathcal{D}(\mathcal{A}) &= \mathcal{D}(\mathcal{F}) \cap ker(\mathcal{B}) = \\ &= \{\phi(z) \in L_2(0,1) : \phi(z), \phi'(z)\ are\ abs.\ cont., \\ &\quad \mathcal{A}\phi(z) \in L_2(0,1)\ and\ \phi'(0) = 0 = \phi'(1)\} \end{aligned} \quad (33)$$

The eigenspectrum and associated eigenfunctions of the symmetric operator $\mathcal{A}$ are given by:

$$\begin{aligned} \lambda_n &= \bar{c} - \bar{b}n^2\pi^2,\ n \geq 0,\ \ \phi_0 = 1, \\ \phi_n(z) &= \sqrt{2}cos(n\pi z),\ \ n = 1, \ldots, \infty \end{aligned} \quad (34)$$

and $\phi_n(z) = \psi_n(z)$. By applying the formulation of Eq.13, we obtain the following modal representation of the infinite-dimensional equation:

$$\dot{\tilde{a}}(t) = \tilde{A}\tilde{a}(t) + \tilde{B}\tilde{u}(t),\ \ \tilde{a}(0) = [u(0)\ a(0)] \quad (35)$$

where $\tilde{a}(t) = [u(t)\ a_1(t)\ a_2(t) \cdots a_l(t)]$, $\tilde{A} = [0\ 0; \mathcal{F}B\ \ A]$ and $\tilde{B} = [I\ \ -\hat{B}]'$, with $\tilde{u}(t)$ being the time derivative of the control $u(t)$, $\mathcal{F}B = (\bar{b}I + \bar{c}B(z), \phi(z))$ and $\hat{B}_j = (B(z), \phi_j(z))$. In our calculations, Eq.35 is solved by using a finite-dimensional approximation with 20 modes (further increase in the number of equations led to identical numerical results). The model used for controller design is of the form:

$$\begin{aligned} \dot{a}_s(t) &= \tilde{A}_s a_s(t) + \tilde{B}_s\tilde{u}(t) \\ \dot{a}_f(t) &= \tilde{A}_f a_f(t) + \tilde{B}_f\tilde{u}(t) \end{aligned} \quad (36)$$

with $a_s(t) = [u(t)\ \ a_{s1}(t)\ \ a_{s2}(t)]'$ ($m = 3$, the number of modes considered in the cost functional), $a_f(t) = [a_{3f}(t) \cdots a_{15}(t)]'$ and the sampling time used is $\delta = 7.0298 * 10^{-4}$. In the case of using an MPC formulation constructed on the basis of the slow subsystem, the state constraints of Eq.31 are expressed as constraints on modal states as follows:

$$\chi^{min} \leq \left[ -\int_0^1 r(z)B(z)dz \ \ \int_0^1 r(z)\phi_0 dz \ \ \int_0^1 r(z)\phi_1 dz \right] \begin{bmatrix} u(\tau) \\ a_{s1}(\tau) \\ a_{s2}(\tau) \end{bmatrix} \leq \chi^{max} \quad (37)$$

Then, the low-dimensional MPC formulation of Eqs.22-23 takes the form:

$$\min_{\tilde{u}} \ \ \int_t^{t+T} a_s(\tau)'Qa_s(\tau)d\tau + a_s(T)'\overline{Q}a_s(T) \quad (38)$$

$$\begin{aligned} s.t.\ \ \dot{a}_s(\tau) &= \tilde{A}_s a_s(\tau) + \tilde{B}_s\tilde{u}(\tau), \quad \tau \in [t, t+T] \\ u^{min} &\leq u(\tau) \leq u^{max} \\ \chi^{min} &\leq \mathcal{C}_s a_s(\tau) \leq \chi^{max} \end{aligned} \quad (39)$$

where the weight $Q = diag\{R, Q_s\}$ is given as $Q_s = 50diag\{m - 1\}$ and $R = 0.01$, the horizon length $T = 0.2109$ and $\mathcal{C}_s$ is given as $\left[ -\int_0^1 r(z)B(z)dz \ \int_0^1 r(z)\phi_0 dz \ \int_0^1 r(z)\phi_1 dz \right]$. A terminal constraint with respect to the slow modes is used of the form $a_s(T) = 0$ and the initial condition is given as $\bar{x}(z,0) = sin(z)$ in all simulation runs. The resulting quadratic program is solved using the MATLAB subroutine QuadProg. The control action is then implemented on the 20-th order model of Eq.35.

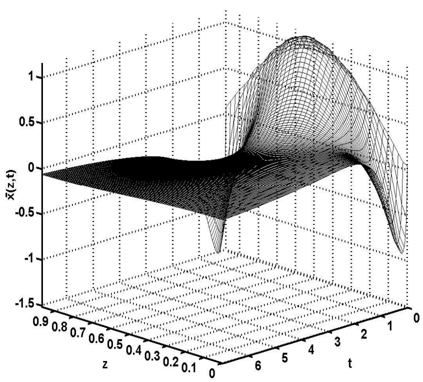


Fig. 1. Closed-loop state profile under the MPC formulation of Eqs.38-39.

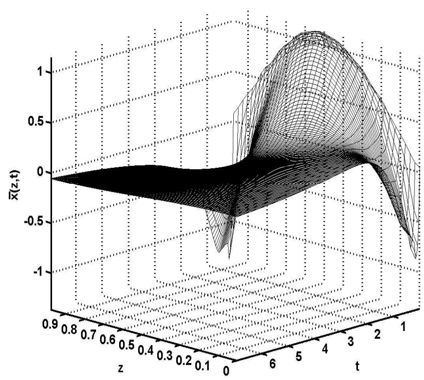


Fig. 2. Closed-loop state profile under the MPC formulation of Eqs.40-41.

Simulation studies demonstrate that the MPC law of Eqs.38-39 stabilizes the PDE state (Fig.1), but it fails to satisfy the state constraint (Fig.3-solid line). In order to appropriately account for the fast states, the MPC formulation given by Eqs.24-25 is considered which takes the form:

$$\min_{\tilde{u}} \int_t^{t+T} a_s(\tau)' Q a_s(\tau) d\tau + a_s(T)' \overline{Q} a_s(T) \quad (40)$$

$$\begin{aligned} s.t.\ \dot{a}_s(\tau) &= \tilde{A}_s a_s(\tau) + \tilde{B}_s \tilde{u}(\tau), \quad \tau \in [t, t+T] \\ \dot{a}_f(\tau) &= \tilde{A}_f a_f(\tau) + \tilde{B}_f \tilde{u}(\tau) \\ u^{min} &\leq u(\tau) \leq u^{max} \\ \chi^{min} + \alpha &\leq \mathcal{C}_s a_s(\tau) + \mathcal{C}_f a_f(\tau) \leq \chi^{max} - \alpha \end{aligned} \quad (41)$$

where $\tilde{\mathcal{A}}_f$ is a matrix of dimensions $(l-m)\times$ $(l-m)$, and $\tilde{B}_f$ is a vector of dimension $(l-m)\times 1$, $\mathcal{C}_{fj} = \int_0^1 r(z)\phi_j(z)dz,\ j = m+1, \cdots, l$, and $\alpha = 0.0151$. Fig.2, Fig.3 (dotted line) and Fig.4 show the closed-loop state, state constraint and manipulated input profiles respectively. It is clear that the MPC formulation attains closed-loop stability and constraint satisfaction for the same initial condition for which the formulation of Eqs.38-39 violates state constraints.

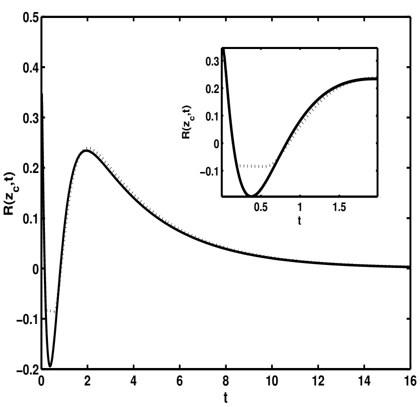


Fig. 3. Closed-loop state constraint $R(z_c,t) = \int_0^1 \delta(z - z_c)\bar{x}(z,t)dz$ at $z_c = 0.11$ under the MPC formulation of Eqs. 38-39 (solid line) and under the MPC formulation of Eqs.40-41(dotted line).

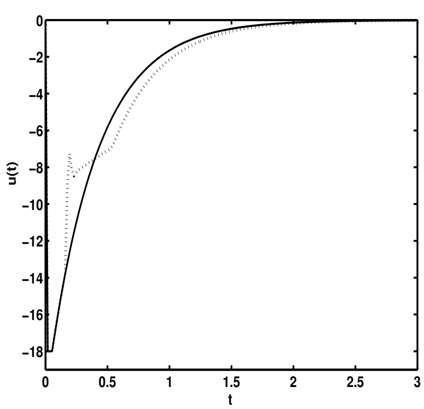


Fig. 4. The manipulated input profile applied at the boundary under the MPC formulations of Eqs.38-39 (solid line) and Eqs.40-41 (dotted line).

In summary, this work focused on boundary model predictive control of linear parabolic partial differential equations (PDEs) with input and state constraints. Various predictive control formulations were discussed and a novel predictive control scheme, designed accounting for the two-time scale behavior of the spectrum of the parabolic PDE, was proposed. Application to a linear parabolic PDE with unstable steady state and flux boundary control demonstrated a successful application of the predictive control algorithm proposed in Theorem 1 in a way that stability and constraint satisfaction were enforced in the infinite-dimensional closed-loop system.

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