Static Anti-windup Controller Synthesis using Simultaneous Convex Design

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***Abstract*— We present a method for the synthesis of a control law which incorporates both a traditional linear output-feedback controller as well as a static anti-windup compensator. Unlike traditional anti-windup controller design in which the linear controller and anti-windup compensator are designed sequentially, our method synthesizes all controller parameters simultaneously, thus providing a priori account of the effects of saturation on the closed-loop dynamics. Moreover, we derive sufficient conditions for the quadratic stability and (possibly) multiple performance bounds on the closed-loop dynamics such that the entire synthesis is cast as an optimization problem over linear matrix inequalities (LMIs).**

I. Introduction

The behavior of linear, time-invariant (LTI) system subject to actuator saturation has been extensively studied over the past several decades. During this time, a neutral division has occurred wherein two general methodologies have emerged for handling input saturation; those methods that account for saturation *a priori* and those which account for saturation *a posteriori*. Among theses techniques, the term anti-windup has been used extensively to describe a large class of input constrained control methodologies.

Originally, anti-windup was used to describe a modified P/PI/PID control law, consisting of the original P/PI/PID control law with an additional anti-windup compensator which modified the the control law only in the event of actuator saturation.[4]. Subsequently, anti-windup control was extended to describe any LTI control law which consisted of an LTI controller and anti-windup compensation[1], [12], [9], [6], [10], [8], [3].However, a majority of these methods remained *a posteriori* technique, following a two step design methodology: *Design first the linear controller ignoring effects of any control input nonlinearity and then add anti-windup compensation to minimize the adverse effects of any control input nonlinearities on closed loop performance.* The main advantage of this design methodology is that no restrictions are placed on the original linear controller design. One can use the wealth of existing linear control theory and design methods to build the linear controller, then subsequently design the anti-windup compensator to minimize any adverse behavior that the linear controller would exhibit during saturation. Moreover, existing linear controller implementation can be retrofitted for saturation using such an anti-windup compensation approach. The main disadvantage is that although the linear controller and anti-windup compensation both affect the closed-loop performance the effect of the linear controller on the performance under the saturation is completely ignored, by definition.

In this paper, we propose a control law with the structure of the traditional anti-windup design: a linear controller and static anti-windup compensation. Unlike the traditional two-step design procedure for anti-windup, we propose a method for the simultaneous synthesis of both the linear controller and static anti-windup compensator. There are several instances where anti-windup retrofits to existing linear controllers involving not only the addition of anti-windup compensation but also retuning/detuning of the original linear controller parameters. Such retuning/detuning of the original linear controller during anti-windup retrofitting is currently carried out using ad-hoc guidelines involving an intuitive understanding of the interactions and trade-offs between linear and constrained closed-loop responses. The proposed method provides a systematic framework for carrying out these trade-offs in a multi-objective settings.

Our method is based on extending the work in [11] to include anti-windup synthesis; an idea originally proposed in [7]. However, this was not possible until an LMI solution was provided for the traditional two-step anti-windup synthesis in [9]. The result of combining the work in [11] with [7] is a multi-objective synthesis with guarantee for stability, unconstrained performance and constrained performance where the overall synthesis is cast as an optimization over LMIs. Thus we provide an efficient and practical method for controller synthesis which provides the ability to design unconstrained and constrained closed-loop response, and their interaction, a *priori*.

II. Problem Statement and Formulation

Consider the multi-input, multi-output (MIMO), LTI plant described by the following state space equations:

$$P\left\{\begin{array}{rcl} \dot{x}_p &=& A_p x_p + B_w w + B_p sat(u) \\ z &=& C_z x_p + D_{zw} w + D_z sat(u) \\ y &=& C_y x_p + D_{yw} w \end{array}\right. \tag{1}$$

where u is the vector of constrained control input(s), w is the vector of exogenous inputs (reference signals, disturbances, noise etc.), y is the vector of plant measurements available to the controller and z is the vector of plant outputs which

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govern the performances of the control system and $sat(\cdot)$ is the standard decentralized saturation function defined as:

$$sat(u) = \begin{bmatrix} sat(u_1) \\ \vdots \\ sat(u_n) \end{bmatrix}$$

$$sat(u_i) = \begin{cases} u_i^{max} & \text{if } u_i > u_i^{max} \\ u_i & \text{if } u_i^{min} \leq u_i \leq u_i^{max} \\ u_i^{min} & \text{if } u_i < u_i^{min} \end{cases}$$

Our goal is the synthesis of a full order, dynamic output-feedback controller with static anti-windup, as shown in Figure (1):

$$K \begin{cases} \dot{x}_k &= A_k x_k + B_k y + \Lambda_1[sat(u) - u] \\ u &= C_k x_k + D_k y + \Lambda_2[sat(u) - u] \end{cases} \quad (2)$$

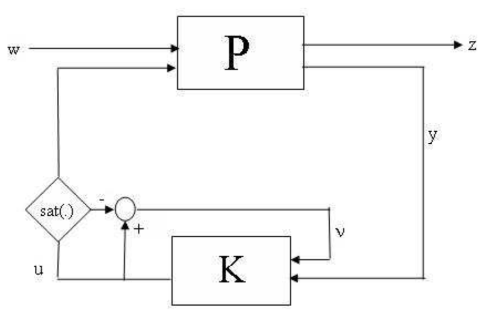

Fig. 1. Standard interconnection for the a nti-windup problem

If we define $\nu = u - sat(u)$, $\xi = -\Lambda\nu = \begin{bmatrix} \Lambda_1 \\ \Lambda_2 \end{bmatrix} \nu$ and $x_{cl} = \begin{bmatrix} x_p \\ x_k \end{bmatrix}$ then we can rewrite P and K as a closed-loop system with the following realization:

$$T \begin{cases} \dot{x}_{cl} &= \mathscr{A} x_{cl} + \mathscr{B}_w w + (\mathscr{B}_\nu - \mathscr{B}_\xi \Lambda)\nu \\ u &= \mathscr{C}_u xcl + \mathscr{D}_{uw} w - \mathscr{D}_{u\xi}\Lambda\nu \\ z &= \mathscr{C}_z x_{cl} + \mathscr{D}_{zw} w + (\mathscr{D}_{z\nu} - \mathscr{D}_{z\xi}\Lambda)\nu \end{cases} \quad (3)$$

where,

$$\mathscr{A} = \begin{bmatrix} A_p + B_p D_k C_y & B_p C_k \\ B_k C_y & A_k \end{bmatrix} \quad \mathscr{B}_w = \begin{bmatrix} B_w + B_p D_k D_{yw} \\ B_k D_{yw} \end{bmatrix}$$

$$\mathscr{B}_\nu = \begin{bmatrix} -B_p \\ 0 \end{bmatrix} \quad \mathscr{B}_\xi = \begin{bmatrix} 0 & B_p \\ I & 0 \end{bmatrix}$$

$$\mathscr{C}_u = \begin{bmatrix} D_k C_y & C_k \end{bmatrix} \quad \mathscr{D}_{uw} = \begin{bmatrix} D_k D_{yw} \end{bmatrix}$$

$$\mathscr{D}_{u\xi} = \begin{bmatrix} 0 & I \end{bmatrix} \quad \mathscr{C}_z = \begin{bmatrix} C_z + D_z D_k C_y & D_z C_k \end{bmatrix}$$

$$\mathscr{D}_{zw} = \begin{bmatrix} D_{zw} + D_z D_k D_{yw} \end{bmatrix} \quad \mathscr{D}_{z\nu} = \begin{bmatrix} -D_z \end{bmatrix}$$

$$\mathscr{D}_{z\xi} = \begin{bmatrix} 0 & D_z \end{bmatrix} \quad \Lambda = \begin{bmatrix} \Lambda_1 \\ \Lambda_2 \end{bmatrix}$$

Also we refer to a particular input-output channel as $T_{ij} = L_i T R_j$ where,

$$T_{ij} \begin{cases} \dot{x}_{cl} &= \mathscr{A} x_{cl} + \mathscr{B}_{w_j} w_j + (\mathscr{B}_\nu - \mathscr{B}_\xi \Lambda)\nu \\ u &= \mathscr{C}_u xcl + \mathscr{D}_{uw_j} w_j - \mathscr{D}_{u\xi}\Lambda\nu \\ z_i &= \mathscr{C}_{z_i} x_{cl} + \mathscr{D}_{z_i w_j} w_j + (\mathscr{D}_{z_i\nu} - \mathscr{D}_{z_i\xi}\Lambda)\nu \end{cases} \quad (4)$$

with,

$$\mathscr{B}_{w_j} = \begin{bmatrix} B_w R_j + B_p D_k D_{yw} R_j \\ B_k D_{yw} R_j \end{bmatrix} \quad \mathscr{D}_{uw_j} = \begin{bmatrix} D_k D_{yw} R_j \end{bmatrix}$$

$$\mathscr{C}_{z_i} = \begin{bmatrix} L_i C_z + L_i D_z D_k C_y & L_i D_z C_k \end{bmatrix}$$

$$\mathscr{D}_{z_i w_j} = \begin{bmatrix} L_i D_{zw} R_j + L_i D_z D_k D_{yw} R_j \end{bmatrix}$$

$$\mathscr{D}_{z_i\nu} = \begin{bmatrix} -L_i D_z \end{bmatrix} \quad \mathscr{D}_{z_i\xi} = \begin{bmatrix} 0 & L_i D_z \end{bmatrix}$$

L_i and $R_j$are block diagonal matrices which isolate the desired input and output channels required for each performance objective. Thus our goal is to formulate LMIs which guarantee that the closed-loop system T is aymptotically stable as well as formulate additional LMIs which provide a variety of performance guarantees for individual input-output channels, T_{ij}.

III. Synthesis of Stabilizing K(s)

A. Synthesis of Globally Stabilizing K for Unconstrained Systems

For the case when the closed loop system (3) is unconstrained ($\nu = 0$), we can use the result from [11] to synthesize A_k, B_k, C_k and D_k such that T($\nu = 0$) is asymptotically stable. We state the following useful result from [11].

Theorem 1 (Stabilizing Unconstrained, Output-feedback Control[11]). *Given the linear plant in (1), there exists a controller, K, of the form in (2) which asymptotically stabilizes P with $\nu = 0$ if and only if there exists the matrices $\hat{A}$, $\hat{B}$, $\hat{C}$, $\hat{D}$,P_{11} and Q_{11} such that the following inequalities are satisfied:*

$$\begin{bmatrix} A_p\mathbf{Q_{11}} + \mathbf{Q_{11}}A_p^T + B_p\hat{\mathbf{C}} + \hat{\mathbf{C}}^T B_P^T & * \\ \hat{\mathbf{A}} + C_y^T \hat{\mathbf{D}}^T B_p^T B_p^T + A_p^T & \mathbf{P_{11}}A_p + A_p^T\mathbf{P_{11}} + \hat{\mathbf{B}}C_y + C_y^T\hat{\mathbf{B}}^T \end{bmatrix} < 0 \quad (5)$$

$$\begin{bmatrix} \mathbf{Q_{11}} & * \\ I & \mathbf{P_{11}} \end{bmatrix} > 0$$

Given such a solution, the parameters A_k, B_k, C_k and D_k can be calculated from the following sequential set of algebraic equations

$$Q_{12}P_{12}^T = I - \mathbf{Q_{11}}\mathbf{P_{11}} \quad (6a)$$

$$\hat{\mathbf{D}} = D_k \quad (6b)$$

$$\hat{\mathbf{C}} = C_k Q_{12}^T + D_k C_y Q_{11} \quad (6c)$$

$$\hat{\mathbf{B}} \ = \ P_{12}B_k + \mathbf{P_{11}}B_pD_k \tag{6d}$$

$$\begin{aligned}\hat{\mathbf{A}} \ = \ & P_{12}A_kQ_{12}^T + P_{12}B_kC_y\mathbf{Q_{11}} + \\ & P_{11}B_pC_kQ_{12}^T + \mathbf{P_{11}}(A_p + B_pD_kC_y)\mathbf{Q_{11}}\end{aligned} \tag{6e}$$

Proof: Let us require $T(\nu = 0)$ admits a quadratic Lyapunov function of the form:

$$V = x_{xl}^T P x_{cl} \tag{7}$$

Requiring that $V(x) > 0$ and $\dot{V}(x) < 0$ for all $t > 0$ guarantees that $T(\nu = 0)$ is asymptotically stable. Next, it is shown in [2] that such conditions are satisfied if and only if there exists a solution to the following set of matrix inequalities:

$$\mathscr{A}^T P + P\mathscr{A} < 0 \qquad P > 0 \tag{8}$$

Now, partition P such that:

$$P \ = \ \begin{bmatrix} \mathbf{P_{11}} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix} \tag{9}$$

and partition its inverse, $Q = P^{-1}$ as:

$$Q \ = \ \begin{bmatrix} \mathbf{Q_{11}} & Q_{12} \\ Q_{12}^T & Q_{22} \end{bmatrix} \tag{10}$$

Note that since $PQ = I$, then:

$$A \begin{bmatrix} \mathbf{Q_{11}} \\ Q_{12}^T \end{bmatrix} \ = \ \begin{bmatrix} I \\ 0 \end{bmatrix} \tag{11}$$

Define:

$$\Pi_1 \ = \ \begin{bmatrix} \mathbf{Q_{11}} & I \\ Q_{12}^T & 0 \end{bmatrix} \ \Pi_2 \ = \ \begin{bmatrix} I & \mathbf{P_{11}} \\ 0 & P_{12}^T \end{bmatrix} \tag{12}$$

and thus $P\Pi_1 = \Pi_2$. Finally, apply the congruence transformation, (Π_1^T), to both matrix inequalities in (**??**) to get the final result in (5), utilizing the change of variables in (6).
Note that we have used boldface letters to emphasize the decision variables in (5). Given a solution to (5), it was shown in [11] that one can always calculate Q_{12} and P_{12} from $Q_{12}P_{12}^T = I - \mathbf{Q_{11}}\mathbf{P_{11}}$ for full order controller synthesis and also Q_{12} and P_{12} are invertible. Thus, one can always reconstruct P, Q as well as A_k, B_k, C_k, D_k. Finally recall that it is well established in linear control theory that a solution is guaranteed to exist for (5) if and only if (A_P, B_p) is stablilizable and (C_y, A_p) is detectable.

B. Synthesis of Globally Stabilizing K for Constrained Systems

For the case of global asymptotic stability of the closed-loop system (when $\nu \neq 0$), we choose to construct a Lyapunov function of the same form $V = x_{cl}^T P x_{cl}$ but now we will require that $V(x) > 0$ and $\dot{V}(x) + [u^T W \nu + \nu^T W u - 2\nu^T W \nu] < 0$ for all $t > 0$ where W is a positive definite diagonal matrix. It was shown in [9] that solving such a Lyapunov problem would guarantee global asymptotic stability of the closed loop system in (3). Moreover, it was shown that a solution exists for such a Lyapunov problem if and only if there exists a solution to the following set of matrix inequalities:

$$\begin{bmatrix} \mathscr{A}^T P + P\mathscr{A} & * \\ \mathscr{B}_\nu^T P - \Lambda^T \mathscr{B}_\xi^T P + W\mathscr{C}_u & -2W - WD_{u\xi}\lambda - \lambda^T \mathscr{D}_{u\xi}^T W \end{bmatrix} < 0 \tag{13}$$

$$P > 0, \ \ W > 0$$

In [9], the authors go on to provide a LMI solution to synthesize Λ for fixed A_k, B_k, C_k and D_k. Moreover, it was shown that this condition can also be derived by application of the multi-loop circle criterion. Here, we wish to provide a linear solution where we simultaneously synthesize A_k, B_k, C_K and D_k and Λ. We state the following theorem here,

Theorem 2 (Stabilizing Constrained, Output-feedback Control). *Given the linear plant P in (1), there exists a controller, K, of the form in 2 which asymptotically stabilizes P with $\nu \neq 0$ if there exist the matrices $\hat{\mathbf{A}}$, $\hat{\mathbf{B}}$, $\hat{\mathbf{C}}$, $\hat{\mathbf{D}}$, $\hat{\Lambda}_1$, $\hat{\Lambda}_2$, $\hat{\mathbf{P_{11}}} and \hat{\mathbf{Q_{11}}}$ such that the following inequalities are satisfied:*

$$\begin{bmatrix} \begin{matrix} A_p\mathbf{Q_{11}} + \mathbf{Q_{11}}A_p^T + \\ B_p\hat{\mathbf{C}} + \hat{\mathbf{C}}^T B_P^T \end{matrix} & * & * \\ \hat{\mathbf{A}} + C_y^T \hat{\mathbf{D}}^T B_p^T + A_p^T & \begin{matrix} \mathbf{P_{11}}A_p + A_p^T \mathbf{P_{11}} + \\ \hat{\mathbf{B}}C_y + C_y^T \hat{\mathbf{B}}^T \end{matrix} & * \\ -\hat{\Lambda}_2^T B_p^T + \hat{\mathbf{C}} & -\hat{\Lambda}_1^T + \hat{\mathbf{D}}C_y & -\hat{\Lambda}_2 - \hat{\Lambda}_2^T \end{bmatrix} < 0 \tag{14}$$

$$\begin{bmatrix} \mathbf{Q_{11}} & * \\ I & \mathbf{P_{11}} \end{bmatrix} > 0$$

Given such a solution, the parameters A_k, B_k, C_k and D_k are calculated from (6) while Λ_1 and Λ_2 satisfy

$$\hat{\Lambda}_1 = P_{12}\Lambda_1 M + \mathbf{P_{11}}B_p(I + \Lambda_2)M \tag{15a}$$

$$\hat{\Lambda}_2 = (I + \Lambda_2)M \tag{15b}$$

Proof: Simply apply the congruence transformation $diag(\Pi_1^T, M)$, where $M = W^{-1}$ to (13) to get (14).

As in [11], given a solution to (14), one can always reconstruct P, Q and thus $A_k, B_k, C_k, D_k, \Lambda_1$ and Λ_2. Next, we would like to determine the conditions on the plant P, such that a solution exists to (14).

Theorem 3 (Stabilizing Constrained, Output-feedback Control). *Given the linear plant P in (1), a necessary and sufficient condition for the existance of a solution to Theorem (2) is that all the eigenvalues of A_P have negative real part.*

Proof: Apply the congruence transformation, $diag(Q,M)$ to (13) to get

$$\begin{bmatrix} Q\mathscr{A}^T + \mathscr{A}Q & * \\ M\mathscr{B}_\nu^T + \mathscr{C}_u Q & -2M \end{bmatrix} + \begin{bmatrix} -\mathscr{B}_\xi \\ \mathscr{D}_{u\xi} \end{bmatrix} X \begin{bmatrix} 0 & I \end{bmatrix}$$

$$+\left[\begin{array}{c} 0 \\ I \end{array}\right] X^T \left[\begin{array}{cc} -\mathscr{B}_\xi^T & -\mathscr{D}_{u\xi}^T \end{array}\right] < 0 \qquad (16)$$

$$Q > 0, \quad M > 0$$

wehre we have introduced the new variable, $X = \Lambda M$. Now, according to [2], we can eliminate the variable X by replacing (16) with the equivalent inequalities:

$$\psi^T \left[\begin{array}{cc} Q\mathscr{A}^T + \mathscr{A}Q & * \\ M\mathscr{B}_v^T + \mathscr{C}_u Q & -2M \end{array}\right] \psi < 0$$

$$\zeta^T \left[\begin{array}{cc} Q\mathscr{A}^T + \mathscr{A}Q & * \\ M\mathscr{B}_v^T + \mathscr{C}_u Q & -2M \end{array}\right] \zeta < 0 \qquad (17)$$

$$Q > 0, \quad M > 0$$

where $\psi = \left[\begin{array}{c} \psi_1 \\ \psi_2 \end{array}\right]$, $\left[\begin{array}{cc} 0 & I \end{array}\right] \psi = 0$ and $\left[\begin{array}{cc} -\mathscr{B}_{xi}^T & -\mathscr{D}_{u\xi}^T \end{array}\right] \zeta = 0$. After some algebra, one can reduce (17) to:

$$\begin{aligned} Q\mathscr{A}^T + \mathscr{A}Q &< 0 \\ Q_{11}A_p^T + A_p Q_{11} &< 0 \end{aligned} \qquad (18)$$

Utilizing Theorem 1(5) and applying Schur complement to the sedond inequality, we can further reduce (18) to:

$$\left[\begin{array}{cc} A_p\mathbf{Q_{11}} + \mathbf{Q_{11}}A_p^T + B_p\hat{\mathbf{C}} + \hat{\mathbf{C}}^T B_p^T & * \\ \hat{\mathbf{A}} + C_y^T \hat{\mathbf{D}}^T B_p^T + A_p^T & \mathbf{P_{11}}A_p + A_p^T\mathbf{P_{11}} + \hat{\mathbf{B}}C_y + C_y^T\hat{\mathbf{B}}^T \end{array}\right] < 0 \qquad (19)$$

$$\mathbf{Q_{11}} - \mathbf{P_{11}}^{-1} > 0, \; ; \; mathbfP_{11} > 0$$

$$\mathbf{Q_{11}}A_p^T + A_p\mathbf{Q_{11}} < 0$$

Now, if all eigenvalues of A_p have negative real part, it is well known (see [2] for example) that for any $Z > 0$, $A_p\mathbf{Q_{11}} + \mathbf{Q_{11}}A_p^T = -Z$ has unique solution with $\mathbf{Q_{11}} > 0$. Now, define $\tilde{Q}_{11}$ such that $A_p\tilde{Q}_{11} + \tilde{Q}_{11}A_P^T = -I$ and $\tilde{P}_{11}$ such that $\tilde{P}_{11}A_p + A_p^T\tilde{P}_{11} = -I$. Then for any scalar $\alpha > 0$ and scalar $\beta > 0$, choose $\mathbf{Q_{11}} = \alpha\tilde{Q}_{11}$ such that $A_p\mathbf{Q_{11}} + \mathbf{Q_{11}}A_p^T = -\alpha I$ and $\mathbf{P_{11}} = \beta\tilde{P}_{11}$ such that $\mathbf{P_{11}}A_p + A_p^T\mathbf{P_{11}} = -\beta I$. Without loss of generality we can rewrite (19) as:

$$\left[\begin{array}{cc} -\alpha I + B_p\hat{\mathbf{C}} + \hat{\mathbf{C}}^T B_p^T & * \\ \hat{\mathbf{A}} + C_y^T\hat{\mathbf{D}}B_p^T + A_p^T & -\beta I + \hat{\mathbf{B}}C_y + C_y^T\hat{\mathbf{B}}^T \end{array}\right] < 0 \quad (20)$$

$$\begin{aligned} \alpha\tilde{Q}_{11} - \frac{1}{\beta}\tilde{P}_{11}^{-1} &> 0 \\ A_p\tilde{Q}_{11} + \tilde{Q}_{11}A_p^T &= -I \\ \tilde{P}_{11}A_p + A_p^T\tilde{P}_{11} &= -I \end{aligned} \qquad (21)$$

Thus (20) can always be solved for appropriately large α and β.

Remark 1: *As shown in [5], the conditions in (18) are not always feasible for the two-step static anti-windup synthesis. However, (18) is always feasible for the one-step static anti-windup synthesis.*

Remark 2: *It is shown from (6) that (20) admits the trivial solution $u = 0$, i.e. $A_k = 0, B_k = 0, C_k = 0, D_k = 0$ for strictly stable plants, i.e. A_p with all eigen values with negative real parts. However, it is true that infinite other nontrivial solutions exist provided the plant is strictly stable. Experience also demonstrates the existance of nontrivial solution.*

IV. Performance Objective and Multi-Objective Synthesis

In designing an anti-windup control system, the designer must choose which control channel to optimize and moreover, in what sense to optimize the chosen channel. Some authors have proposed directly minimizing the gain from reference and disturbance signal to the output error or state error while other authors have suggested minimizing the gain from the controller output error to the output or state error. In our framework, all forms can be handled directly by appropriately defining w and z. Moreover, when deciding in what sense to minimize a particular input-output channel, several choices have been suggested, although the induced $\mathscr{L}_2$ gain is a popular choice. In [11], the authors present an extensive catalog of performance objectives for a linear control design (no saturation). Here, we wish to present in detail two possible performance objectives, the induced $\mathscr{L}_2$ gain and the peak-to-peak gain, for the saturating closed-loop that can be used in an over-all possibly multi-objective synthesis.

A. Minimizing the induced $\mathscr{L}_2$ gain

We begin by defining the induced $\mathscr{L}_2$ gain from the exogenous input w_j to the desired output, z_i as :

$$\sup_{\|w_j\|_2 \neq 0} \frac{\|z_i\|_2}{\|w_j\|_2} \qquad (22)$$

Now, following [2], define the Lyapunov function (7) requiring:

$$V > 0 \qquad (23)$$
$$\dot{V} + z_i^T z_i - \gamma^2 w_j^T w_j + [u^T W v + v^T W u - 2v^T W v] < 0$$

Integration of the second inequality in (23) from $t = 0$ to $t = T$ with $x_{cl}(0) = 0$ reveals that:

$$V(x(T)) + \int_0^T (z_i^T z_i - \gamma^2 w_j^T w_j) d\tau \leq 0 \qquad (24)$$

Since $V(x(T)) > 0$, this implies that:

$$\frac{\|z_i\|_2}{\|w_j\|_2} \leq \gamma$$

Thus existence of a Lyapunov function which satisfies (23) guarantees asymptotic stability and that the induced $\mathscr{L}_2$ gain form the exogenous input w_j to the desired output z_i is always less than γ. We now give sufficient conditions for the existence of a controller K which guarantees an upper bound on the induced $\mathscr{L}_2$ gain from the exogenous input w_j to the desired output z_i.

Theorem 4 (Induced $\mathscr{L}_2$ gain synthesis for Constrained, Output-feedback Control). *Given the linear plant P in (1), there exists a controller K of the form in (2) which asymptotically stabilizes P with $\nu \neq 0$ and has an induced $\mathscr{L}_2$ gain from w_j to z_i less than γ if there exists a scaler $\gamma > 0$ and matrices $\hat{\mathbf{A}}$, $\hat{\mathbf{B}}$, $\hat{\mathbf{C}}$, $\hat{\mathbf{D}}$, $\hat{\Lambda}_1$, $\hat{\Lambda}_2$, $\mathbf{P_{11}}$ and $\mathbf{Q_{11}}$ such that the following inequality holds:*

$$\begin{bmatrix} A_p\mathbf{Q_{11}} + \mathbf{Q_{11}}A_p^T + B_p\hat{\mathbf{C}} + \hat{\mathbf{C}}^T B_P^T & * & * & * & * \\ \hat{\mathbf{A}} + C_y^T \hat{\mathbf{D}}^T B_p^T + A_p^T & \mathbf{P_{11}}A_p + A_p^T\mathbf{P_{11}} + \hat{\mathbf{B}}C_y + C_y^T\hat{\mathbf{B}}^T & * & * & * \\ R_j^T B_w^T + R_j^T D_{yw}^T \hat{\mathbf{D}}^T B_p^T & R_j^T B_w^T \mathbf{P_{11}} + R_j^T D_{yw}^T \hat{\mathbf{B}}^T & -\gamma^2 I & * & * \\ L_i C_z \mathbf{Q11} + L_i D_z \hat{\mathbf{C}} & L_i C_z + L_i D_z \hat{\mathbf{D}} C_y & L_i D_{zw} R_j + L_i D_z \hat{\mathbf{D}} D_{yw} R_j & -I & * \\ -\hat{\Lambda}_2^T B_p^T + \hat{\mathbf{C}} & -\hat{\Lambda}_1^T + \hat{\mathbf{D}} C_y & \hat{\mathbf{D}} D_{yw} R_j & -\hat{\Lambda}_2^T D_z^T L_i^T & -\hat{\Lambda}_2 - \hat{\Lambda}_2^T \end{bmatrix} < 0 \tag{25}$$

$$\begin{bmatrix} \mathbf{Q_{11}} & * \\ I & \mathbf{P_{11}} \end{bmatrix} > 0$$

A well defined upper bound on the induced $\mathscr{L}_2$ gain is obtained by finding the minimum feasible γ subject to (25) which is a standard LMI eigenvalue optimization problem [2].

Proof: It is easily shown that (23) is equivalent to the following matrix inequaltiy problem:

$$\begin{bmatrix} \mathscr{A}^T P + P\mathscr{A} & * & * & * \\ \mathscr{B}_{w_j}^T P & -\gamma^2 I & * & * \\ \mathscr{B}_\nu^T P - \Lambda^T \mathscr{B}_\xi^T + W\mathscr{C}_u^T & W\mathscr{D}_{uw_j} & -2W - W\mathscr{D}_{u\xi}\Lambda - \Lambda^T\mathscr{D}_{u\xi}^T W & * \\ \mathscr{C}_{z_i} & \mathscr{D}_{z_i w_j} & \mathscr{D}_{z_i\nu} - \mathscr{D}_{z_i\xi}\Lambda & -I \end{bmatrix} < 0 \tag{26}$$

$$W > 0, \; P > 0$$

Now, apply a congruence transformation, $diag(I, I, \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix})$, followed by the congruence transformation, $diag(\Pi_1^T, I, I, M)$ to get the result in (25).

B. Minimizing the Peak-to-Peak Gain

We begin by defining the peak-to-peak gain from the exogenous input w_j to the desired output z_i as:

$$\sup_{|w_j(t)| \leq w_{j\,max}} |z_i(T)|, T \geq t \geq 0, x_{cl}(0) = 0 \tag{27}$$

Now, following [2] and [11], define the usual Lyapunov function and require that:

$$\begin{aligned} V &> 0 \\ \lambda &> 0 \\ \dot{V} + \lambda V - \mu w_j^T w_j + [u^T W \nu + \nu^T W u - 2\nu^T W \nu] &< 0 \\ \lambda V + (\rho - \mu) w_j^T w_j - \rho^{-1} z_i^T z_i - [u^T W \nu + \nu^T W u & \\ -2\nu^T W \nu] &> 0 \end{aligned} \tag{28}$$

where $\rho > 0$, $\rho - \mu > 0$. Since $[u^T W \nu + \nu^T W u - 2\nu^T W \nu] \geq 0$ and $|w_j(t)| \leq w_{jmax}$, the third inequality above (28) guarantees that $\dot{V} < 0$ whenever $\lambda V - \mu w_j^T w_j > 0$. Thus, for $V(x_{cl}(0) = 0)$, $V(X_{cl}(t)$ can never exceed the value $(\mu/\lambda) w_j^T w_j$.
Now, since $\rho > 0$ and $(\rho - \mu) > 0$, the fourth inequality above (28) guarantees that:

$$z_i^T z_i \leq \rho(\lambda V + (\rho - \mu) w_{jmax}^2 - [u^T W \nu + \nu^T W u - 2\nu^T W \nu]$$

Recalling that $V(x_{cl}(t))$ can never exceed the value $(\mu/\lambda) w_j^T w_j$, we get:

$$z_i^T z_i \leq \rho^2 w_{jmax}^2$$

Thus, we can use the Lyapunov problem in (28) to derive an upper bound on z_i or on the gain form bounded w_j to the output z_i. We now give the sufficient conditions for the existence of a controller K which guarantees an upper bound on the peak-to-peak gain from exogenous input w_j to the desired output z_i for the system:

Theorem 5 (Peak-to-Peak Gain Synthesis for Constrained, Output-feedback Control). *Given the linear plant P in (1), there exists a controller K, of the form in (2) which asymptotically stabilizes P with $\nu \neq 0$ and has a peak-to-peak gain from w_j to z_i less than ρ if there exists a scalar $\rho > 0$ and matrices $\hat{\mathbf{A}}$, $\hat{\mathbf{B}}$, $\hat{\mathbf{C}}$, $\hat{\mathbf{D}}$, $\hat{\Lambda}_1$, $\hat{\Lambda}_2$, $\mathbf{P_{11}}$, $\mathbf{Q_{11}}$ λ and μ such that the following inequalities are satisfied:*

$$\begin{bmatrix} A_p\mathbf{Q_{11}} + \mathbf{Q_{11}}A_p^T + B_p\hat{\mathbf{C}} + \hat{\mathbf{C}}^T + \lambda\mathbf{Q11} & * & * & * \\ \hat{\mathbf{A}} + C_y^T\hat{\mathbf{D}}^T B_p^T B_p^T + A_p^T + \lambda I & \mathbf{P_{11}}A_p + A_p^T\mathbf{P_{11}} + \hat{\mathbf{B}}C_y + C_y^T\hat{\mathbf{B}}^T\lambda\mathbf{P_{11}} & * & * \\ R_j^T B_w^T + R_j^T D_{yw}^T\hat{\mathbf{D}}^T B_p^T & R_j^T B_w^T\mathbf{P_{11}} + R_j^T D_{yw}^T\hat{\mathbf{B}}^T & -\mu I & * \\ -\hat{\Lambda}_2^T B_p^T + \hat{\mathbf{C}} & -\hat{\Lambda}_1^T + \hat{\mathbf{D}}C_y & \hat{\mathbf{D}}D_{yw}R_j & -\hat{\Lambda}_2 - \hat{\Lambda}_2^T \end{bmatrix} < 0 \tag{29}$$

$$\begin{bmatrix} \lambda\mathbf{Q}_{11} & * \\ \lambda I & \lambda\mathbf{P_{11}} \\ 0 & 0 \\ L_i C_z\mathbf{Q_{11}} + L_i D_z\hat{\mathbf{C}} & L_i C_z\mathbf{Q_{11}} + L_i D_z\hat{\mathbf{D}}C_y \\ \hat{\Lambda}_2^T B_p^T - \hat{\mathbf{C}} & \hat{\Lambda}_1 - \hat{\mathbf{D}}C_y \end{bmatrix}$$

$$\begin{array}{ccc} * & * & * \\ * & * & * \\ (\rho-\mu)I & * & * \\ L_iD_{wz}R_j + L_iD_{zw}\hat{\mathbf{D}}D_{yw}R_j & \rho I & * \\ -\hat{\mathbf{D}}D_{yw}R_j & \hat{\Lambda}_2^T D_z^T L_i^T & -\hat{\Lambda}_2 - \hat{\Lambda}_2^T \end{array} \Bigg] > 0 \tag{30}$$

$$\begin{bmatrix} \mathbf{Q_{11}} & * \\ I & \mathbf{P_{11}} \end{bmatrix} > 0 \ , \lambda > 0 \tag{31}$$

A well defined bound on the peak-to-peak gain from w_j to z_i is obtained by finding the minimum feasible ρ subject to (29), (30) and (31) which is a standard LMI generalized eigenvalue optimization problem [2].

Proof: It is easily shown that (28) is equivalent to the following matrix inequality problem:

$$\begin{bmatrix} \mathscr{A}^T P + P\mathscr{A} + \lambda P & * & * \\ \mathscr{B}_{w_j}^T P & -\mu I & * \\ \mathscr{B}_{\nu}^T P - \Lambda^T \mathscr{B}_{\xi}^T P + W\mathscr{C}_u^T & W\mathscr{D}_{uw_j} & -2W - W\mathscr{D}_{u\xi}\Lambda - \Lambda^T\mathscr{D}_{u\xi}^T W \end{bmatrix} < 0$$

$$\begin{bmatrix} \lambda P & * & * & * \\ 0 & (\rho-\mu)I & * & * \\ -W\mathscr{C}_u & -W\mathscr{D}_{uw_j} & 2W + W\mathscr{D}_{u\xi}\Lambda + \Lambda^T\mathscr{D}_{u\xi}^T W & * \\ \mathscr{C}_{z_i} & \mathscr{D}_{z_iw_j} & \mathscr{D}_{z_i\nu} - \mathscr{D}_{z_i\xi}\Lambda & -\rho I \end{bmatrix} < 0 \tag{32}$$

$$W > 0, \ P > 0, \ \rho > 0$$

Now, apply a congruence transformation, $diag(I, I, \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix})$ to the second inequality followed by the congruence transformation, $diag(\Pi_1^T, I, M)$ and $diag(\Pi_1^T, I, I, M)$ to get the result in (29),(30),(31).

C. Multi-objective synthesis

The primary contribution of this work is the simultaneous synthesis of all controller parameters in K for the ant-windup control systems such that all degrees of freedom from the controller parameters (linear controller parameters and anti-windup compensator parameters) are available to design and explore the trade-offs between an unconstrained and constrained performance objective. However, it is also possible to simultaneously synthesize all parameters in K such that a set of N performance objectives specified by the designer is met/optimized. For any individual performance objective problem, an optimal solution results in a particular set of controller parameters ($A_k, B_k, C_k, D_k, \Lambda_1$ and Λ_2) and a Lyapunov variable, P. For the multi-objective synthesis, as in [11], we require that a single set of variables ($A_k, B_k, C_k, D_k, \Lambda_1, \Lambda_2$ and P) satisfy the combined set of N matrix inequality problems in order to make the multi-objective synthesis problem convex (an LMI). This is natural in the case of the controller parameters. However, requiring a single Lyapunov function to satisfy all objectives (whether unconstrained or constraiend) indroduces conservatism. This is the price we pay for convexity.

For multi-objective unconstained output-feedback controller synthesis, the effect of the conservatism is well understood and methods have been developed to reduce the conservatism. In the case of multi-objective anti-windup output feedback controller synthesis, we encounter a different problem. In our method, we use a stabilty theorem which can be shown to be equivalent to the multi-loop circle criteriion for guaranteeing global asymptotic stabilty [9]. Requiring a single Lyapunov function for both the constrained and unconstrained objectives can lead to more conservative performance bounds from the one-step synthesis presented here when compared to the equivalent two-step synthesis. However, in many case the two-step synthesis can lead to more conservatove results when compared to one-step synthesis, or eve worse. At times the two-step synthesis also fails to provide a solution in some cases while the one-step syntehsis is guaranteed to find a stabilizing solution for all open-loop stable plants However, both the two-step and one-step synthesis are available and computable as LMIs. For those few cases where the two-step synthesis outperforms one-step synthesis presented here, the simultaneous synteheis can be modified such that each performance objective has a unique Lyapunov variable. Although this modification results in a optimaization over bilinear matrix inequalties (BMIs), one can use the solutions from the LMI version of the one-step and two-step synthesis as initial points and easily compute tighter performance bounds form a local BMI optimazation.

V. CONCLUSION

We have presented a framework for the synthesis of a constrained linear control law which incorporates both traditional linear output-feedback controller and static anti-windup compensator. However, our work differs from the traditional anti-windup synthesis where the output-feedback controller is synthesized first, followed by the anti-windup compensator. In our method, all controller parameters are synthesized simultaneously. Thus, our simultaneous approach provides:

- A systematic framework for the synthesis of a control law where the effect of all controller parameters can be utilized to design the unconstrained and constrained closed-loop performance.
- A framework for the synthesis of a control law which provides, a *priori*, bounds on multiple unconstrained/constrained performance objectives and allows insight into the trade-off between each objectives.
- A method for bounding constrained performance with a choice of several objectives. Previously reported LMI anti-windup synthesis techniques are limited to only $\mathscr{L}_2$ gain (or closely related) performance objectives.

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