Oxygen Depleted Region by the Theory of Krogh in Cartesian Coordinates

Dr. Kal Renganathan Sharma PE
Formerly Principal, Sakthi Engineering College
Anna University Affiliated, Ramapuram (opp. MGR Gardens)\
Chennai – 89, India
Email: jyoi_kalpika@yahoo.com

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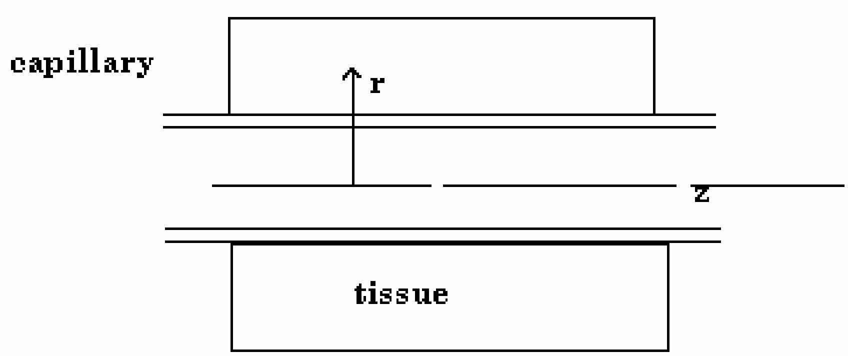

Figure 1.0 Geometry of Krogh Tissue Cylinder

A microscopic view of capillaries in tissue indicates a repetitive arrangement of capillaries surrounded by a cylindrical layer of tissue. A idealized sketch of the capillary bed and the corresponding layer of tissue idealized into a cylinder is shown in Figure 1.0. Let the radius of the tissue layer be r_T. The residence time of the blood in the capillary is in the order of 1 second. The wave diffusion and relaxation time is comparable in magnitude to the residence time in the blood. Krogh [1919] showed this cylindrical capilary tissue model to study the supply of oxygen to muscle. The tissue space surrounding the capillary is considered a continuous phase albeit it consists of discrete cells. An effective diffusivity D_T can be used to represent the diffusion process in the tissue. The driving force for the diffusion is driven by the consumption of the solute by the cells within the tissue space.

The Michaelis-Menten equation can be used to describe the metabolic consumption of the solute in the tissue space. The equation may be written as;

$$R = V_m C_T /(K_m + C_T) \tag{1}$$

Where C_T is the concentration of the solute in the tissue space. For consumption of the solute R will have a positive value and for solute production it will have a negative value. V_m represents the maximum reaction rate. The maximum reaction rate occurs when $<C> >> K_m$. The reaction rate is then in zero order in solute concentration. The blood flows through the capillary with an average velocity of V. A steady state shell balance on the solute in the blood from z to z + Δz can be written as;

$$-V\, dC/dz = 2/r_c \; K_0 \; (C - C_T|_{rc + tm}) \tag{2}$$

where K_o is represented by a overall mass transfer coefficient. The overall mass transfer coefficient represents combined resistance of fluid flowing through the capillary k_m and the permeability of the solute in the capillary wall P_m. A steady state shell balance at a given value of z from r to r + Δr may also be written for the solute concentration in the tissue space;

$$D_T/r \; d/dr\, (r\, dC_T/dr) - R = 0 \tag{3}$$

The boundary conditions for Eqs. [2-3], are;

$$z = 0, \; C = C_o \tag{4}$$

$$r = r_c + t_m \,, \; C_T = C_T|_{rc + tm} \tag{5}$$

$$r = r_T, \qquad dC_T/dr = 0 \tag{6}$$

The axial diffusion is neglected in the tissue space in comparison with the radial diffusion. From the zero order rate of reaction $R = R_o$ a constant. Solving for Eq. [3] And the boundary conditions given in Eqs. [4-6],

$$C_T - C_T|_{rc + tm} = (r^2 - (r_c + t_m)^2)R_o/4D_T - r_T^{\,2}R_o/2D_T \ln(r/(r_c + t_m)) \tag{7}$$

The variation of concentration as a function of z can be calculated by equating the change in solute concentration within the blood to the consumption of solute in the tissue space;

$$C = C_o - R_o/Vr_c^2 (r_T^2 - (r_c + t_m)^2)z \qquad (8)$$

Eq. [7] is combined with Eq. [8];

$$C_T|_{rc+tm} - C_o = - R_o/Vr_c^2 (r_T^2 - (r_c + t_m)^2)z - R_o/2r_cK_o (r_T^2 - (r_c + t_m)^2) \qquad (9)$$

Combining Eqs.[7-9],

$$C_T - C_o = (r^2 - (r_c + t_m)^2)R_o/4D_T - r_T^2R_o/2D_T \ln(r/(r_c + t_m)) - R_o/Vr_c^2 (r_T^2 - (r_c + t_m)^2)z - R_o/2r_cK_o (r_T^2 - (r_c + t_m)^2) \qquad (10)$$

It can be deduced that under certain conditions some regions may not receive any solute. A critical radius of tissue can be idenfied, $r_{critical}$ and defined as the distance beyond which no solute is present in the tissue.

$$\text{At } r = r_{critical}, \quad dC_T/dr = 0 \text{ and } C_T = 0 \qquad (11)$$

This can be solved for from Eq. [10] after replacing r_T with $r_{critical}$. The equation is nonlinear.

Cartesian Coordinates

Idealize Figure 1.0 in the cartesian coordinates and obtain the solution for the concentration of the solute in the tissue space. The governing equations for the concentration of the solute in the capillary and in the tissue can be written after taking the r in Figure 1.0 as x [Sharma, 2005],

$$-V\, dC/dz = 2/r_c\ K_0\ (C - C_T|_{rc+tm}) \qquad (12)$$

Considering the effects of diffusion in x direction only in the tissue and assuming a zeroth order reaction rate

$$D_{AB}\partial^2C_T/\partial x^2 = R_o \qquad (13)$$

Integrating, and substituting for the boundary conditions ;

$$x = x_c + t_m , \ C_T = C_T|_{xc+tm} \qquad (14)$$

$$x = x_T, \qquad dC_T/dx = 0 \tag{15}$$

$$-R_o/D_{AB}x_T = c_1 \tag{16}$$

$$C_T - C_T|_{xc+tm} = (R_o/2D_{AB})(x^2 - (x_c + t_m)^2) - R_o x_T/D_{AB}(x - (x_c + t_m)) \tag{17}$$

The variation of concentration as a function of z can be calculated by equating the change in solute concentration within the blood to the consumption of solute in the tissue space;

$$V A C_o - V A C = R_o z A_T \tag{18}$$

$$C = C_o - R_o z A_T/VA \tag{19}$$

Eq. [19] is combined with Eq. [17];

$$R_o A_T/A = 2/x_c \; K_0 \; (C - C_T|_{rc+tm}) \tag{20}$$

$$C_T|_{rc+tm} = C - K_0 x_c R_o A_T/2A \tag{21}$$

Therefore,

$$C_T - C_o = R_o z A_T/VA + K_0 x_c R_o A_T/2A + (R_o/2D_{AB})(x^2 - (x_c + t_m)^2) - R_o x_T/D_{AB}(x - (x_c + t_m)) \tag{22}$$

At a critical distance from the capillary wall the concentration in the solute will become zero. This can be solved for from the above equation.

At and beyond the critical distance,

$$dC_T/dx = 0 = C_T \tag{23}$$

Replacing x_T with $x_{critical}$,

$$0 = C_0 + R_o z A_T/VA + K_0 x_c R_o A_T/2A + (R_o/2D_{AB})(x^2 - (x_c + t_m)^2) - R_o x_{critical}/D_{AB}(x - (x_c + t_m)) \tag{24}$$

$$x_{critical}^2(-R_o/2D_{AB}) = C_0 + R_o z A_T/VA + K_0 x_c R_o A_T/2A - (R_o/2D_{AB})(x_c + t_m)^2 - R_o x_{critical}/D_{AB}(x - (x_c + t_m)) \tag{25}$$

The quadratic equation in $x_{critical}$ is then,

$$A x_{critical}^2 + B x_{critical} + C = 0 \tag{26}$$

Where,

$$A = -(R_o/2D_{AB}) \quad (27)$$

$$B = + (x_c + t_m)\, R_o/D_{AB} \quad (28)$$

$$C = C_0 + R_o z A_T/VA + K_0 x_c\, R_o A_T/2A - (R_o/2D_{AB}) x_c + t_m)^2 + R_o(x_c + t_m)/D_{AB} \quad (29)$$

When the solution of the quadratic expression for the critical distance in the tissue are real, and found to be less than the thickness of the tissue then the onset of zero concentration will occur prior to the periphery of the tissue. This zone can be seen as the anaroxic or oxygen depleted regions in the tissue.

Bibliography

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