Interior Point Solution of Multilevel QP Problems Arising in Embedded MPC Formulations

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Introduction

This study examines the use of an interior point strategy to solve multilevel quadratic programming problems that arise from including a closed-loop formulation of constrained model predictive control (MPC) within quadratic programming problems. We motivate the formulation through its application to the constraint back-off problem, although the strategy is applicable to several problem types.

Formulation

The difficulty of including constrained MPC within a dynamic optimization problem arises from the need to solve a quadratic optimization problem at every time step. This results in a multi-level problem since multiple optimization problems are embedded within a main optimization problem. A sequential approach can be used, in which the optimization algorithm selects values for the decision variables and the decision variables are then sent to a simulator. However, active constraints would give rise to derivative discontinuities, and if the process has unstable modes then the simulation might become unbounded and the optimizer might be unable to recover. An alternative is the simultaneous approach, in which the dynamic equations of the process are transformed into algebraic equations that are then included within the optimization problem. We have chosen to follow the latter approach.

In order to use the simultaneous approach, it is necessary to transform the quadratic problem that the model predictive controller solves at each step into a set of constraints that can be included within the original optimization problem. The approach used in this paper is to express the quadratic programming sub-problems solved at each time step by the model predictive controller via their Karush-Kuhn-Tucker (KKT) conditions and include these as constraints within the optimization problem.

The general form of a QP problem is:

$$\begin{array}{rcl} \min\limits_{x} & & \frac{1}{2}x^T H x + g^T x \\ \text{Subject to} & & \\ Ax & = & b \\ x & \geq & 0 \end{array}$$

The KKT conditions of this QP problem can be written as:

$$\begin{aligned} Hx - A^T\nu + g - u &= 0 \\ Ax &= b \\ u_i \nu_i &= 0 \\ (u, x) &\geq 0 \end{aligned}$$

From this it can be seen that the transformation of the MPC QP optimization problem into its KKT optimality conditions results in a set of constraints that is linear except for the complementarity constraints.

The inclusion of complementarity constraints within the main optimization problem leads to a mathematical program with complementarity constraints (MPCC). In general, these cannot be solved reliably with standard nonlinear programming solvers. The approach used in this paper is to use an interior point implementation tailored to treat complementarity constraints in the original (primal) problem in the same manner as those that arise from the KKT conditions, rather than treating them as general nonlinear constraints. This is the strategy used by IPOPT-C, an algorithm and software implementation developed by Raghunathan and Biegler (2003). An alternative strategy is to rewrite the complementarity constraints using binary variables, resulting in an mixed integer problem.

Both approaches have their merits. If the complementarity constraints are transformed using binary variables, then, if the remaining equations are linear and the objective is linear (or quadratic), the resulting problem is a mixed integer linear (quadratic) programming problem. The benefit of this formulation is that the optimum found by the solver is guaranteed, under mild conditions, to be the global optimum. The disadvantage of this form is that in the worst case the solution time can grow exponentially with problem size. On the other hand, we have no guarantee yet of global optimality for the solution found by the MPCC approach, but it is the aim of this study to compare the performance of this strategy to the MIP approach with regard to convergence to the global optimum and growth of the solution time with increasing problem size.

The back-off problem

The motivating example for this formulation is the constraint back-off problem, although the strategy is applicable to several problem types.

The steady-state economic optimum in process plants generally lies at the intersection of two or more constraints. However, in order to avoid constraint violations in the presence of unmeasured disturbances, it is necessary that the operating point be moved some distance from the constraints into the feasible region. The calculation of this constraint back-off may be posed as a dynamic optimization problem in which an economic criterion is optimized subject to the closed-loop trajectories satisfying input and output constraints. This problem may be extended to integrated design and control in which the decision space includes equipment design parameters, control structure and controller tuning.

The majority of studies in constraint back-off, as well as the more comprehensive integrated plant and control system design problems, have restricted the choice of control system to linear controllers. Here, the focus is on the constraint back-off problem with a quadratic objective using a constrained model predictive controller. This results in a multilevel quadratic programming problem.

Case study: CSTR

The first case study is of a single-input single-output continuously stirred, isothermal tank reactor, based on an example in Marlin (2000). The goal is to find an operating point that minimizes the gap between the steady state concentration of A and a target concentration. The controlled variable is the concentration of reactant A, the manipulated variable is the inlet feed flow rate and the disturbance is a change in the inlet concentration.

The target setpoint was a concentration of A of 0.85 mol/m^3. Path constraints were imposed to restrict the concentration of reactant A between a lower limit of 0.5 and an upper limit of 0.85 mol/m^3. The inlet flow rate was constrained to remain between a lower limit of 0.05 and an upper limit of 0.90 m^3/min

The tuning parameters used for the MPC controller can be found in Table 1.

Table 1: Tuning parameters for CSTR MPC.

Parameter	Description	Value
M	Manipulated variable moves	2
P	Prediction horizon	10
Q	Controlled variable weight	1
R	Manipulated variable weight	1

The time horizon for the problem was 40 time steps and the disturbance step change was 0.69 m^3/min.

The case study was solved using both the MIQP and MPCC formulations. The optimal steady state setpoint was found to be 0.692 and Figure 1 shows the trajectory obtained. To study the effect on solution time as the problem size is increased, the value of M, and hence the number of complementarity pairs and integer variables required, was increased. The solution times for both formulations are reported in Table 2.

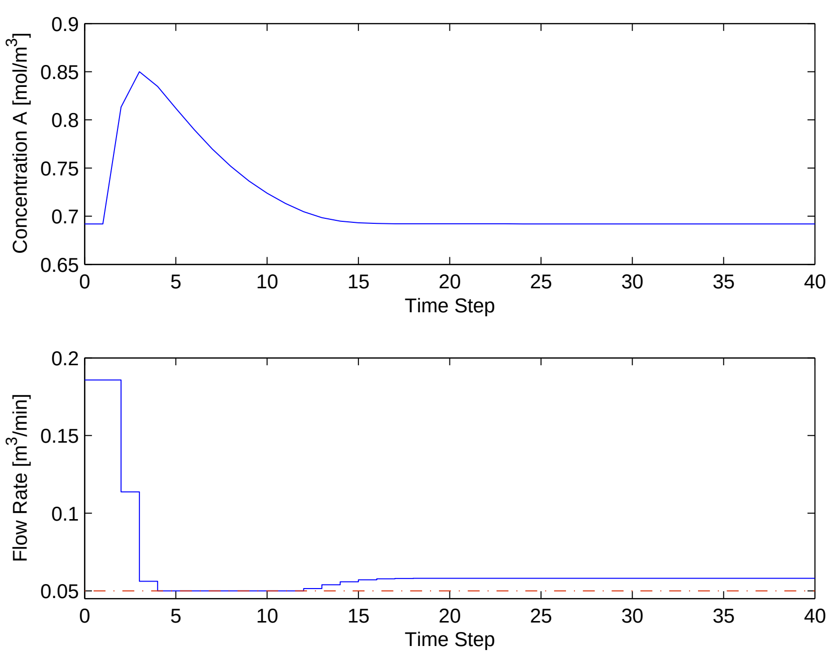

Figure 1: Optimal trajectory obtained for the CSTR problem for a disturbance of 0.69 m^3/min.

Using IPOPT-C to solve the quadratic programming problem with complementarity constraints returned the same objective function value and trajectory as using CPLEX 9 to solve the MIQP problem when CPLEX was able to return a solution.

Table 2: CPU Time needed to solve the CSTR example using CPLEX and IPOPT for increasing M.

M	Complementarities / Integer Variables	CPLEX CPU Time (s)	IPOPT-C CPU Time (s)
2	162	0.1780	0.3250
3	244	0.6929	0.3779
4	326	> 4hrs	1.9697

Case study: FCCU

The next case study was a fluidized catalytic cracking unit (FCCU). The state-space model was derived from the transfer function model identified by Ansari and Tade (2000).

The controlled variables are the oxygen concentration in the outlet flue gas from the regenerator and the regenerator bed temperature. The manipulated variables are the inlet air flow rate to the regenerator and the riser outlet temperature. The disturbance to the process was a change in the inlet feed flow rate.

In addition, the oxygen concentration had to remain between 0.2 and 1.2 vol % and the regenerator temperature had to remain between 705 and 735 ^{o}C. The inlet air flow rate was required to remain between 140 and 155 ton/hr and the riser outlet temperature had to stay between 515 and 535 ^{o}C.

The tuning parameters for the MPC controller for the FCCU problem can be found in Table 3.

Table 3: Tuning parameters for FCCU MPC.

Parameter	Description	Value
M	Manipulated variable moves	2
P	Prediction horizon	10
Q_{O2}	O_2 weight	1
Q_{regen}	Regenerator weight	1
R_{air}	Air flow rate weight	0.1
R_{riser}	Riser weight	1

The target for the oxygen concentration was 1.2 vol % and the regenerator bed temperature target was 735 ^{o}C. The time horizon for the problem was 26 time steps and the disturbance was a step change of -0.95 m^3/hr to the inlet feed flow rate.

The case study was solved using both the MIQP and MPCC formulations. The optimal set-points were 0.602 vol % and 734.999 ^{o}C for the oxygen concentration and regenerator bed temperature respectively. The controlled and manipulated variable trajectories for the FCCU problem can be found in Figures 2 and 3.

We also tested how the solution time increased as the problem size increased. To do this, the FCCU problem was re-solved with the original time-step duration halved and then quartered so that the number of integer variables and complementary pairs were doubled and quadrupled respectively. The solution times when using the MIQP or MPCC formulations for the original and expanded problems can be found in Table 4.

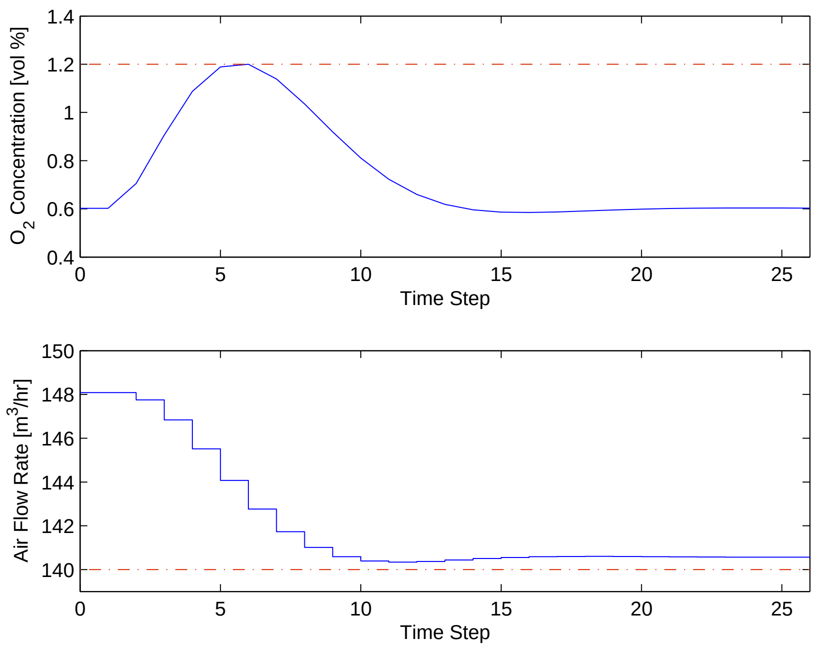

Figure 2: Oxygen and air flow rate trajectories obtained for the FCCU problem for a disturbance of -0.95 m^3/hr.

Table 4: CPU Time needed to solve the FCCU example using CPLEX and IPOPT for increasing time steps.

Time Steps	Complementarities / Integer Variables	CPLEX CPU Time (s)	IPOPT-C CPU Time (s)
26	216	3.9294	2.7386
52	424	406.605	8.079
104	840	$>$ 4hrs	14.9327

In all cases where the MIQP was able to solve, IPOPT-C reported the same objective function values and trajectories for the QP with complementarity constraints.

Conclusion

It was demonstrated that using an interior point approach to solve the multilevel quadratic programming problem with complementarity constraints is an efficient alternative to using the MIQP formulation, especially as the number of binary variables increases. In addition, in all cases where the MIQP was able to solve, IPOPT-C reported the same objective function value. Since a global optimal solution is guaranteed for the MIQP formulation, this provides anecdotal evidence that IPOPT-C is a reliable and efficient method of finding the global optimum for a class of QP problems with complementarity constraints.

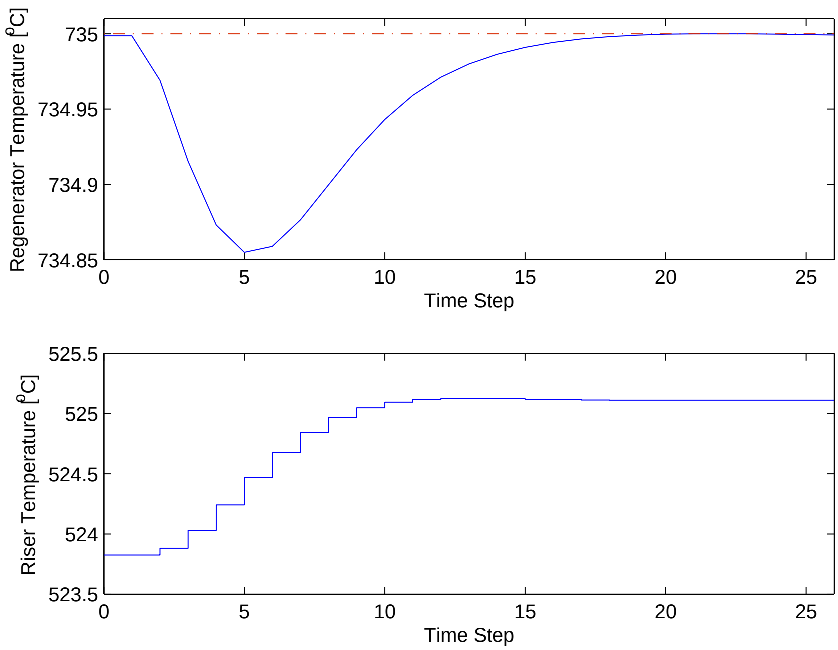

Figure 3: Regenerator and riser trajectories obtained for the FCCU problem for a disturbance of -0.95 m^3/hr.

References *

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