Improving Mixed Integer Linear Programming Formulations

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Abstract

This paper addresses the impact of first level Reformulation Linearization Technique (RLT) and Reduced Reformulation Linearization Technique (RRLT) on Mixed Integer Linear Programs (MILP) with big-M constraints. Sherali and Adams (1994) propose a RLT that generates a hierarchy of relaxations spanning from the ordinary linear programming relaxation to the nth level convex hull relaxation of feasible convex set for mixed integer zero-one programs, where n is the number of binary variables. Later, Sherali et al. (2000) show that there exists a first level representation having nearly half the RLT constraints that yield the same lower bound. The motivation of this paper is based on the computational success of the first level RLT and RRLT on various classes of problems formulated as mixed integer 0-1 programs. In contrast to Sherali and Adams (1994), who multiply all constraints with all binaries present in a formulation, we multiply only Big-M constraints with binaries present within these constraints. We solve a variety of problems such as flow-shop, job-shop and strip-packing to illustrate the performance of the modified RLT and RRLT.

Introduction

MILP formulations find their application in a wide variety of chemical as well as other engineering problems. While the application of MILP is vast, the challenges for the researchers to provide a faster solution still continue to exist. Modeling plays a crucial role on the solution efficiency of any problem. Several modeling approaches (event-based, sequence-based, slot-based etc.) do exist in the literature for MILP formulations. For a given problem, different approaches can have different performance efficiencies on the problem solution. However, relaxed MILP value is definitely a good indicator to clearly show which approach yields a tight formulation (i.e. feasible region is small).

In an attempt to tighten the MILP relaxations, several researchers have worked on various relaxation techniques. Balas (1993), Sherali and Adams (1994) and Lovasz and Schrijver (1991) developed hierarchies of relaxation for MILP formulations. Balas (1983) showed how a hierarchy of relaxations spanning from the linear programming relaxation to convex hull relaxation could be constructed in an inductive fashion for MILPs. Sherali and Adams (1994) proposed a RLT that generates a hierarchy of relaxations for 0-1 mixed integer problems (MIP). The proposed technique first converts the problem into a non-linear, polynomial mixed-integer zero-one problem by multiplying the constraints with some suitable d-degree, d = {0,1,….,n}, polynomial factors involving the n binary variables. Then, it linearizes the resulting nonlinear program through suitable redefinitions of variables. As d varies from 0 to n, we obtain a hierarchy of relaxations spanning from the ordinary linear programming relaxation to the convex hull of feasible solutions. The facets of the convex hull of feasible solutions in terms of the original problem variables are obtained through standard projection operation. Lovasz and Schrijver (1991) generated yet another hierarchy of n relaxations by transforming the representations in the original n variables at each stage to a suitable representation in an n^2 variable space.

Grossmann and Lee (2003) showed that the feasible region of the relaxation resulting from the convex hull reformulation is always as tight as or tighter than that of the big-M reformulation. The tightness of the relaxed feasible region, which is reflected in the lower bound (for minimization problem) is an important criterion when solving the original MILP, as tighter relaxed feasible regions reduce the search space of the solution algorithm. However, the representation of the convex hull requires the introduction of new disaggregated variables and additional constraints that can greatly increase the size of the problem and thus increase in computation time. In order to overcome the aforementioned problem, recently Sawaya and Grossmann (2004) proposed a cutting plane approach which is based on converting the generalized disjunctive program (GDP) into an equivalent big-M reformulation which is successively strengthened by cuts generated from an LP or QP separation problem.

In this paper, we first describe the Big-M relaxation of a disjunction involving convex linear or non-linear inequalities. Based on the work of Sherali and Adams, we examine the first-level representation generated by the RLT for Big-M constraints in Big-M relaxation for mixed integer 0-1 programs. The first-level RLT relaxation is constructed by multiplying the constraints with the bound factors (a binary and its complement) and then linearizing the problem by substituting a new variable for each non-linear product term thus produced. While Sherali and Adams (1994) multiply all constraints with all binaries present in the model, we multiply only the Big-M constraints with the binaries present within these constraints. In the following, we discuss the first level representation of RLT and RRLT for MILP formulations and then we demonstrate the technique on different problems. Finally, we discuss the computational impact of these techniques on a variety of problem instances.

First Level Representation of RLT and RRLT for MILPs

Consider the following MILP problem in the form of disjunctions (Sawaya and Grossmann, 2005):

$$\text{Min} = f(x) + \sum_{k \in SD} c_k$$

$$\text{s.t.} \quad Rx \le r$$

$$\bigvee_{i \in D_k} \begin{bmatrix} Y_{ik} \\ A_{ik} x \le a_{ik} \\ c_k = \gamma_{ik} \end{bmatrix} k \in SD$$

$$\Omega(Y) = \text{True}$$

$$x \in R_+^n,\ y_{ik} \in \{\text{True, False}\} \qquad i \in D_k,\ k \in SD \qquad \text{(A)}$$

where SD is the set of disjunctions and $A_{ik}x \le a_{ik}$ are linear inequalities representing the constraints of the problem, Y_{ik} are boolean variables to represent discrete decisions, γ_{ik} are fixed charges and $c_k \in R_+^n$ are continuous functions. A disjunction $k \in SD$ is composed of various disjuncts $i \in D_k$, each containing a set of linear equations and/or inequalities representing the constraints of this problem, connected together by the logical OR operator (V) that ensures that only one disjunct is true. There are many ways to reformulate the linear GDP problem (LGDP) to a Mixed Integer Program (MIP). The two most common alternatives termed as big-M reformulation (BMR) and convex hull reformulation (CHR).

Big-M Reformulation:
The LGDP can be reformulated as a MIP by replacing the Boolean variables Y_{jk} by binary variables y_{jk} and using Big-M constraints. The logic constraints $\Omega(Y)$ are converted into linear inequalities (Williams, 1985), which leads to following reformulation.

$$Min Z = \sum_{k \in SD} \sum_{i \in D_k} \gamma_{ik} y_{ik} + q' x$$

$$\begin{aligned}
\text{s.t.} \quad & Rx \le r \\
& A_{ik} x \le a_{ik} + M_{ik}(1 - y_{ik}) \qquad i \in D_k,\ k \in SD \qquad (1) \\
& \sum_{i \in D_k} y_{ik} = 1 \ , \qquad k \in SD \\
& Qy \le q \\
& x \in R_+^n \ ,\ y_{ik} \in \{0,1\} \qquad i \in D_k,\ k \in SD
\end{aligned}$$

where y_{ik} are 0-1 variables corresponding to Boolean variables Y_{ik}.
M_{ik} are big-M parameters and the tightest value of M_{ik} can be calculated from
$Mik = \max \{A_{ik}(x) - a_{ik} / x^l \le x \le x^u\}$.
The logic constraints $\Omega(Y)$ are written in the form of conjunctive normal form from which the inequalities $Qy \le q$ are derived.

Convex Hull Reformulation:
In order to obtain the CHR , problem LGDP is transferred into an MIP by replacing the Boolean variables Y_{ik} by binary variables y_{ik} and disaggregating the continuous variables $x \in R_+^n$ into new variables $v \in R_+^n$. Using the convex hull constraints for each disjunction (Balas 1998; Raman and Grossmann, 1994), this leads to the following reformulation (Raman and Grossmann, 1994):

$$Min Z = \sum_{k \in SD} \sum_{i \in D_k} \gamma_{ik} y_{ik} + q' x$$

$$\begin{aligned}
\text{s.t.} \quad & Rx \le r \\
& A_{ik} v_{ik} \le a_{ik} y_{ik} \qquad i \in D_k,\ k \in SD \\
& x = \sum_{i \in D_k} v_{ik} \qquad x \in R^n \\
& 0 \le v_{ik} \le y_{ik} U \qquad i \in D_k \\
& \sum_{i \in D_k} y_{ik} = 1 \qquad \forall k \in SD \\
& Qy \le q \qquad (B) \\
& x, v \in R_+^n \ ,\ y_{ik} \in \{0,1\} \quad i \in D_k,\ k \in SD
\end{aligned}$$

where y_{ik} are relaxed 0-1 variables and v_i are disaggregated variables for continuous variables x. Eq. (B) defines a convex set in (x,v,y) space provided the inequalities $A_{ik}(x) \le a_{ik}$, $i \in D_k$ are convex and bounded. The convex hull in equation (B) can be provided to be tighter than or at least as tight as the Big-M relaxation.

The new variables v$\in R^n$ in CHR are the disaggregated variables, while the parameters U serve as their upper bounds and are chosen so as to match the upper bounds on the continuous variables x$\in R^n$. The advantage of the convex hull formulation is that it provides a tighter lower bound, which results in the reduction of search effort for the branch and bound algorithm. This reduction in feasible region is very expensive due to the tremendous increase in the number of variables and constraints in the original problem. The big-M formulation, on the other hand, is more convenient to use since the problem size is smaller compared to the convex hull formulation. The poorer lower bound of big-M, however, can demand more CPU time than the convex hull. Thus, there exists a clear trade-off between the tighter lower bound and the problem size.

RLT for Big-M Reformulation (RLT-BMR):

Sherali and Adams(1994) described the RLT process for constructing a relaxation X_d of the feasible region X defined in (A) corresponding to any level d$\in\{0,1,\ldots,n\}$, where n being the number of binary variables. For d = 0, the relaxation X_0 is simply the LP relaxation obtained by deleting the integrality restrictions on the binary variables. To construct the relaxation for any level d$\in\{1,\ldots,n\}$, they consider the bound factors $y_{ik}\geq 0$ and $(1-y_{ik})\geq 0$ for $i\in D_k$, $k\in SD$ and compose the bound factors products of degree d by selecting some d distinct variables from the set y_{ik}, $i\in D_k$, $k\in SD$ and by using either the bound factor y_{ik} or $(1-y_{ik})$ for each selected variable in a product of these terms. Then they linearize the resulting relaxation by substituting the appropriate variables say w and v. Denote the projection of X_d onto space of the original variables (x, y_{ik}) by
$X_{p0}\equiv X_0\supseteq X_{p1}\supseteq X_{p2}\supseteq\ldots\ldots\supseteq X_{pn}\equiv \text{conv}(X)$

where $X_{p0}\equiv X_0$ (for d=0) denoted the ordinary linear programming relaxation and conv(X) denotes the convex hull of X.

In this section we consider the first level, d=1, RLT for BMR by multiplying the disjunctive constraints of big-M relaxation with the binaries y_{ik} present in the particular constraint and linearizing the resulting problem by replacing y_{ik}^2 by y_{ik} and $p_{ik} = xy_{ik}$, i$\in D_k$, k$\in$SD. We need to multiply only the big-M constraints with the binaries present within the constraints else there will be increase in lot of constraints and variables which may increase the computation time.

We obtain the first level RLT reformulation for big-M given below:

(RLT-BMR) $$MinZ=\sum_{k\in SD}\sum_{i\in D_k}\gamma_{ik}y_{ik}+q'x$$

s.t. $Rx\leq r$

$$A_{ik}p_{ik}\leq a_{ik}y_{ik} \qquad i\in D_k,\ k\in SD \tag{1a}$$

$$A_{ik}(x-p_{ik})\leq a_{ik}(1-y_{ik})+M_{ik}(1-y_{ik}) \quad i\in D_k,\ k\in SD \tag{1b}$$

$$\sum_{i\in D_k}y_{ik}=1 \qquad \forall k\in SD$$

$Qy\leq q$

$x \in R_+^n$, $y_{ik} \in \{0,1\}$ $\quad\quad i \in D_k,\ k \in SD$

$0 \le p_{ik} \le (UB_{ik}) y_{ik}$ $\quad\quad i \in D_k, k \in SD$

In contrast to Sherali and Adams (1994), we multiply only with the binary present within the constraint, thus we do not have any variable defining the product of two different binaries.

Theorem: Let RLT-BMR be feasible and define RLT-BMR-1 as the formulation in which the original big-M constraints (1) have been added. Then there exists a dual optimal solution to RLT-BMR-1 such that for each $\mathrm{i} \in \mathrm{D_k},\ \mathrm{k} \in SD$, the dual variable associated with at least one of the (1a) or (1b) for each $\mathrm{i} \in \mathrm{D_k},\ \mathrm{k} \in SD$ is zero. Thus we can delete such constraints having zero dual multiplier from RLT-BMR-1 which results in reduced first level RLT big-M relaxation having the same lower bound as RLT-BMR.

Remark: Theorem asserts that if we add the original big-M constraints (1) to RLT-BMR, then we can delete some of the constraints from (1a) and (1b), which have zero dual multipliers and yet retain the same lower bound of RLT-BMR. For the case of single binary the reduced first level RLT-BMR is of the same size as of RLT-BMR and yields the convex hull of feasible solution to MIP.

Towards this end, we find that whenever the coefficient of binary y_{ik} in the constraint (1) is positive, then we should try retaining (1b) and we should retain (1a), whenever coefficient of binary is negative or zero.

RRLT for Big-M Reformulation (RRLT-BMR):

We construct RRLT-BMR-1 by appropriately deleting one of the each pair of the constraints (1a) and (1b) $\forall\, \mathrm{i} \in \mathrm{D_k},\ \mathrm{k} \in SD$ and adding the original constraints (1) $\forall\, \mathrm{i} \in \mathrm{D_k},\ \mathrm{k} \in SD$.

(RRLT-BMR-1) $MinZ = \sum_{k \in SD} \sum_{i \in D_k} \gamma_{ik} y_{ik} + q'x$

s.t. $\mathrm{Rx} \le \mathrm{r}$

$\mathrm{A_{ik}(x)} \le \mathrm{a_{ik}} + \mathrm{M_{ik}}(1 - \mathrm{y_{ik}})$ $\quad\quad \mathrm{i} \in \mathrm{D_k},\ \mathrm{k} \in SD$

$\mathrm{A_{ik} p_{ik}} \le \mathrm{a_{ik} y_{ik}}$ $\quad\quad \mathrm{i} \in \mathrm{D_k},\ \mathrm{k} \in SD$ $\quad\quad$ (1a)

$\sum_{i \in D_k} y_{ik} = 1 \quad\quad \forall k \in SD$

$\mathrm{Qy} \le \mathrm{q}$

$x \in R_+^n$, $y_{ik} \in \{0,1\}$ $\quad\quad i \in D_k,\ k \in SD$

$0 \le p_{ik} \le (UB_{ik}) y_{ik}$ $\quad\quad i \in D_k, k \in SD$

Remark: The optimal objective value for RRLT-BMR-1 is same as of RLT-BMR-1.

Now, we demonstrate the procedures to formulate the first level RLT and RRLT technique on big-M reformulations of flow-shop problems. The same procedures apply to strip-packing and job-shop problems.

Flow shop problem (FS):

In a flow shop problem, there are i , i' = 1, 2, … , I products to be processed on j = 1, 2, …, J units placed serially according to the recipe that is identical for all I products. A flow shop with I products and J units with only one unit in each stage is characterized as [$I \times J$] flow shop. We consider a flow shop with No intermediate storage(NIS) configuration in which a product up on completion can wait in a unit for an unlimited period of time (Unlimited storage(UW) policy) until the subsequent unit becomes free. The flow shop model without setup considerations and NIS policy is given as follows:

Min ms

s.t. $ms \geq C_{ij} \qquad j = J$

$$C_{ij} \geq C_{i(j-1)} + t_{ij} \tag{2}$$

$$\begin{bmatrix} Y_{ij}^1 \\ C_{i(j+1)} \leq C_{i'j} \quad \forall i < i', j \neq J \\ C_{ij} + t_{i'j} \leq C_{i'j} \quad \forall i < i', j = 1 \end{bmatrix} \vee \begin{bmatrix} Y_{ij}^2 \\ C_{i'(j+1)} \leq C_{ij} \quad \forall i < i', j \neq J \\ C_{i'j} + t_{ij} \leq C_{ij} \quad \forall i < i', j = 1 \end{bmatrix} \tag{3}$$

ms, $t_{ij} \in R_+^1$, $Y_{ij}^1, Y_{ij}^2 \in \{True, False\}$

where C_{ij} is the completion time, t_{ij} is the processing time of a product i in the unit j, ms is the makespan to be minimized. Eq. (2) ensures that the completion time of a product in a unit exceeds its completion time in the previous unit and its processing time in the unit under consideration. Each clause in disjunction (3) forms a pair of disjunctive constraints which govern the NIS Storage policy. The disjunctive constraints behave in such a way that if one binds ensuring the precedence relation, the other relaxes with the help of a large penalty H. Second inequality in each clause forms another pair of disjunctive constraints which ensure the product transitions in the first unit. It is to be noted that it is written for only the first unit as the product transitions in all other units are handled by first inequality in each clause.

BMR for FS:

Min ms

$ms \geq C_{ij} \qquad j = J$

$C_{ij} \geq C_{i(j-1)} + t_{ij}$

$C_{i'j} \geq C_{i(j+1)} - H(1 - Y_{ii'}^1) \qquad \forall i < i', j \neq J$

$C_{ij} \geq C_{i'(j+1)} - H(1 - Y_{ii'}^2) \qquad \forall i < i', j \neq J$

$C_{i'j} \geq C_{ij} + Y_{ii'}^1 . t_{i'j} - H(1 - Y_{ii'}^1) \qquad \forall i < i', j = 1$

$$C_{ij} \geq C_{i'j} + Y_{ii'}^{2} t_{ij} - H.(1 - Y_{ii'}^{2}) \quad \forall i < i', j = 1$$

$$Y_{ii'}^{1} + Y_{i'i}^{2} = 1$$

ms, $t_{ij} \in R_{+}^{1}$, $Y_{ij}^{1}, Y_{ij}^{2} \in \{0,1\}$

CHR for FS:

Min ms

$$ms \geq C_{ij} \qquad j = J$$

$$C_{ij} \geq C_{i(j-1)} + t_{ij} \qquad \forall i \in I, j \in J$$

$$v_{iii'(j+1)}^{1} - v_{i'ii'j}^{1} \leq 0 \quad \forall i < i',\ j \neq J$$

$$v_{i'ii'(j+1)}^{2} - v_{iii'j}^{2} \leq 0 \quad \forall i < i',\ j \neq J$$

$$v_{iii'j}^{1} + t_{i'j} Y_{ii'}^{1} - v_{i'ii'j}^{1} \leq 0 \quad \forall i < i', j = 1$$

$$v_{i'ii'j}^{2} + t_{ij} Y_{ii'}^{2} - v_{iii'j}^{2} \leq 0 \quad \forall i < i', j = 1$$

$$Y_{ii'}^{1} + Y_{i'i}^{2} = 1$$

$$v_{nii'j}^{1} \leq H Y_{ij}^{1} \qquad \forall i < i', n = i \vee i'$$

$$v_{nii'j}^{2} \leq H Y_{ij}^{2} \qquad \forall i < i', n = i \vee i'$$

ms, $t_{ij} \in R_{+}^{1}$, $Y_{ij}^{1}, Y_{ij}^{2} \in \{0,1\} \quad \forall i \in I, j \in J$

RLT for BMR of FS:

Min ms

$$ms \geq C_{ij} \qquad j = J$$

$$C_{ij} \geq C_{i(j-1)} + t_{ij}$$

$$C_{i'j} \geq C_{i(j+1)} - H(1 - Y_{ii'}^{1}) \qquad \forall i < i', j \neq J$$

$$C_{ij} \geq C_{i'(j+1)} - H(1 - Y_{ii'}^{2}) \qquad \forall i < i', j \neq J$$

$$C_{i'j} \geq C_{ij} + Y_{ii'}^{1}.t_{i'j} - H(1 - Y_{ii'}^{1}) \quad \forall i < i', j = 1$$

$$C_{ij} \geq C_{i'j} + Y_{ii'}^{2} t_{ij} - H.(1 - Y_{ii'}^{2}) \quad \forall i < i', j = 1$$

$$Y_{ii'}^{1} + Y_{i'i}^{2} = 1$$

$$p_{ii'j}^{1} \geq q_{ii'j+1}^{1} \qquad \forall i < i', j \neq J$$

$$q^2_{ii'j} \geq p^2_{ii'j+1} \qquad \forall i < i', j \neq J$$

$$p^1_{ii'j} \geq q^1_{ii'j+1} + t_{i'j} Y^1_{ij} \qquad \forall i < i', j \neq J$$

$$q^2_{ii'j} \geq p^2_{ii'j+1} + t_{ij} Y^2_{ij} \qquad \forall i < i', j \neq J$$

$$C_{i'j} - p^1_{ii'j} \geq C_{i(j+1)} - q^1_{ii'(j+1)} - H(1 - Y^1_{ii'}) \qquad \forall i < i', j \neq J$$

$$C_{ij} - q^2_{ii'j} \geq C_{i'(j+1)} - p^2_{ii'(j+1)} - H(1 - Y^2_{ii'}) \qquad \forall i < i', j \neq J$$

$$C_{i'j} - p^1_{ii'j} \geq C_{ij} - q^1_{ii'j} - H(1 - Y^1_{ii'}) \qquad \forall i < i', j = 1$$

$$C_{ij} - q^2_{ii'j} \geq C_{i'j} - p^2_{ii'j} - H(1 - Y^1_{ii'}) \qquad \forall i < i', j = 1$$

$$p^d_{ii'j} \leq HY^d_{ij} \qquad \forall i < i', j \neq J \qquad , \text{d} = 1, 2$$

$$q^d_{ii'(j+1)} \leq HY^d_{ij} \quad \forall i < i', j \neq J \qquad , \text{d} = 1, 2$$

ms, $t_{ij} \in R^1_+$, $Y^1_{ij}, Y^2_{ij} \in \{0,1\} \quad \forall i \in I, j \in J$

RRLT for BMR of FS:

Min ms

$$ms \geq C_{ij} \qquad j = J$$

$$C_{ij} \geq C_{i(j-1)} + t_{ij}$$

$$C_{i'j} \geq C_{i(j+1)} - H(1 - Y^1_{ii'}) \qquad \forall i < i', j \neq J$$

$$C_{ij} \geq C_{i'(j+1)} - H(1 - Y^2_{ii'}) \qquad \forall i < i', j \neq J$$

$$C_{i'j} \geq C_{ij} + Y^1_{ii'}.t_{i'j} - H(1 - Y^1_{ii'}) \qquad \forall i < i', j = 1$$

$$C_{ij} \geq C_{i'j} + Y^2_{ii'} t_{ij} - H.(1 - Y^2_{ii'}) \quad \forall i < i', j = 1$$

$$Y^1_{ii'} + Y^2_{i'i} = 1$$

$$p^1_{ii'j} \geq q^1_{ii'j+1} \qquad \forall i < i', j \neq J$$

$$p^2_{ii'j} \geq q^2_{ii'j+1} \qquad \forall i < i', j \neq J$$

$$p^1_{ii'j} \geq q^1_{ii'j+1} + t_{i'j} Y^1_{ij} \qquad \forall i < i', j \neq J$$

$$p^2_{ii'j} \geq q^2_{ii'j+1} + t_{ij} Y^2_{ij} \qquad \forall i < i', j \neq J$$

$$p^d_{ii'j} \leq HY^d_{ij} \qquad \forall i < i', j \neq J \qquad , \text{d} = 1, 2$$

$$q^{d}_{ii'(j+1)} \leq HY^{d}_{ij} \quad \forall i < i', j \neq J \qquad , \mathrm{d} = 1, 2$$

$$\mathrm{ms},\ t_{ij} \in R^{1}_{+}\ ,\ \ Y^{1}_{ij}, Y^{2}_{ij} \in \{0,1\} \quad \forall i \in I, j \in J$$

As discussed above we remove the block of constraints obtained by multiplying (1-Y_{ik}) for RRLT.

Examples

We solved each of the three problems (flow-shop, zero-wait job-shop and strip-packing) using different data sets. From the computational results, we observe a significant reduction in the solution time for most of the problems. For strip-packing and job-shop problems, RRLT performed better than RLT. On the other hand, RLT technique consistently improved the solution time for all the instances of flow-shop problem significantly (almost 50% reduction).

CONCLUSION

In this paper, we proposed a modified first level representation of RLT and RRLT for MILP formulations with big-M constraints. From the rigorous computational tests on various problems, we observed that the techniques could significantly impact the solution times up to 50%. It is clear form the above results that the first level RLT and RRLT improve the computational performance of MILP formulations.

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